

NEW ESTIMATION OF THE REMAINDER IN TAYLOR'S FORMULA USING GRÜSS' TYPE INEQUALITIES AND APPLICATIONS

SEVER SILVESTRU DRAGOMIR

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Abstract. Some perturbed Taylor's expansions via Grüss' type integral inequalities and applications for elementary mappings are given.

1. Introduction

The following theorem is well known in the literature as Taylor's formula or Taylor's theorem with the integral remainder:

THEOREM 1.1. *Let $I \subset \mathbf{R}$ be a closed interval, let $a \in I$ and let n be a positive integer. If $f : I \rightarrow \mathbf{R}$ is such that $f^{(n)}$ is absolutely continuous, then for each $x \in I$*

$$f(x) = T_n(f; a, x) + R_n(f; a, x) \quad (1.1)$$

where $T_n(f; a, x)$ is Taylor's polynomial, i.e.,

$$T_n(f; a, x) := \sum_{k=0}^n \frac{(x-a)^k}{k!} f^{(k)}(a) \quad (1.2)$$

(note that $f^{(0)} = f$ and $0! = 1$), and the remainder is given by:

$$R(f; a, x) := \frac{1}{n!} \int_a^x (x-t)^n f^{(n+1)}(t) dt. \quad (1.3)$$

A simple proof of this theorem can be done by mathematical induction using the integration by parts formula.

The following corollary concerning the estimation of the remainder is useful when we want to approximate concrete functions by their Taylor's expansions.

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COROLLARY 1.2. *With the above assumptions, we have the estimations:*

$$|R(f; a, x)| \leq \frac{(x-a)^n}{n!} \int_a^x |f^{(n+1)}(t)| dt; \quad (1.4)$$

or

$$|R_n(f; a, x)| \leq \frac{1}{n!} \left[\frac{(p-1)(x-a)^{[(n+1)p-1]/(p-1)}}{(n+1)p-1} \right]^{\frac{p-1}{p}} \left(\int_a^x |f^{(n+1)}(t)|^p dt \right)^{\frac{1}{p}} \quad (1.5)$$

when $p > 1$, or, the estimation:

$$|R_n(f; a, x)| \leq \frac{(x-a)^{n+1}}{(n+1)!} \max_{t \in (a,x)} |f^{(n+1)}(t)| \quad (1.6)$$

for all $x \geq a, x \in I$.

Proof. The estimation (1.4) and (1.6) are obvious.

Using Hölder's integral inequality we have that:

$$\begin{aligned} \left| \int_a^x (x-t)^n f^{(n+1)}(t) dt \right| &\leq \left(\int_a^x |f^{(n+1)}(t)|^p dt \right)^{\frac{1}{p}} \left(\int_a^x (x-t)^{nq} dt \right)^{\frac{1}{q}} \\ &= \left[\frac{(x-a)^{nq+1}}{nq+1} \right]^{1/q} \left(\int_a^x |f^{(n+1)}(t)|^p dt \right)^{\frac{1}{p}}. \end{aligned}$$

But $q = \frac{p}{p-1}$, and then

$$\left[\frac{(x-a)^{nq+1}}{nq+1} \right]^{1/q} = \left[\frac{(p-1)(x-a)^{[(n+1)p-1]/(p-1)}}{(n+1)p-1} \right]^{(p-1)/p}$$

and the corollary is proven. \square

2. The results

We can state and prove the following theorem which provides another type of approximation for the mappings using their Taylor's expansions perturbed by some new terms as follows:

THEOREM 2.1. *Let $f : I \rightarrow \mathbf{R}$ be as above and $a \in I$. Then we have the Taylor's perturbed formula:*

$$f(x) = T_n(f; a, x) + \frac{(x-a)^{n+1}}{(n+1)!} \cdot [f^{(n)}; a, x] + G_n(f; a, x); \quad (2.1)$$

the remainder $G_n(f; a, x)$ satisfies the estimation

$$|G_n(f; a, x)| \leq \frac{1}{4} \cdot \frac{(x-a)^{n+1}}{n!} [\Gamma(x) - \gamma(x)] \tag{2.2}$$

where $[f^{(n)}; a, x]$ is the divided difference of $f^{(n)}$ in the points a and x , i.e.,

$$[f^{(n)}; a, x] = \frac{f^{(n)}(x) - f^{(n)}(a)}{x - a}$$

and

$$\Gamma(x) := \sup_{t \in [a, x]} f^{(n+1)}(t), \gamma(x) := \inf_{t \in [a, x]} f^{(n+1)}(t) \tag{2.3}$$

for all $x \geq a, x \in I$.

Proof. We shall use the following well known Grüss' inequality (see for example [1, p. 296])

$$\left| \frac{1}{b-a} \int_a^b g(x) h(x) dx - \frac{1}{b-a} \int_a^b g(x) \cdot \frac{1}{b-a} \int_a^b h(x) dx \right| \tag{2.4}$$

$$\leq \frac{1}{4} (\Phi - \varphi) (\Theta - \eta)$$

where g, h are integrable on $[a, b]$ and $\varphi \leq g(x) \leq \Phi, \eta \leq h(x) \leq \Theta$ for all $x \in [a, b]$.

Thus, by Grüss' inequality we have:

$$\left| \frac{1}{x-a} \int_a^x (x-t)^n f^{(n+1)}(t) dt - \frac{1}{x-a} \int_a^x (x-t)^n dt \cdot \frac{1}{x-a} \int_a^x f^{(n+1)}(t) dt \right|$$

$$\leq \frac{1}{4} (x-a)^n [\Gamma(x) - \gamma(x)]$$

and as

$$\int_a^x (x-t)^n dt = \frac{(x-a)^{n+1}}{n+1}, \int_a^x f^{(n+1)}(t) dt = f^{(n)}(x) - f^{(n)}(a)$$

we get

$$\left| \int_a^x (x-t)^n f^{(n+1)}(t) dt - \frac{(x-a)^{n+1}}{n+1} \cdot \frac{f^{(n)}(x) - f^{(n)}(a)}{x-a} \right| \tag{2.5}$$

$$\leq \frac{1}{4} (x-a)^{n+1} [\Gamma(x) - \gamma(x)]$$

and as

$$n!R_n(f; a, x) = \int_a^x (x-t)^n f^{(n+1)}(t) dt$$

then by (2.5) we have:

$$\left| R_n(f; a, x) - \frac{(x-a)^{n+1}}{(n+1)!} \cdot \frac{f^{(n)}(x) - f^{(n)}(a)}{x-a} \right| \leq \frac{(x-a)^{n+1} [\Gamma(x) - \gamma(x)]}{4n!}. \quad (2.6)$$

Using the classical Taylor's formula (1.1) we deduce the representation (2.1) and the estimation (2.2) for the remainder term $G_n(f; a, x)$. \square

Before we shall prove the second estimate of the remainder $G_n(f; a, x)$ in terms of $f^{(n+2)}$, we need the following lemma which is interesting in itself too (see for example [1] or its extension for positive functionals from [2]):

LEMMA 2.2. *Let $g, h : [a, b] \rightarrow \mathbf{R}$ be two differentiable mappings and*

$$M_1^{(1)} := \sup_{x \in [a, b]} |h'(x)| < \infty, M_2^{(1)} := \sup_{x \in [a, b]} |g'(x)| < \infty.$$

Then we have the following Grüss' type inequality:

$$\left| (b-a) \int_a^b h(x) g(x) dx - \int_a^b h(x) dx \cdot \int_a^b g(x) dx \right| \quad (2.7)$$

$$\leq \frac{M_1^{(1)} M_2^{(1)}}{12} (b-a)^4.$$

Proof. By Lagrange's theorem we have that:

$$|h(x) - h(y)| \leq M_1^{(1)} |x - y|,$$

$$|g(x) - g(y)| \leq M_2^{(1)} |x - y|$$

for all $x, y \in [a, b]$, from where we deduce:

$$|(h(x) - h(y))(g(x) - g(y))| \leq M_1^{(1)} M_2^{(1)} (x - y)^2$$

for all $x, y \in [a, b]$.

Integrating on $[a, b]$ over x and y we deduce

$$\left| \int_a^b \int_a^b (h(x) - h(y))(g(x) - g(y)) dx dy \right| \leq M_1^{(1)} M_2^{(1)} \int_a^b \int_a^b (x - y)^2 dx dy. \quad (2.8)$$

But:

$$\int_a^b \int_a^b (h(x) - h(y))(g(x) - g(y)) dx dy$$

$$= 2 \left[(b-a) \int_a^b h(x) g(x) dx - \int_a^b h(x) dx \cdot \int_a^b g(x) dx \right]$$

and

$$\int_a^b \int_a^b (x-y)^2 dx dy = 2 \left[(b-a) \int_a^b x^2 dx - \left(\int_a^b x dx \right)^2 \right] = \frac{(b-a)^4}{12}$$

and then by (2.8) we deduce (2.7). \square

REMARK 2.3. For some other similar results holding for positive linear functionals see the paper [2].

The following theorem holds:

THEOREM 2.4. Suppose that $f^{(n+1)}$ is differentiable on I and

$$M^{(n+2)}(x) := \sup_{t \in [a, x]} |f^{(n+2)}(t)| < \infty.$$

Then we have the representation (2.1) where $G_n(f; a, x)$ satisfies the estimation:

$$|G_n(f; a, x)| \leq \frac{(x-a)^{n+2} M^{(n+2)}(x)}{12(n-1)!} \quad (2.9)$$

for all $x \geq a$, $x \in I$.

Proof. If we shall apply the above lemma for

$$h(t) := (x-t)^n, g(t) := f^{(n+1)}(t), t \in [a, x]$$

we get

$$\begin{aligned} |h'(t)| &\leq n(x-a)^{n-1} \quad \text{for all } t \in [a, x]; \\ |g'(t)| &\leq M^{(n+2)}(x) \quad \text{for all } t \in [a, x] \end{aligned}$$

and, obviously

$$\begin{aligned} &\left| (x-a) \int_a^x (x-t)^n f^{(n+1)}(t) dt - \int_a^x (x-t)^n dt \cdot \int_a^x f^{(n+1)}(t) dt \right| \\ &\leq \frac{n(x-a)^{n-1} M^{(n+2)}(x) (x-a)^4}{12} \end{aligned}$$

from where we get:

$$\left| \int_a^x (x-t)^n f^{(n+1)}(t) dt - \frac{(x-a)^{n+1}}{n+1} \cdot \frac{f^{(n)}(x) - f^{(n)}(a)}{x-a} \right| \leq \frac{n(x-a)^{n+2} M^{(n+2)}(x)}{12}.$$

Now, if we use the Taylor's formula (1.1) we deduce the corresponding representation (2.1) and the estimation (2.9).

We shall omit the details. \square

To prove the next estimation for the remainder term in the representation (2.1), we need the following lemma which is interesting in itself too (see for example the paper [2]).

LEMMA 2.5. Assume that h, g are differentiable on $[a, b]$, $g'(x) \neq 0$ on (a, b) and

$$M^{(1)} := \sup_{x \in [a, b]} \left| \frac{h'(x)}{g'(x)} \right| < \infty.$$

Then we have the inequality:

$$\begin{aligned} & \left| (b-a) \int_a^b h(x) g(x) dx - \int_a^b h(x) dx \cdot \int_a^b g(x) dx \right| \\ & \leq M^{(1)} \left[(b-a) \int_a^b g^2(x) dx - \left(\int_a^b g(x) dx \right)^2 \right]. \end{aligned} \quad (2.10)$$

Proof. Using Cauchy's mean value theorem we have:

$$|h(x) - h(y)| \leq M^{(1)} |g(x) - g(y)|$$

for all $x, y \in [a, b]$, from where we deduce:

$$|(h(x) - h(y))(g(x) - g(y))| \leq M^{(1)} (g(x) - g(y))^2$$

for all $x, y \in [a, b]$.

Integrating this inequality over x and y on $[a, b]$ we get:

$$\left| \int_a^b \int_a^b (h(x) - h(y))(g(x) - g(y)) dx dy \right| \leq M^{(1)} \int_a^b \int_a^b (g(x) - g(y))^2 dx dy, \quad (2.11)$$

and as:

$$\begin{aligned} & \left| \int_a^b \int_a^b (h(x) - h(y))(g(x) - g(y)) dx dy \right| \\ & = 2 \left[(b-a) \int_a^b h(x) g(x) dx - \int_a^b h(x) dx \cdot \int_a^b g(x) dx \right] \end{aligned}$$

and

$$\int_a^b \int_a^b (g(x) - g(y))^2 dx dy = 2 \left[(b-a) \int_a^b g^2(x) dx - \left(\int_a^b g(x) dx \right)^2 \right]$$

then from (2.11) we deduce (2.10). \square

REMARK 2.6. For some results in connection with the above lemma holding for positive linear functionals see the paper [2].

Now, we can state and prove another estimation result for the remainder term in the representation formula (2.1).

THEOREM 2.7. Assume that $f^{(n+1)}$ is differentiable on I and satisfies the condition:

$$\left| f^{(n+2)}(t) \right| \leq M_1^{(n+2)}(x) (x-t)^{n-1}, t \in [a, x], x \in I, x \geq a. \quad (2.12)$$

Then we have the representation (2.1) and the remainder term $G_n(f; a, x)$ satisfies the inequality:

$$|G_n(f; a, x)| \leq \frac{M_1^{(n+2)}(x) (x-a)^{2n+1}}{(n-1)!(2n+1)(n+1)^2} \quad (2.13)$$

for all $x \in I, x \geq a$.

Proof. Let $h(t) := f^{(n+1)}(t)$ and $g(t) := (x-t)^n; t \in [a, x], x \in I, x \geq a$. Then:

$$\left| \frac{h'(t)}{g'(t)} \right| = \frac{|f^{(n+2)}(t)|}{n(x-t)^{n-1}} \leq \frac{M_1^{(n+2)}(x)}{n}.$$

Applying Lemma 2.5 we have that:

$$\begin{aligned} & \left| (x-a) \int_a^x f^{(n+1)}(t) (x-t)^n dt - \int_a^x (x-t)^n dt \cdot \int_a^x f^{(n+1)}(t) dt \right| \\ & \leq \frac{M_1^{(n+1)}(x)}{n} \left[(x-a) \int_a^x (x-t)^{2n} dt - \left(\int_a^x (x-t)^n dt \right)^2 \right] \\ & = \frac{M_1^{(n+1)}(x)}{n} \left[\frac{(x-a)(x-a)^{2n+1}}{2n+1} - \left[\frac{(x-a)^{n+1}}{n+1} \right]^2 \right] \\ & = \frac{M_1^{(n+1)}(x) (x-a)^{2n+2}}{n} \cdot \frac{n^2}{(2n+1)(n+1)^2} = \frac{nM_1^{(n+2)}(x) (x-a)^{2n+2}}{(2n+1)(n+1)^2}, \end{aligned}$$

from where we deduce:

$$\begin{aligned} & \left| \int_a^x f^{(n+1)}(t) (x-t)^n dt - \frac{(x-a)^{n+1}}{n+1} \cdot \frac{f^{(n)}(x) - f^{(n)}(a)}{x-a} \right| \\ & \leq \frac{nM_1^{(n+2)}(x) (x-a)^{2n+2}}{(2n+1)(n+1)^2}. \end{aligned}$$

Using Taylor's formula (1.1), we deduce the representation (2.1) and the estimation for the remainder term $G_n(f; a, x)$ given in (2.13). \square

3. Applications for the exponential mapping

Consider $f : \mathbf{R} \rightarrow \mathbf{R}$, $f(x) = e^x$. Then obvious $f^{(n)}(x) = e^x$, $n \in \mathbf{N}$ and

$$T_n(f; a, x) = e^a \sum_{k=0}^n \frac{(x-a)^k}{k!}.$$

Using the estimation (1.4) we get:

$$0 \leq e^x - e^a \sum_{k=0}^n \frac{(x-a)^k}{k!} \leq \frac{(x-a)^n}{n!} (e^x - e^a)$$

for all $x \geq a$ and particularly:

$$0 \leq e^x - \sum_{k=0}^n \frac{x^k}{k!} \leq \frac{x^n}{n!} (e^x - 1)$$

for all $x \geq 0$.

By the estimate (1.5) we deduce:

$$0 \leq e^x - e^a \sum_{k=0}^n \frac{(x-a)^k}{k!} \leq \frac{1}{n!} \left[\frac{(p-1)(x-a)^{[(n+1)p-1]/(p-1)}}{(n+1)p-1} \right]^{\frac{p-1}{p}} \left(\frac{e^{px} - e^{pa}}{p} \right)^{1/p}$$

for all $x \geq a$ and $p > 1$ and, particularly,

$$0 \leq e^x - e^a \sum_{k=0}^n \frac{x^k}{k!} \leq \frac{1}{n!} \left[\frac{(p-1)x^{[(n+1)p-1]/(p-1)}}{(n+1)p-1} \right]^{\frac{p-1}{p}} \left(\frac{e^{px} - 1}{p} \right)^{1/p}$$

for all $x \geq 0$.

Using the estimation (1.6) we get the well known inequality

$$0 \leq e^x - e^a \sum_{k=0}^n \frac{(x-a)^k}{k!} \leq \frac{(x-a)^{n+1}}{(n+1)!} e^x$$

for all $x \geq a$ and, particularly,

$$0 \leq e^x - \sum_{k=0}^n \frac{x^k}{k!} \leq \frac{x^{n+1}}{(n+1)!} e^x$$

for all $x \geq 0$.

From the estimate (2.1) we also get that:

$$\left| e^x - e^a \sum_{k=0}^n \frac{(x-a)^k}{k!} - \frac{(x-a)^n (e^x - e^a)}{(n+1)!} \right| \leq \frac{(x-a)^{n+1} (e^x - e^a)}{4n!}$$

for all $x \geq a$ and, particularly,

$$\left| e^x - \sum_{k=0}^n \frac{x^k}{k!} - \frac{x^n (e^x - 1)}{(n+1)!} \right| \leq \frac{x^{n+1} (e^x - 1)}{4n!}$$

for all $x \geq 0$.

By the inequality (2.9) we get that:

$$\left| e^x - e^a \sum_{k=0}^n \frac{(x-a)^k}{k!} - \frac{(x-a)^n (e^x - e^a)}{(n+1)!} \right| \leq \frac{(x-a)^{n+2} e^x}{12(n-1)!} \quad (3.1)$$

for all $x \geq a$ and, particularly,

$$\left| e^x - \sum_{k=0}^n \frac{x^k}{k!} - \frac{x^n (e^x - 1)}{(n+1)!} \right| \leq \frac{x^{n+2} e^x}{12(n-1)!} \quad (3.2)$$

for all $x \geq 0$.

From the estimate (2.13) we get that:

$$\left| e^x - e^a \sum_{k=0}^n \frac{(x-a)^k}{k!} - \frac{(x-a)^n (e^x - e^a)}{(n+1)!} \right| \leq \frac{(x-a)^{n+2} e^x}{(n-1)! (2n+1) (n+1)^2} \quad (3.3)$$

for all $x \geq a$ and, particularly,

$$\left| e^x - \sum_{k=0}^n \frac{x^k}{k!} - \frac{x^n (e^x - 1)}{(n+1)!} \right| \leq \frac{x^{n+2} e^x}{(n-1)! (2n+1) (n+1)^2}. \quad (3.4)$$

REMARK. Note that the estimation given by (3.3) is better than the estimation (3.1) and the same for (3.4) and (3.2).

4. Applications for the logarithmic mapping

Consider the mapping $f : (0, \infty) \rightarrow \mathbf{R}$, $f(x) = \ln x$. Then

$$f^{(n)}(x) = \frac{(-1)^{n-1} (n-1)!}{x^n}, n \geq 1, x > 0$$

and

$$T_n(f; a, x) = \ln a + \sum_{k=1}^n \frac{(-1)^{k-1} (x-a)^k}{ka^k}, \quad a > 0.$$

Using the inequality (1.4) we get :

$$\left| \ln x - \ln a - \sum_{k=1}^n \frac{(-1)^{k-1} (x-a)^k}{ka^k} \right| \leq \frac{(x-a)^n}{n!} \int_a^x \frac{n!}{t^{n+1}} dt, x \geq a$$

which is equivalent with

$$\left| \ln \left(\frac{x}{a} \right) - \sum_{k=1}^n \frac{(-1)^{k-1} (x-a)^k}{ka^k} \right| \leq \frac{(x-a)^n}{n} \cdot \frac{x^n - a^n}{a^n x^n} \quad (4.1)$$

for all $x \geq a$, and, particularly,

$$\left| \ln(x+1) - \sum_{k=1}^n \frac{(-1)^{k-1} x^k}{k} \right| \leq \frac{x^n [(x+1)^n - 1]}{n(x+1)^n} \quad (4.2)$$

for all $x \geq 0$.

By the inequality (1.5) we get:

$$\begin{aligned} & \left| \ln x - \ln a - \sum_{k=1}^n \frac{(-1)^{k-1} (x-a)^k}{ka^k} \right| \\ & \leq \frac{1}{n!} \left[\frac{(p-1)(x-a)^{[(n+1)p-1]/(p-1)}}{(n+1)p-1} \right]^{\frac{p-1}{p}} \left(\int_a^b \frac{[n!]^p dx}{x^{(n+1)p}} \right)^{1/p} \end{aligned}$$

which is equivalent with:

$$\begin{aligned} & \left| \ln \left(\frac{x}{a} \right) - \sum_{k=1}^n \frac{(-1)^{k-1} (x-a)^k}{ka^k} \right| \quad (4.3) \\ & \leq \frac{1}{[(n+1)p-1]} \left[(p-1)(x-a)^{[(n+1)p-1]/(p-1)} \right]^{\frac{p-1}{p}} \left[\frac{x^{(n+1)p-1} - a^{(n+1)p-1}}{a^{(n+1)p-1} x^{(n+1)p-1}} \right]^{\frac{1}{p}} \end{aligned}$$

for all $x \geq a$ and $p > 1$, and, particularly,

$$\begin{aligned} & \left| \ln x - \sum_{k=1}^n \frac{(-1)^{k-1} x^k}{k} \right| \quad (4.4) \\ & \leq \frac{1}{[(n+1)p-1]} \left[(p-1)x^{[(n+1)p-1]/(p-1)} \right]^{\frac{p-1}{p}} \left[\frac{(x+1)^{(n+1)p-1} - 1}{(x+1)^{(n+1)p-1}} \right]^{\frac{1}{p}} \end{aligned}$$

for all $x \geq 1$ and $p > 1$.

Using the inequality (1.6) we deduce the classical result:

$$\left| \ln \left(\frac{x}{a} \right) - \sum_{k=1}^n \frac{(-1)^{k-1} (x-a)^k}{ka^k} \right| \leq \frac{(x+1)^{n+1}}{(n+1)a^{n+1}} \quad (4.5)$$

for all $x \geq a$, and particularly

$$\left| \ln(x+1) - \sum_{k=1}^n \frac{(-1)^{k-1} x^k}{k} \right| \leq \frac{x^{n+1}}{n+1} \quad (4.6)$$

for all $x \geq 0$.

We shall apply now Theorem 2.1. We have:

$$f^{(n)}(x) - f^{(n)}(a) = (-1)^{n-1} (n-1)! \left[\frac{1}{x^n} - \frac{1}{a^n} \right] = \frac{(-1)^{n-1} (n-1)! (x^n - a^n)}{a^n x^n}$$

$$\Gamma(x) - \gamma(x) = n! \left[\frac{1}{a^{n+1}} - \frac{1}{x^{n+1}} \right] = \frac{n! (x^{n+1} - a^{n+1})}{a^{n+1} x^{n+1}}$$

and thus

$$\begin{aligned} & \left| \ln x - \ln a - \sum_{k=1}^n \frac{(-1)^{k-1} (x-a)^k}{ka^k} - \frac{(-1)^n (x^n - a^n)}{n(n+1)a^n x^n} \right| \\ & \leq \frac{1}{4} \frac{(x-a)^{n+1}}{n!} \cdot \frac{n! (x^{n+1} - a^{n+1})}{a^{n+1} x^{n+1}}, x \geq a \end{aligned}$$

which is equivalent with:

$$\left| \ln \left(\frac{x}{a} \right) - \sum_{k=1}^n \frac{(-1)^{k-1} (x-a)^k}{ka^k} - \frac{(-1)^n (x^n - a^n)}{n(n+1)a^n x^n} \right| \leq \frac{(x-a)^{n+1} (x^{n+1} - a^{n+1})}{4a^{n+1} x^{n+1}}$$

for all $x \geq a$ and, particularly,

$$\left| \ln(x+1) - \sum_{k=1}^n \frac{(-1)^{k-1} x^k}{k} - \frac{(-1)^n [(x+1)^n - 1]}{n(n+1)a^n x^n} \right| \leq \frac{x^n [(x+1)^{n+1} - 1]}{4(x+1)^{n+1}}$$

for all $x \geq 0$.

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Sever Silvestru Dragomir
School of Communications and Informatics
Victoria University of Technology
PO Box 14428, MCMC, Melbourne
Victoria, 8001, Australia
e-mail: sever@matilda.vut.edu.au