Mathematical
Inequalities
& Applications
Volume 2, Number 1 (1999), 151–163

TRIANGLES FROM AREAS

ZVONKO ČERIN

(*communicated by V. Volenec*)

Abstract. We consider the problem to determine for which central points X of the triangle ABC will the areas of triangles BCX, CAX, and ABX be sides of a triangle. We shall prove that only nine out of hundred and one central points from Kimberling's list have this property. The algebraic method of proof for this result is also used to obtain some new examples of three areas that are sides of a triangle and are build from elements of a given triangle.

1. Introduction

The present article is looking for ways of associating to a triangle ABC a point X of the plane such that areas of triangles BCX, CAX, and ABX are always sides of a triangle.

The centroid G and the incenter I are easy examples of such points X. Indeed, the triangles BCG, CAG, and ABG having equal areas are sides of an equilateral triangle while the areas of the triangles BCI, CAI, and ABI being proportional to the sides a, b, and c are obviously sides of a triangle.

Since G and I are just two of central points of a triangle ABC listed in Table 1 of [1], we can state a problem that we completely answer in this paper.

PROBLEM. *For what natural numbers i less than 102 will the central point X_i of the triangle ABC from the Kimberling's list have the property that areas of the triangles BCX_i, CAX_i, and ABX_i are sides of a triangle?*

We shall get the solution of this problem with an entirely algebraic proof in an analytic approach. Our main result is the following theorem.

THEOREM 1. *From 101 centres X_i of the triangle ABC from Kimberling's Table 1, only values 1, 2, 9, 10, 37, 38, 39, 45, and 86 of the index i have the property that areas of triangles BCX_i, CAX_i, and ABX_i are sides of a triangle regardless of the shape of ABC.*

Mathematics subject classification (1991): 51M16, 52A40, 51N20, 51M04.
Key words and phrases: triangle, area, triangular triple, inequality, central point, pedal, antipedal, areal coordinates.

Paper MIA-02-13

This result is related to the basic problems in triangle geometry which asks when three given segments are sides of a triangle. The opening chapter of the book *Recent Advances in Geometric Inequalities* by Mitrinović, Pečarić, and Volenec [3] gives an extensive survey of results on this question. Our goal is to further extend the list of these contributions.

The present article considers first the problem to determine the locus of points X in the plane of a triangle ABC such that the triangles BCX, CAX, and ABX are sides of a triangle.

The sidelines B_mC_m, C_mA_m, and A_mB_m of the complementary or medial triangle $A_mB_mC_m$ (the cevian triangle of the centroid) of ABC provide the solution to this problem. Indeed, they partition the plane into seven convex regions and the union of four of them is the required locus. In particular, the interior of the complementary triangle is precisely the set of points X of the triangle ABC whose areal coordinates are sides of a triangle.

Then we study when will the triangle from the areas of the triangles BCX, CAX, and ABX for a point X in the interior of the complementary triangle be acute, right, or obtuse. The answer utilises three interesting hyperbolas.

Next is a search for central points X from the Table 1 in [1] which have the property that the areas of BCX, CAX, and ABX are sides of a triangle regardless of the shape of the triangle ABC. In other words, we want to see which central points are always inside the complementary triangle. The answer gives the above Theorem 1.

Our method of proof could be applied to find several new ways of building triangles from elements of the base triangle.

2. Preliminaries

For an expression f, let $[f]$ denote a triple $(f, \varphi(f), \psi(f))$, where $\varphi(f)$ and $\psi(f)$ are cyclic permutations of f. For example, if $f = \sin A$ and $g = b + c$, then

$$[f] = (\sin A, \sin B, \sin C) \qquad \text{and} \qquad [g] = (b+c, c+a, a+b).$$

Let us call a triple $[a]$ of real numbers *triangular* provided a, b, and c are sides of a triangle. The letter Ω is reserved for the set of all triangular triples.

Let T denote a function that maps each triple $[a]$ of real numbers to a number

$$2\,a^2\,b^2 + 2\,a^2\,c^2 + 2\,b^2\,c^2 - a^4 - b^4 - c^4.$$

Since $T([a]) = (a+b+c)(b+c-a)(a-b+c)(a+b-c)$, it is clear that $[a] \in \Omega$ if and only if $T([a]) > 0$. Let T_i be a short notation for $T(|BCX_i|)$, where X_i is the i-th central point of ABC and $i = 1, \ldots, 101$ and $|BCX_i|$ denotes the (oriented) area of the triangle BCX_i.

3. Triangles from areas

Let (α, β, γ) be the actual areal or barycentric coordinates of a point P in the plane of the triangle ABC. Then $|\alpha|\,S$, $|\beta|\,S$, $|\gamma|\,S$ are areas of triangles BCP, CAP, and ABP, where S denotes the area of ABC. These products are sides of a triangle if and only if $T([\,|\alpha|\,S\,]) > 0$. However, this inequality is easily seen to be equivalent with the inequality

$$(\alpha+\beta+\gamma)(-\alpha+\beta+\gamma)(\alpha-\beta+\gamma)(\alpha+\beta-\gamma) > 0$$

by looking separately at seven regions determined by the sidelines of ABC. But,

$$-\alpha+\beta+\gamma = 0$$

is the equation of the sideline B_mC_m of the complementary triangle $A_mB_mC_m$ on midpoints of sides while $\alpha-\beta+\gamma=0$ and $\alpha+\beta-\gamma=0$ are the equations of C_mA_m and A_mB_m. Since $\alpha+\beta+\gamma=1$, the lines B_mC_m, C_mA_m, and A_mB_m partition the plane into seven convex regions and on four of them which do not contain the vertices of ABC the above inequality holds.

In particular, we proved the following result.

COROLLARY 1. *The areals of a point P with respect to a triangle ABC are sides of a triangle if and only if P lies in the interior of the complementary triangle of ABC.*

4. Shape of triangles from areals

We can now easily describe conditions on a point P in the interior of the complementary triangle of ABC such that the areals α, β, and γ of P are sides of either an acute, right, or obtuse triangle. Indeed, it is well-known that the triangle ABC is acute, right, or obtuse if and only if the product $(-a^2+b^2+c^2)(a^2-b^2+c^2)(a^2+b^2-c^2)$ is positive, zero, or negative. In our situation, this product is

$$(-\alpha^2+\beta^2+\gamma^2)(\alpha^2-\beta^2+\gamma^2)(\alpha^2+\beta^2-\gamma^2).$$

But, $-\alpha^2+\beta^2+\gamma^2=0$ is the equation of a hyperbola Γ_a through the vertices B_m and C_m of the complementary triangle with centre at the vertex A_a of the anticomplementary triangle $A_aB_aC_a$ of ABC and sidelines A_aB_a and C_aA_a as asymptotes. Hence, for a point P in the interior of the complementary triangle, the areals of P are sides of either an acute, right, or obtuse triangle provided P lies either in the "triangular" region determined by hyperbolas Γ_a, Γ_b, and Γ_c, the point P lies on their union, or P is in one of the three remaining petals.

5. Placement of ABC

We shall position the triangle ABC in the following fashion with respect to the rectangular coordinate system in order to simplify our calculations. The vertex A is the origin with coordinates $(0, 0)$, the vertex B is on the x-axis and has coordinates $(r h, 0)$, and the vertex C has coordinates $(g q r/k, 2f g r/k)$, where $h = f + g$, $k = f g - 1$, $\varphi = f^2 + 1$, $u = f^2 - 1$, $\psi = g^2 + 1$, $v = g^2 - 1$, $\Phi = f^4 + 1$, and $\Psi = g^4 + 1$. The three parameters r, f, and g are the inradius and the cotangents of half of angles at vertices A and B. Without loss of generality, we can assume that both f and g are larger than 1 (i. e., that angles A and B are acute).

Nice features of this placement are that all central points from Table 1 in [1] have rational functions in f, g, and r as coordinates and that we can easily switch from f, g, and r to side lengths a, b, and c and back with substitutions

$$a = \frac{r f \psi}{k}, \qquad b = \frac{r g \varphi}{k}, \qquad c = r h,$$

$$f = \frac{(b+c)^2 - a^2}{\sqrt{T([a])}}, \quad g = \frac{(a+c)^2 - b^2}{\sqrt{T([a])}}, \quad r = \frac{\sqrt{T([a])}}{2\,(a+b+c)}.$$

Moreover, since we use the Cartesian coordinate system, computation of distances of points and all other formulas and techniques of analytic geometry are available and well-known to widest audience. A price to pay for these conveniences is that symmetry has been lost.

The third advantage of the above position of the base triangle is that we can easily find coordinates of a point with given areals. More precisely, if a point P has coordinates x and y and $\lambda = |BCP|/|CAP|$ and $\mu = |CAP|/|ABP|$, then

$$x = \frac{(h k \mu + g u)\, r}{k\,(\lambda \mu + \mu + 1)}, \qquad y = \frac{2 f g r}{k\,(\lambda \mu + \mu + 1)}.$$

This formulas will greatly simplify our exposition because there will be no need to give explicitly coordinates of points but only its first barycentric coordinate. For example, we write $X_6[a^2]$ to indicate that the symmedian point X_6 has areals equal to $a^2 : b^2 : c^2$. Then we use the above formulas with $\lambda = a^2/b^2$ and $\mu = b^2/c^2$ to get the coordinates

$$\left(\frac{(f u v + 2 g \Phi)\, g h r}{2\,(f^2 \Psi + f g u v + g^2 \Phi)}, \frac{f g h^2 k r}{f^2 \Psi + f g u v + g^2 \Phi} \right)$$

of X_6 in our coordinate system.

6. Elimination of 92 central points

An easy task is to eliminate 92 central points X_i by exhibiting a triangle for which $T_i \leqslant 0$. In fact, for most cases only five triangles all with $r = 1$ and

triangle	t_1	t_2	t_3	t_4	t_5
f	2	1.01	2	60	5
g	20	1.02	5	5	5

will suffice. Indeed, $T_i \leqslant 0$ for the triangle t_j and $i \in I_j$, where $j = 1, \dots, 5$,

$$I_0 = \{1, \dots, 101\}, \quad I_2 = \{20, 40, 64, 66, 67, 70, 71, 74, 93, 100\}, \quad I_4 = \{21, 55\},$$

$$I_3 = \{3, 43, 48, 63, 87, 92\}, \quad I_5 = \{83\}, \quad I_6 = \{1, 2, 9, 10, 37, 38, 39, 45, 86\},$$

and $I_1 = I_0 - I_2 - I_3 - I_4 - I_5 - I_6$.

The above statement is simple to state but the reader should be aware that there is a lot of work behind it because we must know coordinates of each central point from Kimberling's list. Under the assumption that one believes that the above claim is true, we can proceed to show that for indices in the set I_6 the triangle test T_i is positive regardless of the shape of ABC.

7. Indices from I_7

Let us consider I_6 as the union of sets $I_7 = \{1, 2, 9, 10, 37, 39, 86\}$ and $I_8 = \{38, 45\}$. For indices i in I_7 we can easily get the conclusion that $T_i > 0$ from known results listed in the last section of the first chapter in [3].

For example, since $X_{86}[\frac{1}{b+c}]$ and the triple $[\frac{1}{b+c}]$ is according to item 34 on page 22 of [3] triangular, we get $T_{86} > 0$. Of course, we can arrive to the same conclusion by direct computation of T_{86} in terms of f, g, and r. In the usual way we find that

$$T_{86} = \frac{(f\, g\, h\, r)^4\, m_1\, m_2\, m_3}{k^4\, m_4^3},$$

where

$$\begin{aligned}
m_1 &= (k+1)(2\,k+1)\,h^2 + (f-g)\,h\,k\,(k+2) + k^2\,(k+1),\\
m_2 &= (k+1)(2\,k+1)\,h^2 - (f-g)\,h\,k\,(k+2) + k^2\,(k+1),\\
m_3 &= (k^2 + 5\,k + 3)\,h^2 - k^2\,(k+1),\\
m_4 &= (5\,k^2 + 11\,k + 5)\,h^2 + k^2\,(k+1).
\end{aligned}$$

Since the replacement of f and g with $1+f$ and $1+g$ in polynomials $m_1, \dots, m_4$ gives polynomials with all coefficients positive, we conclude that $T_{86} > 0$ and thus get an alternative proof of item 34.

Similarly, for the Mittenpunkt $X_9[a\,(b+c-a)]$, we can use item 35 on page 22 of [3] or check directly that $T_9 = f^6\, g^6\, h^6\, r^8 / (h^2\, k + k^3 + k^2)^3$, is obviously always positive.

For $X_1[a]$ (incenter), $X_2[1]$ (centroid), $X_{10}[b+c]$ (Spieker centre), $X_{37}[a(b+c)]$, and $X_{39}[a^2\,(b^2+c^2)]$ (Brocard midpoint), argument is even simpler because triples $[a]$, $[1]$, $[b+c]$, and $[a^2\,(b^2+c^2)]$ are clearly triangular.

In this way we conclude that only the remaining two cases X_{38} and X_{45} will give new interesting examples of triangular triples.

8. Central points $X_{38}[a\,(b^2+c^2)]$ and $X_{45}[a\,(2\,b+2\,c-a)]$

An easy calculation shows that $T_{38} = f^2\, g^2\, h^2\, r^8\, m_1\, m_+\, m_- \,/\, (4\,k^2\, m_4^3)$, where

$$m_1 = (k^2+k+2)\,h^2 + k^2\,(k+1)\,(3\,k+2),$$
$$m_4 = (2\,k^2+3\,k+2)\,h^2 - k^2\,(k-2),$$
$$m_\pm = 6f\,h^3 + (3\,k+2)(k-2)\,h^2 \pm (f-g)\,h\,k^2 - 4\,k^2\,(k+1).$$

Since the replacement of f and g with $1+f$ and $1+g$ in polynomials m_1, m_+, m_-, and m_4 gives polynomials with all coefficients positive, we conclude that $T_{38} > 0$.

COROLLARY 2. *For every triangle ABC with sides a, b, and c, the triple $[a\,(b^2+c^2)]$ is triangular.*

As above, we can show that $T_{45} = f^6\, g^6\, h^6\, r^8\, m_1\, m_2\, m_3/(k^2\, m_4^3)$, where

$$m_1 = f^2+3, \quad m_2 = g^2+3, \quad m_3 = h^2+3\,k^2, \quad m_4 = (k^2+5\,k+1)\,h^2+3\,k^2\,(k+1),$$

are clearly all positive.

COROLLARY 3. *For every triangle ABC with sides a, b, and c, the triple $[a\,(2\,b+2\,c-a)]$ is triangular.*

9. $X_3[\,a^2\,(b^2+c^2-a^2)\,]$ - circumcentre

Though we have excluded the central point X_3 (circumcentre), it gives an interesting triangular triple. One can easily find that

$$T_3 = \frac{(k^2-h^2)^2\,(f^2-1)^2\,(g^2-1)^2 f^2\, g^2\, h^2\, r^8}{64\,k^6},$$

It follows that T_3 is always positive except when ABC has an angle of $\pi/2$ radians.

COROLLARY 4. *For every triangle ABC with sides a, b, and c, and angles A, B, and C different from $\pi/2$ radians, the triples $[\,|\sin 2A|\,]$ and $[\,a^2\,|b^2+c^2-a^2|\,]$ are triangular.*

10. Peculiar property of the circumcentre

In this section we shall search for central points X_i from the Kimberling's list which have the property that areas of triangles $AX_{ib}X_{ic}$, $BX_{ic}X_{ia}$, and $CX_{ia}X_{ib}$ are sides of a triangle, where X_{ia}, X_{ib}, and X_{ic} are projections of X_i onto the side lines BC, CA, and AB, respectively.

THEOREM 2. *Among 101 central points X_i from Table 1 in [1], the circumcentre X_3 is the only with the property that areas of triangles $AX_{ib}X_{ic}$, $BX_{ic}X_{ia}$, and $CX_{ia}X_{ib}$ are sides of a triangle regardless of the shape of ABC.*

Proof. Any triangle with $f = 101/100$, $g = 101/100$ eliminates the point X_{21}, those with $f = 2$, $g = 20$ eliminates the point X_{18}, while those with $f = 2$, $g = 5$ eliminates all other points X_i except X_3. For X_3 the triangle test $\frac{3}{256}\,(f\,g\,h\,r^2)^4/k^4$ is clearly positive. □

Much more satisfying is the related problem where we look for central points X_i which have the property that areas of triangles $X_iX_{ib}X_{ic}$, $X_iX_{ic}X_{ia}$, and $X_iX_{ia}X_{ib}$ are sides of a triangle. Using the same method one can prove the following theorem.

THEOREM 3. *From 101 centres X_i of the triangle ABC from Kimberling's Table 1, only values 1, 6, 42, 57, 58, 81, 82, 83, and 89 of the index i have the property that areas of triangles $X_iX_{ib}X_{ic}$, $X_iX_{ic}X_{ia}$, and $X_iX_{ia}X_{ib}$ are sides of a triangle for every triangle ABC.*

We can now replace vertices of the pedal triangle of X_i with vertices of its antipedal triangle in the previous two theorems. It is interesting that only the orthocentre X_4 in the first theorem and only the circumcentre X_3 and the orthocentre X_4 in the second theorem lead to triangular triples of areas.

11. Points and their isogonal conjugates

For a central point X_i of ABC, let Y_i denote its isogonal conjugate.

THEOREM 4. *For triangles ABC that are not isosceles only central points X_{37}, X_{45}, X_{81}, and X_{89} have the property that areas of triangles AX_iY_i, BX_iY_i, and CX_iY_i are sides of a triangle regardless of the shape of ABC.*

Proof. The method of proof is the same only the details are a bit more complicated to write down. We first check on several concrete triangles to eliminate most of the points. The four remaining points from the statement have rational functions in f, g, and r as triangle tests for areas of three triangles. Since factors in numerators and denominators of these rational functions are polynomials that have all coefficients positive following the substitution $f = f + 1$ and $g = g + 1$ and factors which vanish only when ABC is isosceles are complete squares, we conclude that triangle tests of areas are always positive for triangles that are not isosceles. □

COROLLARY 5. *For every triangle ABC which is not isosceles and whose sides are a, b, and c, the triples*

$$\left[\frac{(2\,a+b+c)\,|b^2-c^2|}{a}\right] \qquad \textit{and} \qquad \left[\frac{(4\,a+b+c)(2\,b+2\,c-a)\,|b-c|}{a}\right]$$

are triangular.

Proof. The areas of triangles $AX_{37}Y_{37}$ and $AX_{81}Y_{81}$ are proportional to the first element of the first triple while the areas of triangles $AX_{45}Y_{45}$ and $AX_{89}Y_{89}$ are proportional to the first element of the second triple. □

The triangular triples in the above corollary have led the author to the following result.

THEOREM 5. *For every triangle ABC which is not isosceles and whose sides are a, b, and c, the triple $[\,|b-c|/a\,]$ is triangular.*

Proof. Let $x = \sqrt{\frac{(b-c)^2}{a^2}}$, $y = \sqrt{\frac{(c-a)^2}{b^2}}$, and $z = \sqrt{\frac{(a-b)^2}{c^2}}$. Then

$$T(x,\, y,\, z) = \frac{(b-c)^2\,(c-a)^2\,(a-b)^2\,(w-a^2)\,(w-b^2)\,(w-c^2)}{a^4\,b^4\,c^4},$$

where $w = a\,b + a\,c + b\,c$.

Let us check that $w - a^2 > 0$. Indeed, since $b + c > a$ and $a > 0$ we get $a\,(b+c) > a^2$ and finally $w > a^2$ because $b\,c > 0$.

In a similar fashion we see that $w > b^2$ and $w > c^2$ so that $T(x,\, y,\, z) > 0$. □

12. Triangles from products of areals and powers of sides

The Corollary 1 is just one of a whole series of results. More precisely, for each integer k we can ask for the locus of all points P such that $w_1\,a^k$, $w_2\,b^k$, and $w_3\,c^k$ are sides of a triangle, where w_1, w_2, and w_3 are areal coordinates of P with respect to ABC. We shall see shortly that this locus is the interior of a suitable triangle whose vertices can be described as follows.

Let A_k, B_k, and C_k be points on the sides BC, CA, and AB of ABC such that

$$\frac{CA_k}{A_kB} = \left(\frac{c}{b}\right)^k, \qquad \frac{AB_k}{B_kC} = \left(\frac{a}{c}\right)^k, \qquad \frac{BC_k}{C_kA} = \left(\frac{b}{a}\right)^k.$$

Notice that $A_kB_kC_k$ for $k = -4,\, -3,\, -2,\, -1,\, 0,\, 1,\, 2$ are the cevian triangle of the Third Power point (the central point X_{32}), the cevian triangle of the Second Power point (the central point X_{31}), the cevian triangle of the Grebe-Lemoine point K or X_6, the incentral triangle $A_IB_IC_I$ (the cevian triangle of the incentre I or X_1), the complementary triangle $A_mB_mC_m$ with vertices at midpoints of sides (the cevian triangle of the centroid G or X_2), the cevian triangle of the isogonal conjugate of the Second Power point (the central point X_{75}), and the cevian triangle of the isogonal conjugate of the Third Power point (the central point X_{76} – the Third Brocard point).

THEOREM 6. *Let k be an integer. The locus of all points P such that $w_1\,a^k$, $w_2\,b^k$, and $w_3\,c^k$ are sides of a triangle is the interior of the triangle $A_kB_kC_k$.*

Proof. The case $k = 0$ was proved earlier. We shall now prove the cases $k = -1$ and $k = 1$. The statement for arbitrary k is proved similarly.

The actual areal coordinates of the point P inside the triangle ABC are $w_1 = r f\,(2\,r\,g\,h - 2\,g\,x - v\,y)/(2\,k)$, $w_2 = r\,g\,(2 f\,x - u\,y)/(2\,k)$, and $w_3 = r\,h\,y/2$, where x and y are coordinates of P. It follows that

$$T([w_1/a]) = [2\,k\,(f\,-g)\,x - (2\,\varphi + 2\,\psi - 3\,\varphi\,\psi)\,y - 2\,r\,g\,h\,\varphi]$$
$$[2\,h\,(k+2)\,x - (4\,g^2 + u\,v)\,y - 2\,r\,g\,h\,\varphi][2\,h\,(k+2)\,x - (4f^{2} + u\,v)\,y - 2\,r\,g\,h\,\varphi]$$
$$[-2\,k\,(f\,-g)\,x - (4 - u\,v)\,y - 2\,r\,g\,h\,\varphi]/(2\,\varphi\,\psi)^4$$

and

$$T([w_1\,a]) = r^8 f^{2}\,g^2\,h^2\,k^{-8}\,[f\,g\,k\,(f\,-g)\,x + k\,(f^{2}\,v - g\,h)\,y + rf^{2}\,g\,h\,\psi]$$
$$[f\,(k+2)\,x + (f^{2}\,v - k)\,y - rf^{2}\,\psi][g\,(k+2)\,x + (k - g^2\,u)\,y - rf\,g\,\psi]$$
$$[k\,(g-f\,)\,x + k\,(f\,h + v)\,y - rf\,h\,\psi].$$

The first three parenthesis in $T([w_1/a])$ are equations of sidelines of the incentral triangle $A_I B_I C_I$ and the last three parenthesis in $T([w_1\,a])$ are equations of sidelines of $A_1B_1C_1$ (the cevian triangle of the isogonal conjugate of the Second Power point). From this our claim follows immediately because the last parenthesis in $T([w_1/a])$ and the first parenthesis in $T([w_1\,a])$ are always positive having positive values in A, B, and C. □

13. Triangles from areas of triangles on cevians

Let P_a, P_b, and P_c denote vertices of the cevian triangle of a point P with respect to the base triangle ABC.

THEOREM 7. *The areas of triangles P_bP_cP, P_cP_aP, and P_aP_bP are sides of a triangle if and only if the point P lies in the four components of the complement of the sidelines of ABC which does not contain its excircles.*

Proof. The areas of triangles P_bP_cP, P_cP_aP, and P_aP_bP are absolute values of $f\,m/(m_b\,m_c)$, $g\,m/(m_c\,m_a)$, and $h\,m/(m_a\,m_b)$, where m is $r\,q\,(u\,q - 2f\,p)(v\,q + 2\,g\,p - 2\,r\,g\,h)/2$ (the product of equations of sidelines of ABC) and $m_a = v\,q + 2\,g\,p$, $m_b = u\,q - 2f\,p + 2\,rf\,h$, and $m_c = 2\,rf\,g - k\,q$ are equations of the sidelines of the anticomplementary triangle $A_aB_aC_a$.

The triangle test for these areas is $T[P_bP_cP] = 64f^{2}\,g^2\,h^2\,k\,m^5/(m_a\,m_b\,m_c)^4$. Its sign depends on the sign of m. But, the factors $v\,q + 2\,g\,p - 2\,r\,g\,h$ and $u\,q - 2f\,p$ of m evaluated at points A and B have negative values while the value of the third factor $r\,q/2$ at point C is positive so that their product m is positive if and only if the point P is in the subset indicated in the statement of the theorem. □

We can now compute areas from the above theorem and get the following family of triangular triples that depends on two parameters.

Let $s = a+b+c$, $s_a = b+c-a$, $s_b = \varphi(s_a)$, $s_c = \psi(s_a)$, $s_{2a} = b^2 + c^2 - a^2$, $s_{2b} = \varphi(s_{2a})$, and $s_{2c} = \psi(s_{2a})$.

COROLLARY 6. *Let* $[a]$ *be a triangular triple, let* $S = \frac{1}{4}\sqrt{s\, s_a\, s_b\, s_c}$, *and let* x *and* y *be any real numbers such that* $y > 0$, $4Sx - s_{2a}\,y > 0$, *and* $4S(c-x) - s_{2b}\,y > 0$. *Then the following triple is triangular*

$$[\,4Sx + s_{2b}\,y,\; 4S(c-x) + s_{2a}\,y,\; 2c(2S - cy)\,].$$

THEOREM 8. *The triple* $[\,|P_bP_cP|/|P_bP_cA|\,]$ *from quotients of areas is triangular if and only if the point* P *is from the four components of the complement of the sidelines of the complementary triangle* $A_mB_mC_m$ *of* ABC *that does not contain excircles of* $A_mB_mC_m$.

Proof. Since the areas $|P_bP_cA|$, $|P_cP_aB|$, and $|P_aP_bC|$ are absolute values of

$$f\,g\,h\,r^2\,q\,m_7/(m_b\,m_c), \qquad f\,g\,h\,r^2\,q\,m_8/(m_c\,m_a), \quad \text{and} \quad f\,g\,h\,r^2\,m_7\,m_8/(k\,m_a\,m_b),$$

the triangle test for the triple of quotients is $k_a\,k_b\,k_c/(f^2\,g^2\,h^2\,r^3)$, where

$$k_a = 2gp + vq - rgh, \qquad k_b = rfh - 2fp + uq, \quad \text{and} \quad k_c = rfg - kq$$

are equations of the sidelines of $A_mB_mC_m$. Their values at its vertices are positive so that their product is positive only on the set described in the statement of the theorem. □

THEOREM 9. *For every point* P *in the interior of the triangle* ABC, *the triple*

$$[\,|P_aP_cB|/|P_cP_aP| + |P_bP_aC|/|P_aP_bP|\,]$$

of sums of quotients of oriented areas is triangular.

Proof. The triangle test for sums of quotients of oriented areas is $16f^2g^2h^2r^6m_9/(3m^2)$, where $m_9 = 4(2fghr - 4fgp + (k+2)(f-g)q)^2 - h^2(2fgr - 3kq)^2 + 16f^2g^2h^2r^2$. It is obvious that m_9 is a conic and since it passes through the vertices A, B, and C, the Steiner point (X_{99} in [1]), and the Yff parabolic point (X_{190} in [1]), we conclude that this is the equation of the Steiner ellipse. Since its value at the centroid G of ABC is positive, it follows that it is positive in all points of its interior. We must exclude sidelines of ABC because there the areas in denominators can be zero. But, we must also find out when the sum $[\,|P_aP_cB|/|P_cP_aP| + |P_bP_aC|/|P_aP_bP|\,]$ and the other two of its cyclic permutations are positive. An easy calculation shows that the sign of this sum depends on the sign of equations of CA, AB, and the tangent to the Steiner ellipse at the point A. Since at the centroid this sum is 6 we conclude that it is positive in the interior of ABC and the conclusion of the theorem has been established. □

COROLLARY 7. *Let* $[a]$ *be a triangular triple, let* $S = \frac{1}{4}\sqrt{s\, s_a\, s_b\, s_c}$, *and let* x *and* y *be any real numbers such that* $y > 0$, $4Sx - s_{2a}\,y > 0$, *and* $4S(c-x) - s_{2b}\,y > 0$. *Then the following triple is triangular*

$$\left[\frac{4Sx + s_{2b}\,y}{4Sx - s_{2a}\,y},\; \frac{4S(x-c) - s_{2a}\,y}{4S(x-c) + s_{2b}\,y},\; \frac{2c^3\,y\,(cy - 2S)}{(4Sx - s_{2a}\,y)(4S(x-c) + s_{2b}\,y)}\right].$$

THEOREM 10. *For every point P in the interior of the triangle ABC, the triple*

$$[\,|P_cP_aP|/|P_aP_cB| + |P_aP_bP|/|P_bP_aC|\,]$$

of sums of quotients of oriented areas is triangular.

Proof. The sign of the triangle test for these sums of quotients of oriented areas depends only on the sign of the product of values of equations of sidelines evaluated at the point P. But, $|P_cP_aP|/|P_aP_cB| + |P_aP_bP|/|P_bP_aC|$ and its cyclic permutations are positive only inside the anticomplementary triangle of ABC so that the conclusion of the theorem follows. □

14. Triangles from areas of triangles on orthocentres

For a point P in the plane of the base triangle ABC, let H_a, H_b, and H_c denote orthocentres of triangles BCP, CAP, and ABP.

THEOREM 11. *For every point P from the intersection of the interiors of the triangle ABC and the circles with centres at midpoints of sides and with half of lengths of sides as radii, the triple $[\,|H_cH_aP| + |H_aH_bP|\,]$ from sums of oriented areas is triangular.*

Proof. The triangle test for this triple is $4f^2 g^2 h^2 r^4 q_a^2 q_b^2 q_c^2/m^2$, where $q_c = p^2+q^2 - h\,r\,p$, $q_b = k\,(p^2+q^2) - r\,g\,u\,p - 2f\,g\,r\,q$, and $q_a = k\,(p^2+q^2) - r\,(f\,v + 2\,g\,u)\,p - 2f\,g\,r\,q$ are equations of the above three circles. From this the conclusion of the theorem is immediate once we determine conditions for $|H_cH_aP| + |H_aH_bP|$, $|H_aH_bP| + |H_bH_cP|$, and $|H_bH_cP| + |H_cH_aP|$ to be positive. But,

$$|H_bH_cP| + |H_cH_aP| = \frac{h\,q_c\,((g\,r\,u - k\,p)^2 + (2f\,g\,r - k\,q)^2)}{k\,(u\,q - 2f\,p)(2\,g\,h\,r - 2\,g\,p - v\,q)}$$

is positive at the centre of q_c, so that this sum of oriented areas is positive on the intersection of the interiors of ABC and the circle whose diameter is AB. The other two sums are positive on analogous sets. Hence, all three are surely positive on the intersection of interiors of ABC with three circles whose diameters are BC, CA, and AB. □

Let us observe that the incentre, the Gergonne point, and both isogonic centres of ABC are in the intersection from the above theorem. In fact, when P is an isogonic centre of ABC, then $|H_cH_aP| + |H_aH_bP|$, $|H_aH_bP| + |H_bH_cP|$, and $|H_bH_cP| + |H_cH_aP|$ have the same value (two thirds of the area of ABC).

15. Triangles from areas of triangles on projections

For a point P in the plane of the base triangle ABC, let P_a, P_b, and P_c denote projections of P into sidelines BC, CA, and AB.

THEOREM 12. *For each point P in the interior of a triangle ABC, the triple*

$$[\,|P_c P_a P| + |P_a P_b P|\,]$$

of sums of oriented areas is triangular.

Proof. The triangle test for this triple is $128 f^2 g^2 h^2 k\, m^2 q_0/(r^2 \varphi^6 \psi^6)$, where

$$q_0 = k\,(p^2 + q^2) - r\,(f\,v + 2\,g\,u)\,p - 2 f\,g\,r\,q + r^2\,g\,h\,u$$

is the equation of the circumcircle. This equation has positive value at its centre, so that the test is positive if and only if P is the interior point of the circumcircle outside the sidelines. However,

$$|P_c P_a P| + |P_a P_b P| = \frac{h\,q\,j_c}{(\varphi^2\,\psi^2)},$$

where j_c is the equation of the tangent to the circumcircle at the vertex C, is positive on the part of plane determined by the line AB and this tangent which contains the interior of ABC. Since similar claims hold for the other two sums of oriented areas, we conclude that our statement in the theorem is true. □

COROLLARY 8. *Let $[\,a\,]$ be a triangular triple, let $S = \frac{1}{4}\sqrt{s\,s_a\,s_b\,s_c}$, and let x and y be any real numbers such that $y > 0$, $4\,S\,x - s_{2a}\,y > 0$, and $4\,S\,(c - x) - s_{2b}\,y > 0$. Then the following triple is triangular*

$$[(4\,S\,(c-x) - s_{2b}\,y)(4\,S\,x + s_{2c}\,y),\ (4\,S\,(c-x) + s_{2c}\,y)(4\,S\,x - s_{2a}\,y),$$
$$2\,y\,(4\,S\,(a^2 - b^2)\,x + ((a^2 - b^2)^2 - c^2\,(a^2 + b^2))\,y + 4\,S\,b^2\,c)]\,.$$

16. Triangles from areas of triangles on centroids

For a point P in the plane of the base triangle ABC, let G_a, G_b, and G_c denote centroids of triangles BCP, CAP, and ABP.

THEOREM 13. *The triple $[\,|G_b G_c P|\,]$ from areas is triangular if and only if the point P is from the interior of four parts of the plane determined by the sidelines of the complementary triangle of the complementary triangle of ABC which contain the centroid and the vertices of ABC. The triple $[\,|G_b G_c P|\,]$ from oriented areas is triangular if and only if the point P is from the interior of the complementary triangle of the complementary triangle of ABC.*

Proof. The triangle test for the triple from both areas and oriented areas is

$$f^2\,g^2\,h^2\,r^5\,q_a^2\,q_b^2\,q_c^2/(6561\,k^4),$$

where $q_c = f\,g\,r - 2\,k\,q$, $q_b = 4 f\,p - 2\,u\,q - f\,h\,r$, and $q_a = 4\,g\,p + 2\,v\,q - 3\,g\,h\,r$ are equations of the sidelines of the complementary triangle of the complementary triangle of ABC. From this the conclusion of the theorem is immediate once we determine conditions for the oriented areas $|G_b G_c P|$, $|G_c G_a P|$, and $|G_a G_b P|$, to be positive. But, $|G_a G_b P| = h\,r\,(f\,g\,r - k\,q)/(9\,k)$ is a positive constant multiple of the equation of the sideline of the complementary triangle of ABC. The same is true for $|G_b G_c P|$ and $|G_c G_a P|$ so that all three oriented areas are positive only in the interior of the complementary triangle of ABC. □

For a triangle ABC, let $A_0B_0C_0$ denote a triangle such that the centroid G of ABC divides segments AA_0, BB_0, and CC_0 in the ratio 1:5.

THEOREM 14. *The triple* $[\,|AG_bG_c|\,]$ *from areas is triangular if and only if the point* P *is from the complement of the sidelines of the triangle* $A_0B_0C_0$ *in the interior of four parts of the plane determined by the sidelines of the complementary triangle of the triangle* $A_0B_0C_0$ *which do not contain its vertices.*

Proof. The triangle test for the triple from areas is $5 f^2 g^2 h^2 r^5 q_a^2 q_b^2 q_c^2/(6561\,k^4)$, where $q_c = f\,g\,r + k\,q$, $q_b = 2 f\,p - u\,q + f\,h\,r$, and $q_a = 3\,g\,h\,r - 2\,g\,p - v\,q$ are equations of the sidelines of the complementary triangle of the triangle $A_0B_0C_0$. From this the conclusion of the theorem is immediate once we observe that sidelines of $A_0B_0C_0$ are locus of points where one of the areas $|AG_bG_c|$, $|BG_cG_a|$, and $|CG_aG_b|$ is zero. □

Notice that we can never get all three oriented areas $|AG_bG_c|$, $|BG_cG_a|$, and $|CG_aG_b|$ to be positive. This is a reason why there is no version of the above theorem for oriented areas.

REFERENCES

[1] CLARK KIMBERLING, *Central points and central lines in the plane of a triangle*, Mathematics Magazine, **67** (1994), 163–187.

[2] CLARK KIMBERLING, *Point-to-line distances in the plane of a triangle*, Rocky Mountain Journal of Math., **24** (1994), 1009–1026.

[3] D. S. MITRINOVIĆ, J. E. PEČARIĆ, AND V. VOLENEC, *Recent Advances in Geometric Inequalities*, Kluwer Academic Publishers, Dordrecht, 1989.

(Received April 1, 1998)

Zvonko Čerin
Kopernikova 7
10010 Zagreb
Croatia
e-mail: cerin@math.hr