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AN INEQUALITY FOR MIXED POWER MEANS

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(*communicated by J. Pečarić*)

Abstract. In 1992 Frank Hollad [1] stated the following inequality

$$(A_1A_2\dots A_n)^{\frac{1}{n}} \geqslant \frac{1}{n}(G_1G_2\dots G_n) \tag{1}$$

where A_k, G_k, $k = 1,2,\dots,n$ are arithmetic and geometric means, respectively, of positive numbers $a_1, a_2, ,\dots, a_k$.

In 1994 Kiran Kedlaya [2] gave a combinatorial proof of (1). In 1995 Takashi Matsuda [3] gave another proof of (1).

In 1996 B. Mond and J. Pečarić [5] proved the following generalization of inequality (1) involving power means:

$$\text{if} \quad s > r \quad \text{then} \quad m_{r,s}(a) \geqslant m_{s,r}(a), \tag{2}$$

where $m_{r,s}(a)$ is defined by the following definition (1.2).

In this article a more general inequality, which concern weighted power means, is proved.

1. Introduction

Let $a = (a_1, a_2, \dots, a_n)$ and $w = (w_1, w_2, \dots, w_n)$ be positive n-tuples, then the arithmetic, geometric and harmonic means of a with weights w are defined by:

$$A_n(a;w)=\frac{1}{W_n}\sum_{k=1}^{n} w_k a_k, \quad G_n(a;w)=\left(\prod_{k=1}^{n} a_k^{w_k}\right)^{\frac{1}{W_n}}, \quad H_n(a;w)=W_n\left(\sum_{k=1}^{n}\frac{w_k}{a_k}\right)^{-1}$$

where

$$W_k = \sum_{i=1}^{k} w_i, \quad k = 1,2,\dots,n.$$

When all weights are equal we write respectively A_n, G_n and H_n.

If r is a real number, then the weighted power mean of order r is defined as

$$M_n^{[r]}(a;w) = \left(\frac{1}{W_n}\sum_{k=1}^{n} w_k a_k^r\right)^{\frac{1}{r}}, \quad r \neq 0 \tag{1.1}$$

$$M_n^{[0]}(a;w) = G_n(a;w)$$

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When all weights are equal we write $M_n^{[r]}(a)$ or $M_n^{[r]}$.

We enter the following definition analogue of (1.1).

If r, s is arbitrary real numbers we define

$$m_{r,s}(a;w) = \left\{ \frac{1}{W_n} \sum_{k=1}^{n} w_k \left[M_k^{[s]}(a;w) \right]^r \right\}^{\frac{1}{r}}. \tag{1.2}$$

When all weights are equal we write $m_{r,s}(a)$.

We need the following Minkowski's inequality, which can be found in [4, p. 170–171].

Let $a = (a_1, a_2, \dots, a_n)$, $b = (b_1, b_2, \dots b_n)$ and $w = (w_1, w_2, \dots, w_n)$ be positive n-tuples, and define the n-tuple $a+b = (a_1+b_1, a_2+b_2, \dots, a_n+b_n)$. Then

$$M_n^{[p]}(a;w) + M_n^{[p]}(b;w) \geqslant M_n^{[p]}(a+b;w) \quad (p \geqslant 1) \tag{1.3}$$

$$G_n(a;w) + G_n(b;w) \leqslant G_n(a+b;w) \quad (p = 0) \tag{1.4}$$

if $p < 1$ then the inequality (1.3) is reversed.

2. The Main Result

REMARK 1. The following simple identities can be easily established [4, p. 132–134]

a) $G_n(a^s;w) = \left\{ G_n(a;w) \right\}^s$,

b) $M_n^{[s]}(a;w) = \left\{ A_n(a^s;w) \right\}^{\frac{1}{s}}, \quad s \neq 0$,

c) $M^{[rs]}(a;w) = \left\{ M_n^{[r]}(a^s;w) \right\}^{\frac{1}{s}}, \quad s \neq 0$,

where $a^s = (a_1^s, a_2^s, \dots, a_n^s)$.

First we prove the following Lemma

LEMMA. *Let $f : R \to R$ be a convex function and $a = (a_1, a_2, \dots, a_n)$, $w = (w_1, w_2, \dots, w_n)$ be two positive n-tuples $(n \geqslant 2)$ such that*

$$W_n w_k - W_k w_n > 0 \quad \textit{for} \quad 2 \leqslant k \leqslant n-1 \tag{2.1}$$

then

$$\frac{1}{W_{n-1}} \sum_{k=1}^{n-1} w_k f\left[W_{n-1} A_k(a;w) \right] \geqslant \frac{1}{W_n} \sum_{k=1}^{n} w_k f\left[W_n A_k(a;w) - w_n a_k \right]. \tag{2.2}$$

The equality holds in case $n = 2$ or if $f(x)$ is strictly convex in case $a_1 = a_2 = \dots = a_n$. If $f(x)$ is concave then the inequality (2.2) is reversed.

REMARK 2. In the special case where all weights are equal the condition (2.1) is satisfied and (2.2) is written

$$\frac{1}{n-1} \sum_{k=1}^{n-1} f(nA_k - A_k) \geqslant \frac{1}{n} \sum_{k=1}^{n} f(nA_k - a_k).$$

Proof of Lemma. For $n=2$ (2.2) is equality.

Let $n \geqslant 3$. Since

$$\frac{w_k W_n}{W_{n-1}} = w_k \frac{W_n w_k - w_n W_k}{w_k W_{n-1}} + w_{k+1} \frac{w_n W_k}{w_{k+1} W_{n-1}}, \quad k = 2,3,\ldots,n-1$$

the left hand of (2.2) is equal to

$$\frac{1}{W_n}\left\{w_1 f\left[W_{n-1}A_1(a;w)\right] + S + w_n f\left[W_{n-1}A_{n-1}(a;w)\right]\right\}$$
$$= \frac{1}{W_n}\left\{w_1 f\left[W_n A_1(a;w) - w_n a_1\right] + S + w_n f\left[W_n A_n(a;w) - w_n a_n\right]\right\}$$

where

$$S = \sum_{k=2}^{n-1} w_k \left\{ \frac{w_n W_{k-1}}{w_k W_{n-1}} f\left[W_{n-1}A_{k-1}(a;w)\right] + \frac{W_n w_k - w_n W_k}{w_k W_{n-1}} f\left[W_{n-1}A_k(a;w)\right] \right\}.$$

Since

$$\frac{w_n W_{k-1}}{w_k W_{n-1}} > 0, \quad \frac{W_n w_k - w_n W_k}{w_k W_{n-1}} > 0$$

and

$$\frac{w_n W_{k-1}}{w_k W_{n-1}} + \frac{W_n w_k - w_n W_k}{w_k W_{n-1}} = 1, \quad 2 \leqslant k \leqslant n-1,$$

we apply the convexity of f inside the braces of sum S, we get

$$S \geqslant \sum_{k=2}^{n-1} w_k f\left(\frac{w_n W_{k-1}}{w_k W_{n-1}} W_{n-1}A_{k-1}(a;w) + \frac{W_n w_k - w_n W_k}{w_k W_{n-1}} W_{n-1}A_k(a;w) \right)$$
$$= \sum_{k=2}^{n-1} w_k f\left[W_n A_k(a;w) - w_n a_k \right].$$

This proves the lemma. The case of equality is a simple consequence of the convexity of f.

COROLLARY. *If we choose* $f(x) = x^p$ *(* $x>0$, $p>1$ *or* $p<0$ *), we obtain*

$$\frac{1}{W_{n-1}} \sum_{k=1}^{n-1} w_k \left[W_{n-1}A_k(a;w)\right]^p \geqslant \frac{1}{W_n} \sum_{k=1}^{n} w_k \left[W_n A_k(a;w) - w_n a_k\right]^p. \tag{2.3}$$

If $0<p<1$ *then the inequality* (2.3) *is reversed.*

For $f(x) = \ln x$, $x>0$ *from* (2.2) *we obtain*

$$\left\{ \prod_{k=1}^{n-1} \left[W_{n-1}A_k(a;w)\right]^{w_k} \right\}^{\frac{1}{W_{n-1}}} \leqslant \left\{ \prod_{k=1}^{n} \left[W_n A_k(a;w) - w_n a_k\right]^{w_k} \right\}^{\frac{1}{W_n}}. \tag{2.4}$$

The equality for (2.4) *and* (2.3) *holds when* $n=2$ *or* $a_1 = a_2 = \cdots = a_n$.

THEOREM. *If $s > r$ and if $w = (w_1, w_2, \dots, w_n)$ satisfy (2.1) of Lemma, then*

$$m_{r,s}(a;w) \geqslant m_{s,r}(a;w) \tag{2.5}$$

with the equality if and only if $a_1 = a_2 = \cdots = a_n$.

Proof of Theorem. We consider five cases.

Case 1. $0 < r < s$ write $s = pr$ ($p > 1$).

Using remark 1 the inequality (2.5) is equivalent to

$$\left\{\frac{1}{W_n}\sum_{k=1}^{n} w_k [M_k^{[pr]}(a;w)]^r\right\}^{\frac{1}{r}} \geqslant \left\{\frac{1}{W_n}\sum_{k=1}^{n} w_k [M_k^{[r]}(a;w)]^{pr}\right\}^{\frac{1}{pr}}$$

or

$$\left\{\frac{1}{W_n}\sum_{k=1}^{n} w_k M_k^{[p]}(a^r;w)\right\}^{\frac{1}{r}} \geqslant \left\{\frac{1}{W_n}\sum_{k=1}^{n} w_k [A_k(a^r;w)]^{p}\right\}^{\frac{1}{pr}}.$$

Since $r > 0$ replacing a^r by a we get

$$\frac{1}{W_n}\sum_{k=1}^{n} w_k M_k^{[p]}(a;w) \geqslant \left\{\frac{1}{W_n}\sum_{k=1}^{n} w_k [A_k(a;w)]^{p}\right\}^{\frac{1}{p}}$$

or

$$\sum_{k=1}^{n} w_k M_k^{[p]}(a;w) \geqslant \left\{\frac{1}{W_n}\sum_{k=1}^{n} w_k [W_n A_k(a;w)]^{p}\right\}^{\frac{1}{p}}. \tag{2.6}$$

We prove the inequality (2.6) by induction on n. Let $n = 2$. Since

$$w_1 M_1^{[p]}(a;w) = \left\{\frac{1}{W_2}\sum_{k=1}^{2} w_k [W_2 A_k(a;w) - w_2 a_k]^{p}\right\}^{\frac{1}{p}}.$$

We have

$$\begin{aligned}
&w_1 M_1^{[p]}(a;w) + w_2 M_2^{[p]}(a;w)\\
&\quad = \left\{\frac{1}{W_2}\sum_{k=1}^{2} w_k [W_2 A_k(a;w) - w_2 a_k]^{p}\right\}^{\frac{1}{p}} + \left\{\frac{1}{W_2}\sum_{k=1}^{2} w_k (w_2 a_k)^{p}\right\}^{\frac{1}{p}}\\
&\quad \geqslant \left\{\frac{1}{W_2}\sum_{k=1}^{2} w_k [W_2 A_k(a;w) - w_2 a_k + w_2 a_k]^{p}\right\}^{\frac{1}{p}}\\
&\quad = \left\{\frac{1}{W_2}\sum_{k=1}^{2} w_k [W_2 A_k(a;w)]^{p}\right\}^{\frac{1}{p}}
\end{aligned}$$

where the last inequality follows from (1.3) by $n = 2$. Therefore the inequality (2.6) is valid for $n = 2$.

We assume that the inequality (2.6) is valid for $n - 1$ then

$$\sum_{k=1}^{n-1} w_k M_k^{[p]}(a;w) \geqslant \left\{\frac{1}{W_{n-1}}\sum_{k=1}^{n-1} w_k [W_{n-1} A_k(a;w)]^{p}\right\}^{\frac{1}{p}}. \tag{2.7}$$

Now we have

$$\begin{aligned}\sum_{k=1}^{n} w_k M_k^{[p]}(a;w) &= \sum_{k=1}^{n-1} w_k M_k^{[p]}(a;w) + w_n M_n^{[p]}(a;w)\\ &\geqslant \left\{\frac{1}{W_{n-1}}\sum_{k=1}^{n-1} w_k [W_{n-1}A_k(a;w)]^p\right\}^{\frac{1}{p}} + w_n\left\{\frac{1}{W_n}\sum_{k=1}^{n} w_k a_k^p\right\}^{\frac{1}{p}}\\ &\geqslant \left\{\frac{1}{W_n}\sum_{k=1}^{n} w_k [W_nA_k(a;w) - w_n a_k]^p\right\}^{\frac{1}{p}} + \left\{\frac{1}{W_n}\sum_{k=1}^{n} w_k (w_n a_k)^p\right\}^{\frac{1}{p}}\\ &\geqslant \left\{\frac{1}{W_n}\sum_{k=1}^{n} w_k [W_nA_k(a;w) - w_n a_k + w_n a_k]^p\right\}^{\frac{1}{p}}\\ &= \left\{\frac{1}{W_n}\sum_{k=1}^{n} w_k [W_nA_k(a;w)]^p\right\}^{\frac{1}{p}}\end{aligned}$$

where the first inequality follows from (2.7), the second inequality follows from (2.3) and the third inequality from (1.3). This completes the inductive proof.

CASE 2. $r < 0 < s$ we write $s = pr$ $(p < 0)$.
Using remark 1 and replacing a^r by a the inequality (2.5) is equivalent to

$$\sum_{k=1}^{n} w_k M_k^{[p]}(a;w) \leqslant \left\{\frac{1}{W_n}\sum_{k=1}^{n} w_k [W_nA_k(a;w)]^p\right\}^{\frac{1}{p}}. \tag{2.8}$$

The proof of (2.8) becomes by induction on n. The result follows the same way, as in case 1, by application of the corresponding cases (2.3) and (1.3) for $p < 0$. Namely all inequalities of *case* 1 are valid here reversed.

CASE 3. Let $0 = r < s$. Using remark 1 the inequality (2.5) is equivalent to

$$\left\{\prod_{k=1}^{n}[M_k^{[s]}(a;w)]^{w_k}\right\}^{\frac{1}{W_n}} \geqslant \left\{\frac{1}{W_n}\sum_{k=1}^{n} w_k [G_k(a;w)]^s\right\}^{\frac{1}{s}}$$

or

$$\left\{\prod_{k=1}^{n}[A_k(a^s;w)]^{\frac{w_k}{s}}\right\}^{\frac{1}{W_n}} \geqslant \left\{\frac{1}{W_n}\sum_{k=1}^{n} w_k G_k(a^s;w)\right\}^{\frac{1}{s}}.$$

Since $s > 0$ replacing a^s by a we get

$$\left\{\prod_{k=1}^{n}[A_k(a;w)]^{w_k}\right\}^{\frac{1}{W_n}} \geqslant \frac{1}{W_n}\sum_{k=1}^{n} w_k G_k(a;w). \tag{2.9}$$

We prove the inequality (2.9) by induction on n. Let $n = 2$. Since

$$w_1 G_1(a;w) = \left\{\prod_{k=1}^{2}[W_2A_k(a;w) - w_2 a_k]^{w_k}\right\}^{\frac{1}{W_2}}$$

we have

$$\begin{aligned}
w_1G_1(a;w)+w_2G_2(a;w)\\
&=\Big\{\prod_{k=1}^{2}[W_2A_k(a;w)-w_2a_k]^{w_k}\Big\}^{\frac{1}{W_2}}+w_2\Big\{\prod_{k=1}^{2}a_k^{w_k}\Big\}^{\frac{1}{W_2}}\\
&=\Big\{\prod_{k=1}^{2}[W_2A_k(a;w)-w_2a_k]^{w_k}\Big\}^{\frac{1}{W_2}}+\Big\{\prod_{k=1}^{2}(w_2a_k)^{w_k}\Big\}^{\frac{1}{W_2}}\\
&\leqslant\Big\{\prod_{k=1}^{2}[W_2A_k(a;w)-w_2a_k+w_2a_k]^{w_k}\Big\}^{\frac{1}{W_2}}=W_2\Big\{\prod_{k=1}^{2}A_k(a;w)]^{w_k}\Big\}^{\frac{1}{W_2}}
\end{aligned}$$

where the last inequality follows from (1.4) by $n=2$. Therefore the inequality (2.9) is valid for $n=2$.

We assume that the inequality (2.9) is valid for $n-1$ then

$$\Big\{\prod_{k=1}^{n-1}[A_k(a;w)]^{w_k}\Big\}^{\frac{1}{W_{n-1}}}\geqslant\frac{1}{W_{n-1}}\sum_{k=1}^{n-1}w_kG_k(a;w). \tag{2.10}$$

Now we have

$$\begin{aligned}
\sum_{k=1}^{n}w_kG_k(a;w)&=\sum_{k=1}^{n-1}w_kG_k(a;w)+w_nG_n(a;w)\\
&\leqslant W_{n-1}\Big\{\prod_{k=1}^{n-1}[A_k(a;w)]^{w_k}\Big\}^{\frac{1}{W_{n-1}}}+w_n\Big\{\prod_{k=1}^{n}a_k^{w_k}\Big\}^{\frac{1}{W_n}}\\
&=\Big\{\prod_{k=1}^{n-1}[W_{n-1}A_k(a;w)]^{w_k}\Big\}^{\frac{1}{W_{n-1}}}+\Big\{\prod_{k=1}^{n}(w_na_k)^{w_k}\Big\}^{\frac{1}{W_n}}\\
&\leqslant\Big\{\prod_{k=1}^{n}[W_nA_k(a;w)-w_na_k]^{w_k}\Big\}^{\frac{1}{W_n}}+\Big\{\prod_{k=1}^{n}(w_na_k)^{w_k}\Big\}^{\frac{1}{W_n}}\\
&\leqslant\Big\{\prod_{k=1}^{n}[W_nA_k(a;w)]^{w_k}\Big\}^{\frac{1}{W_n}}=W_n\Big\{\prod_{k=1}^{n}[A_k(a;w)]^{w_k}\Big\}^{\frac{1}{W_n}}
\end{aligned}$$

where the first inequality follows from (2.10), the second inequality follows from (2.4) and the third inequality from (1.4). This completes the inductive proof.

CASE 4. Let $r<s<0$. We write $s=pr$ $(0<p<1)$.

Using remark 1 and replacing a^r by a the inequality (2.5) is equivalent to

$$\sum_{k=1}^{n}w_kM_k^{[p]}(a;w)\leqslant\Big\{\frac{1}{W_n}\sum_{k=1}^{n}w_k[W_nA_k(a;w)]^p\Big\}^{\frac{1}{p}}. \tag{2.11}$$

The prove of (2.11) becomes by induction on n. The result follows the same way, as in case 1, by application of the corresponding case (2.3) and (1.3) for $0 < p < 1$.

CASE 5. Let $r < s = 0$.

Using Remark 1 and replacing a^r by a the inequality (2.5) is again equivalent to (2.9). This has been proved in case 3.

This completes the proof of Theorem.

NOTATION. If $w_1 \geqslant w_2 \geqslant \ldots \geqslant w_n$ the condition (2.1) is satisfied and (2.5) holds in this case too.

Applications of theorems

Let

$$A(a;w) = (A_1(a;w), A_2(a;w), \ldots, A_n(a;w))$$
$$G(a;w) = (G_1(a;w), G_2(a;w), \ldots, G_n(a;w))$$
$$H(a;w) = (H_1(a;w), H_2(a;w), \ldots, H_n(a;w)).$$

1. For $s = 1$ and $r = 0$ we have a mixed arithmetic-geometric mean inequality

$$G_n(A(a;w);w) \geqslant A_n(G(a;w);w).$$

2. For $s = 0$ and $r = -1$ we have a mixed geometric-harmonic mean inequality

$$H_n(G(a;w);w) \geqslant G_n(H(a;w);w).$$

3. For $s = 1$ and $r = -1$ we have a mixed arithmetic-hannonic mean inequality

$$H_n(A(a;w);w) \geqslant A_n(H(a;w);w).$$

REFERENCES

[1] FRANK HOLLAND, *On a mixed arithmetic-mean, geometric-mean inequality*, Mathematics Competitions **5** (1992), 60–64.
[2] KIRAN KEDLAYA, *Proof of a Mixed Arithmetic-Mean, Geometric-Means Inequality,*, Amer. Math. Monthly **101** (1994), 355–357.
[3] TAKASHI MATSUDA, *An Inductive Proof of a Mixed Arithmetic-Geometric Mean Inequality*, Amer. Math. Monthly **102** (1995), 634–637.
[4] BULLEN, P. S., D. S. MITRINOVIC AND P. M. VASIC, *Means and Their Inequalities*, D. Reidel Publishing Company, Dordrecht, 1988.
[5] B.MOND AND J.PECARIC, *A mixed means inequality*, Austral. Math. Soc. Gazette **23** (1996), 67–70.

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