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GROWTH OF MAXIMUM MODULUS OF RATIONAL FUNCTIONS WITH PRESCRIBED POLES

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Abstract. In this paper we prove several sharp growth estimates for rational functions with prescribed poles in the Chebyshev norm on the unit circle in the complex plane. In particular, our results generalize and sharpen certain polynomial inequalities due to Rahman, Ankeny and Rivlin for such rational functions with restricted zeros.

1. Introduction and statement of results

Let $\mathbf{P}_n$ denote the class of all complex algebraic polynomials of degree at most n. For $k > 0$, let $D_{k-} = \{z; |z| < k\}$, $D_{k+} = \{z; |z| > k\}$ and $T_k = \{z; |z| = k\}$. For f defined on the circle T_k, we set

$$M(f,k) = \sup_{z \in T_k} |f(z)| \quad \text{and} \quad m(f,k) = \inf_{z \in T_k} |f(z)|.$$

For $a_j \in \mathbf{C}$ with $j = 1, 2, \dots, n$, we write

$$W(z) = \prod_{j=1}^{n} (z - a_j) \quad \text{and} \quad B(z) = \prod_{j=1}^{n} \left(\frac{1 - \overline{a}_j z}{z - a_j} \right) \tag{1}$$

and

$$\mathbf{R}_n = \mathbf{R}_n(a_1, a_2, \dots, a_n) = \left\{ \frac{P(z)}{W(z)}; p \in \mathbf{P}_n \right\}.$$

Then $\mathbf{R}_n$ is the set of all rational functions with at most n poles $a_1, a_2, \dots, a_n$ and with finite limit at ∞. We observe that $B(z) \in \mathbf{R}_n$. Throughout our discussion, we shall always assume that all poles $a_1, a_2, \dots, a_n$ lie in D_{1+}. Analogous results can be obtained when we assume all poles lie in D_{1-}.

If $P \in \mathbf{P}_n$, then we have

$$M(P', 1) \leqslant nM(P, 1) \tag{2}$$

and

$$M(P, R \geqslant 1) \leqslant R^n M(P, 1). \tag{3}$$

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Inequality (2) is an immediate consequence of S. Bernstein's theorem on the derivative of a trigonometric polynomial (for reference see [12]). Inequality (3) is a simple deduction from the maximum modulus principle [(see 11, P. 346) or (8, vol. I, p. 132)].

In both (2) and (3) equality holds only for $P(z) = \alpha z^n$, $\alpha \neq 0$ is any real or complex number. If $P \in \mathbf{P}_n$ and $P^*(z) = z^n\overline{P(1/\overline{z})}$ is the conjugate transpose of P, then

$$|P(Rz)| + |P^*(Rz)| \leqslant (R^n + 1)M(P, 1) \quad \text{for } z \in T_1 \text{ and } R \geqslant 1. \tag{4}$$

Inequality (4) is essentially due to Rahman [10, inequality (5.3) for the special operator $B(P(z)) = P(z)$]. Inequalities (2) and (3) can be sharpened if we restrict ourselves to the class of polynomials having no zero in D_{1-}. In fact, if $P \in \mathbf{P}_n$ does not vanish in D_{1-}, then (2) and (3) can be respectively replaced by

$$M(P', 1) \leqslant \frac{n}{2}M(P, 1), \tag{5}$$

$$M(P, R \geqslant 1) \leqslant \left(\frac{R^n + 1}{2}\right)M(P, 1). \tag{6}$$

Inequality (5) was conjectured by Erdös and later proved by Lax [6] (see also [3]) whereas Ankeny and Rivlin [5] used (5) to prove inequality (6). In both (5) and (6) equality holds for $P(z) = \alpha z^n + \beta$ where $\alpha, \beta \in T_1$.

Recently Li, Mahapatra and Rodriguez [7] have proved Bernstein-type inequalities similar to (2) and (4) for rational functions $r \in \mathbf{R}_n$ with poles $a_1, a_2, \ldots, a_n$ all lying in D_{1+} and replaced z^n by Blashke product $B(z)$. If $r \in \mathbf{R}_n$, then for $z \in T_1$ and $R \geqslant 1$,

$$|r(Rz)| \leqslant |B(Rz)|M(r, 1). \tag{7}$$

Equality in (7) holds for $r(z) = \lambda B(z)$ where $\lambda \in T_1$. The inequality (7) is due to Walsh [13, P. 236, Lemma II].

The purpose of this paper is to obtain several sharp inequalities similar to (4) and (6) for rational functions with prescribed poles. Here we first prove the following generalization of (4) for rational functions $r \in \mathbf{R}_n$, which is a refinement of (7), from which one can easily deduce that the inequality (4) remains true for $0 \leqslant R < 1$ as well.

THEOREM 1. *If $r \in \mathbf{R}_n$ and $z \in T_1$, then for every $R \geqslant 0$,*

$$|r(Rz)| + |r^*(Rz)| \leqslant (|B(Rz)| + 1)M(r, 1) \tag{8}$$

where $r^*(z) = B(z)\overline{r(1/\overline{z})}$. Equality in (8) holds for $r(z) = \lambda B(z)$ where $\lambda \in T_1$.

Next we establish the following results for rational functions with restricted zeros which generalize polynomial inequalities of Ankeny and Rivlin [5] and Aziz [1, Theorem 4].

THEOREM 2. *Suppose $r \in \mathbf{R}_n$ and all the zeros r lie in $T_1 \cup D_{1+}$, then for $z \in T_1$ and $R \geqslant 1$, we have*

$$|r(Rz)| \leqslant \left(\frac{|B(Rz)| + 1}{2}\right)M(r, 1). \tag{9}$$

Equality in (9) holds for $r(z) = B(z) + \lambda$ where $\lambda \in T_1$ is choosen suitably.

THEOREM 3. *Suppose $r \in \mathbf{R}_n$ and all the zeros of r lie in $T_1 \cup D_{1+}$, then for $z \in T_1$ and $R \geqslant 1$, we have*

$$|r(Rz)| \leqslant \Big(\frac{|B(Rz)|+1}{2}\Big)M(r,1) - \Big(\frac{|B(Rz)|-1}{2}\Big)m(r,1). \tag{10}$$

Equality in (10) holds for $r(z) = B(z) + \lambda k$ where $k \geqslant 1$ and $\lambda \in T_1$ is choosen suitably.

If $P \in \mathbf{P}_n$ is self-inversive polynomial, that is, if $p^*(z) = \beta P(z)$, $\beta \in T_1$ and $P^*(z) = z^n\overline{P(1/\overline{z})}$, then it is known [9] that

$$M(P,R \geqslant 1) \leqslant \Big(\frac{R^n+1}{2}\Big)M(P,1). \tag{11}$$

Analogously, a rational function $r \in \mathbf{R}_n$ is called self-inversive if $r^*(z) = \beta r(z)$ for some $\beta \in T_1$ where $r^*(z) = B(z)\overline{r(1/\overline{z})}$. Here we prove the following gneralization of (11) for self-inversive rational functions.

THEOREM 4. *If $r \in \mathbf{R}_n$ is self-inversive and $z \in T_1$, then for $R \geqslant 0$, we have*

$$|r(Rz)| \leqslant \Big(\frac{|B(Rz)|+1}{2}\Big)M(r,1). \tag{12}$$

Equality in (12) holds for $r(z) = B(z) + \lambda$ where $\lambda \in T_1$ is choosen suitably.

Finally, we present following result which generalizes a polynomial inequality due to Aziz and Dawood [2].

THEOREM 5. *If $r \in \mathbf{R}_n$ and all the zeros of the r lie in $T_1 \cup D_{1-}$, then for $z \in T_1$ and $R \geqslant 1$, we have*

$$|r(Rz)| \geqslant |B(Rz)|m(r,1). \tag{13}$$

Equality in (13) holds for $r(z) = uB(z)$ where $u \in T_1$.

REMARK. By Theorem 1, we have for $z \in T_1$, $R \geqslant 0$ and $a_j = a > 1$, $j = 1,2,\ldots,n$,

$$\Big|\frac{P(Rz)}{W(Rz)}\Big| + \Big|\frac{P^*(Rz)}{W(Rz)}\Big| \leqslant \Big\{\Big|\Big(\frac{1-aRz}{Rz-a}\Big)^n\Big| + 1\Big\} \sup_{z\in T_1}\Big|\frac{P(z)}{(z-a)^n}\Big|,$$

or,

$$|P(Rz)| + |P^*(Rz)| \leqslant \{|(1-aRz)^n| + |(Rz-a)^n|\} \sup_{z\in T_1}\Big|\frac{P(z)}{(z-a)^n}\Big|. \tag{14}$$

If $\sup_{z\in T_1} |P(z)/(z-a)^n|$ on T_1 is attained at $z = e^{i\alpha}$, $0 \leqslant \alpha \leqslant 2\pi$, then clearly

$$\sup_{z\in T_1}\Big|\frac{P(z)}{(z-a)^n}\Big| = \Big|\frac{P(e^{i\alpha})}{(e^{i\alpha}-a)^n}\Big| \leqslant \frac{\sup_{z\in T_1}|P(z)|}{|(e^{i\alpha}-a)^n|}$$

and from (14), we get for $z \in T_1$,

$$|P(Rz)| + |P^*(Rz)| \leqslant \Big\{\Big|\Big(\frac{1-aRz}{Rz-a}\Big)^n\Big| + 1\Big\}\Big|\Big(\frac{Rz-a}{e^{i\alpha}-a}\Big)^n\Big|M(P,1).$$

Letting $a \to \infty$, we obtain for $z \in T_1$ and $R \geqslant 0$,

$$|P(Rz)| + |P^*(Rz)| \leqslant (R^n + 1)M(P,1),$$

which in particular includes (4). Similarly all other polynomial inequalities mentioned in this paper are limiting casses of our results.

2. Lemmas

To show the equality holds in Theorem 3, we need following lemmas. The first result was recently proved by Aziz and Shah [4].

LEMMA 1. *If* $B(z)$ *is defined by* (1) *and* α *real,* $0 \leqslant \alpha \leqslant 2\pi$*, then*
(i) $B(z) + ke^{i\alpha}$ *has all its zeros in* $T_1 \cup D_{1+}$ *for every* $k \geqslant 1$*;*
(ii) $B(z) + ke^{i\alpha}$ *has all its zeros in* $T_1 \cup D_{1-}$ *for every* $k \leqslant 1$.

LEMMA 2. *If* $B(z)$ *is defined by* (1) *and* α *real,* $0 \leqslant \alpha \leqslant 2\pi$*, then for* $k \geqslant 1$*, we have*
(i) $\sup_{z \in T_1} |B(z) + ke^{i\alpha}| = k + 1$*;*
(ii) $\inf_{z \in T_1} |B(z) + ke^{i\alpha}| = k - 1$.

Proof of Lemma 2. By Lemma 1 (with $k = 1$), it follows that all the zeros of rational function $B(z) - e^{i\alpha}$, $0 \leqslant \alpha \leqslant 2\pi$, lie on T_1, therefore, if $z = t$ is any zero of $B(z) - e^{i\alpha}$, then

$$|B(t) + ke^{i\alpha}| = |e^{i\alpha} + ke^{i\alpha}| = k + 1, \quad t \in T_1. \tag{15}$$

Now by (1), it follows that $|B(z)| = 1$ for $z \in T_1$, which gives

$$|B(z) + ke^{i\alpha}| \leqslant 1 + k, \quad \text{for every } z \in T_1. \tag{16}$$

From (15) and (16), we conclude that

$$\sup_{z \in T_1} |B(z) + ke^{i\alpha}| = |B(t) + ke^{i\alpha}| = k + 1.$$

This proves (i). To establish (ii), we observe by Lemma 1 that all the zeros of $B(z) + e^{i\alpha}$ also lie on T_1. Therefore, if $z = s$ is a zero of $B(z) + e^{i\alpha}$, $0 \leqslant \alpha \leqslant 2\pi$, then for $k \geqslant 1$,

$$|B(s) + ke^{i\alpha}| = |-e^{i\alpha} + ke^{i\alpha}| = k - 1, \quad s \in T_1.$$

Since $|B(z) + ke^{i\alpha}| \geqslant k - |B(z)| = k - 1$ for every $z \in T_1$, it follows as before that

$$\inf_{z \in T_1} |B(z) + ke^{i\alpha}| = |B(z) + ke^{i\alpha}| = k - 1.$$

This completes the proof of Lemma 2

Proof of Theorem 1. We first suppose that $R \geqslant 1$. Since all the poles of $r(z)$ lie in D_{1+}, it follows that $r(z)$ is analytic for $T_1 \cup D_{1-}$. Moreover $|r(z)| \leqslant M(r,1)$ for

$z \in T_1$, therefore, for every complex number $\alpha \in D_{1+}$, we have $|r(z)| < |\alpha M(r,1)|$ for $z \in T_1$. Applying Rouche's theorem, it follows that the analytic function $F(z) = r(z) + \alpha M(r,1)$ does not vanish for $z \in T_1 \cup D_{1-}$. If $F^*(z) = B(z)\overline{F(1/\overline{z})}$, then

$$|F^*(z)| = |B(z)\overline{F(1/\overline{z})} = |F(z)| \quad \text{for } z \in T_1$$

and

$$\begin{aligned} F^*(z)| &= B(z)\big(\overline{r(1/\overline{z})} + \overline{\alpha} M(r,1)\big), \\ &= B(z)\overline{r(1/\overline{z})} + \overline{\alpha} B(z) M(r,1), \\ &= r^*(z) + \overline{\alpha} B(z) M(r,1), \\ &= (P^*(z)/W(z)) + \overline{\alpha} B(z) M(r,1), \\ &= (P^*(z) + \overline{\alpha} B(z) W(z) M(r,1))/W(z), \\ &= (P^*(z) + \overline{\alpha} W^*(z) M(r,1))/W(z) \end{aligned}$$

so that $F^*(z)$ is analytic for $z \in T_1 \cup D_{1-}$. Therefore, $F^*(z)/F(z)$ is also analytic for $z \in T_1 \cup D_{1-}$ and

$$|F^*(z)/F(z)| = 1 \quad \text{for } z \in T_1.$$

Hence by the Maximum Modulus Principle, it follows that

$$|F^*(z) \leqslant |F(z)| \quad \text{for } z \in T_1 \cup D_{1-}.$$

Replacing z by $1/\overline{z}$, it can be easily seen that

$$|F(z) \leqslant |F^*(z)| \quad \text{for } z \in T_1 \cup D_{1+}.$$

That is, for $z \in T_1 \cup D_{1+}$, we have

$$|r(z) + \alpha M(r,1)| \leqslant |r^*(z) + \overline{\alpha} B(z) M(r,1)|. \tag{17}$$

Choosing the argument α such that

$$|r^*(z) + \overline{\alpha} B(z) M(r,1)| = |\alpha|\,|B(z)| M(r,1) - |r^*(z)|,$$

for $z \in T_1$ (which is possible by inequality (7)), from (17) we obtain

$$|r(z)| - |\alpha| M(r,1) \leqslant |\alpha|\,|B(z)| M(r,1) - |r^*(z)|$$

for $z \in T_1 \cup D_{1+}$ and $\alpha \in D_{1+}$. This gives

$$|r(z)| + |r^*(z)| \leqslant |\alpha| \{|B(z)| + 1\} M(r,1).$$

Letting $|\alpha| \to 1$, we get

$$|r(z)| + |r^*(z)| \leqslant \{|B(z)| + 1\} M(r,1) \text{ for } z \in T_1 \cup D_{1+}. \tag{18}$$

Equivalently,

$$|r(Rz)| + |r^*(Rz)| \leqslant \{|B(Rz) + 1\} M(r,1) \tag{19}$$

for every $R \geqslant 1$ and $z \in T_1$.

Next we suppose $R \leqslant 1$ and $z \in T_1$. Replacing z by $1/\overline{z}$ in (18), we get for $z \in T_1 \cup D_{1-}$

$$|\overline{r(1/\overline{z})}| + |\overline{r^*(1/\overline{z})}| \leqslant \{|\overline{B(1/\overline{z})}| + 1\}M(r,1),$$

which implies, for $z \in T_1 \cup D_{1-}$

$$|B(z)\overline{r(1/\overline{z})} + |B(z)\overline{r^*(1/\overline{z})} \leqslant \{|B^*(z)| + |B(z)|\}M(r,1).$$

That is,

$$|r^*(z)| + |r(z)| \leqslant \{|B(z)| + 1\}M(r,1) \quad \text{for } z \in T_1 \cup D_{1-}.$$

Equivalently, for $z \in T_1$ and $R \leqslant 1$, we have

$$|r(Rz)| + |r^*(Rz)| \leqslant \{|B(Rz)| + 1\}M(r,1). \tag{20}$$

Combining (19) and (20), we get the desired result. This completes the proof of Theorem 1.

Proof of Theorem 2. Since $r(z) = P(z)/W(z)$ and $r^*(z) = P^*(z)/W(z)$, we have

$$|r(z)/r^*(z)| = |P(z)/P^*(z)|. \tag{21}$$

By hypothesis all the zeros of $r(z) = P(z)/W(z)$ lie in $T_1 \cup D_{1+}$ and $|P^*(z)/P(z)| = 1$ for $z \in T_1$, it follows that the function $P^*(z)/P(z)$ is analytic in $T_1 \cup D_{1-}$. By the Maximum Modulus Principle, we have

$$|P^*(z)/P(z)| \leqslant 1 \quad \text{for } z \in T_1 \cup D_{1-}.$$

Replacing z by $1/\overline{z}$, we conclude with the help of (21) that

$$|r(z)/r^*(z)| = |P(z)/P^*(z)| \leqslant 1 \quad \text{for } z \in T_1 \cup D_{1+}.$$

This implies

$$|r(z)| \leqslant |r^*(z)| \quad \text{for } z \in T_1 \cup D_{1+}.$$

Equivalently,

$$|r(Rz)| \leqslant |r^*(Rz)| \quad \text{for } z \in T_1 \text{ and for every } R \geqslant 1.$$

Combining this with the conclusion of Theorem 1, we obtain

$$2|r(Rz)| \leqslant (|B(Rz)| + 1)M(r,1)$$

for $z \in T_1$ and every $R \geqslant 1$, which is equivalent to (9) and this completes the proof of Theorem 2.

Proof of Theorem 3. We have $m(r,1) = \inf_{z \in T} |r(z)|$, so that $m(r,1) \leqslant |r(z)|$ for $z \in T_1$. If $r(z)$ has a zero on T_1, then $m(r,1) = 0$ and the result follows from Theorem 2. Henceforth, we suppose all the zeros of $r(z)$ lie in D_{1+}, so that $m(r,1) > 0$ and all the zeros of $r^*(z) = B(z)\overline{r(1/\overline{z})}$ lie in D_{1-}. Hence, if α is any complex number such that $\alpha \in D_{1-}$, then for $z \in T_1$

$$|r^*(z)| \geqslant m(r,1) > |\alpha| m(r,1),$$

therefore, it follows by Rouche's theorem that all the zeros of rational function $F(z) = r^*(z) + \alpha m(r,1)$ lie in D_{1-}, so that the rational function

$$\begin{aligned} F^*(z) = B(z)\overline{F(1/\overline{z})} &= B(z)\big(\overline{r^*(1/\overline{z})} + \alpha m(r,1)\big), \\ &= B(z)\overline{r^*(1/\overline{z})} + \overline{\alpha} m(r,1)B(z), \\ &= r(z) + \overline{\alpha} m(r,1)B(z) \end{aligned}$$

has all its zeros in D_{1+}. Hence the function $F(z)/F^*(z)$ is analytic in $T_1 \cup D_{1-}$ and $|F(z)/F^*(z)| = 1$ for $z \in T_1$. Therefore, by the Maximum Modulus Principle, we have

$$|F(z)| \leqslant |F^*(z)| \quad \text{for } z \in T_1 \cup D_{1-}.$$

Replacing z by $1/\overline{z}$, we get

$$|F^*(z)| \leqslant |F(z)| \quad \text{for } z \in T_1 \cup D_{1+}.$$

Equivalently,

$$|r(z) + \overline{\alpha} m(r,1)B(z)| \leqslant |r^*(z) + \alpha m(r,1)|$$

for $z \in T_1 \cup D_{1+}$. This gives for every $R \geqslant 1$ and $z \in T_1$,

$$|r(Rz) + \overline{\alpha} m(r,1)B(Rz)| \leqslant |r^*(Rz) + \alpha m(r,1)|. \tag{22}$$

Choosing argument of α such that

$$|r(Rz) + \overline{\alpha} m(r,1)B(Rz)| = |r(Rz)| + |\alpha| m(r,1)|B(Rz)|,$$

we obtain from (22), for $z \in T_1$ and $R \geqslant 1$

$$\begin{aligned} |r(Rz)| + |\alpha| m(r,1)|B(rz)| &\leqslant |r^*(Rz) + \alpha m(r,1)| \\ &\leqslant |r^*(Rz)| + |\alpha| m(r,1), \end{aligned}$$

or,

$$|r(Rz)| + |\alpha|\big(|B(Rz)| - 1\big)m(r,1) \leqslant |r^*(Rz)|$$

for $z \in T_1$ and $R \geqslant 1$. Letting $|\alpha| \to 1$, we get for $z \in T_1$ and $R \geqslant 1$,

$$|r(Rz)| + \big(|B(Rz) - 1\big)m(r,1) \leqslant |r^*(Rz)|,$$

which gives with the help of Theorem 1, for $z \in T_1$ and $R \geqslant 1$

$$2|r(Rz)| + \big(|B(Rz) - 1\big)m(r,1) \leqslant \big(|B(Rz)| + 1\big)M(r,1) \leqslant |r^*(Rz)| + |r(Rz)|.$$

This implies for $z \in T_1$ and $R \geqslant 1$,

$$|r(Rz)| \leqslant \Big(\frac{|B(Rz)| + 1}{2}\Big)M(r,1) - \Big(\frac{|B(Rz) - 1}{2}\Big)m(r,1).$$

This proves inequality (10). To show that equality in (10) holds for $r(z) = B(z) + \lambda k$, $k \geqslant 1$ for suitably choosen $\lambda \in T_1$, we observe by Lemma 1 that all the zeros of $B(z) + \lambda k$, $k \geqslant 1$ lie in $T_1 \cup D_{1+}$. Also by Lemma 2, we have

$$M(r,1) = \sup_{z \in T_1} |B(z) + \lambda k| = k + 1$$

and

$$m(r,1) = \inf_{z\in T_1} |B(z)+\lambda k| = k-1$$

so that for fixed $\theta = \theta_0$, $0 \leqslant \theta_0 < 2\pi$ and for

$$\lambda = B(Re^{i\theta_0})/|B(Re^{i\theta_0})|,$$

we have

$$|r(Re^{i\theta_0})| = \left|B(Re^{i\theta_0}) + k\big(B(Re^{i\theta_0})/|B(Re^{i\theta_0})|\big)\right| = |B(Re^{i\theta_0})| + k.$$

Also,

$$\begin{aligned}&\Big(\frac{|B(Re^{i\theta_0})|+1}{2}\Big)M(r,1) - \Big(\frac{|B(Re^{i\theta_0})|-1}{2}\Big)m(r,1)\\ &\quad = \Big(\frac{|B(Re^{i\theta_0})|+1}{2}\Big)(k+1) - \Big(\frac{|B(Re^{i\theta_0})|-1}{2}\Big)(k-1) = |B(Re^{i\theta_0})| + k.\end{aligned}$$

Hence two sides of (10) are equal. This completes the proof of Theoem 3.

Proof of Theorem 4. Since $r(z)$ is self-inversive rational function, it follows that

$$|r^*(Rz)| = |r(Rz)| \quad \text{for } z \in T_1 \text{ and } R \geqslant 0.$$

Combining this with the conclusion of Theorem 1, we readily obtain (12) and this proves Theorem 4.

Proof of Theorem 5. If $r(z)$ has a zero on T_1, then $m(r,1) = 0$ and inequality (13) is trivial. Henceforth, we suppose $r(z)$ has all its zeros in D_{1-}. Therefore $r^*(z) = B(z)\overline{r(1/\overline{z})}$ has all its zeros in D_{1+} and

$$m(r,1) \leqslant |r(z)| = |r^*(z)| \quad \text{for } z \in T_1,$$

so that $m(r,1)/r^*(z)$ is analytic for $z \in T_1 \cup D_{1-}$ and by the Maximum Modulus Principle, it follows that

$$m(r,1) \leqslant |r^*(z)| = |P^*(z)/W(z)|$$

for $z \in T_1 \cup D_{1-}$. Replacing z by $1/\overline{z}$, we get

$$m(r,1) \leqslant |P(z)/W^*(z)| = |r(z)/B(z)| \quad \text{for } z \in T_1 \cup D_{1+}.$$

Equivalently

$$|r(z)| \geqslant |B(z)|m(r,1) \quad \text{for } z \in T_1 \cup D_{1+},$$

which implies for $z \in T_1$ and for every $R \geqslant 1$,

$$|r(Rz)| \geqslant |B(Rz)|m(r,1).$$

This completes the proof of Theorem 5.

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