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INEQUALITIES FOR A POLYNOMIAL AND ITS DERIVATIVE

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Abstract. Improved version of the Rivlin's theorem has been stated and proved.

1. Introduction awd statement of results

Let $p(z)$ be a polynomial of degree n then the following inequality is well known:

$$\max_{|z|=r} |p(z)| \geqslant r^n \max_{|z|=1} |p(z)| \quad \text{for} \quad r \leqslant 1. \tag{1.1}$$

Inequality (1.1) is due to Zarantonello and Varga [6] and equality holds for all polynomials whose zeros lie at the origin.

Rivlin [5] considered the class of polynomials not vanishing in $[z] < 1$ and proved the following inequality analogous to (1.1)

$$\max_{|z|=r} |p(z)| \geqslant \left(\frac{1+r}{2}\right)^n \max_{|z|=1} |p(z)| \quad \text{for } r \leqslant 1, \tag{1.2}$$

which is much better than (1.1). Besides equality in (1.2) holds for $p(z) = \left(\frac{\alpha+\beta z}{2}\right)^n$ where $|\alpha| = |\beta|$.

The class of polynomials $p(z) = a_0 + \sum\limits_{v=\mu}^{n} a_v z^v$ having no zeros in $|z| < K$, $K \geqslant 1$ was considered by Dewan [3] (see also [2]) who generalized inequality (1.2) by proving the following

THEOREM A. *If* $p(z) = a_0 + \sum\limits_{v=\mu}^{n} a_v z^v$ *has no zeros in* $|z| < K$, $K \geqslant 1$, *then for* $0 \leqslant r \leqslant \lambda \leqslant 1$

$$\max_{|z|=r} |p(z)| \geqslant \left(\frac{K^\mu + r^\mu}{K^\mu + \lambda^\mu}\right)^{n/\mu} \max_{|z|=\lambda} |p(z)|. \tag{1.3}$$

The equality holds for polynomials of the form $p(z) = (z^\mu + K^\mu)^{n/\mu}$, *where* n *is a multiple of* μ.

In this paper we shall prove the following extension and generalization of Theorem A.

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THEOREM. *Let* $p(z) = a_0 + \sum_{v=\mu}^{n} a_v z^v$ *be a polynomial of degree* n, *such that* $p(z) \neq 0$ *in* $|z| < K$, $K > 0$, *then for* $0 < r \leqslant R \leqslant K$,

$$\max_{|z|=r} |p(z)| \geqslant \left(\frac{K^\mu + r^\mu}{K^\mu + R^\mu}\right)^{n/\mu} \max_{|z|=R} |p(z)| + \left[1 - \left(\frac{K^\mu + r^\mu}{K^\mu + R^\mu}\right)^{n/\mu}\right] \min_{|z|=K} |p(z)|. \quad (1.4)$$

The equality holds for polynomials of the form $p(z) = (z^\mu + K^\mu)^{n/\mu}$ *where* n *is a multiple of* μ.

REMARK. For $\mu = 1$ the above Theorem also provides, in general, an improvement over a result due to Bidkham and Dewan [2] and for $K = 1$, $\mu = 1$ it improves upon a result due to Govil [4].

2. A Lemma

We need the following result for the proof of our theorem.

LEMMA. *Let* $p(z) = a_0 + \sum_{v=\mu}^{n} a_v z^v$ *be a polynomial of degree* n, *such that* $p(z) \neq 0$ *in* $|z| < K$, $K \geqslant 1$, *then*

$$\max_{|z|=1} |p'(z)| \leqslant \frac{n}{1 + K^\mu} \left\{ \max_{|z|=1} |p(z)| - \min_{|z|=K} |p(z)| \right\}.$$

The equality holds for polynomials of the form $p(z) = \left(z^\mu + K^\mu\right)^{n/\mu}$, *where* n *is a multiple of* μ.

The above lemma is a particular case of a result due to Aziz and Rathar [1,Theorem 3, equation (14)].

3. Proof of Theorem

Since $p(z)$ has no zeros in $|z| < K$, $K > 0$ then for $0 < t \leqslant K$, $P(z) = p(tz)$ has no zeros in $|z| < K/t$, $K/t \geqslant 1$. Hence on using the Lemma, we get

$$\max_{|z|=1} |P'(z)| \leqslant \frac{n}{1 + \frac{K^\mu}{t^\mu}} \left\{ \max_{|z|=1} |P(z)| - \min_{|z|=K/t} |P(z)| \right\}$$

or

$$\max_{|z|=t} |p'(z)| \leqslant \frac{nt^{\mu-1}}{K^\mu + t^\mu} \left\{ \max_{|z|=t} |p(z)| - \min_{|z|=K} |p(z)| \right\}. \quad (3.1)$$

Let $M(p, r) = \max_{|z|=r} |p(z)|$, and $m(p, r) = \min_{|z|=r} |p(z)|$, then (3.1) is equivalent to

$$M(p', t) \leqslant \frac{nt^{\mu-1}}{K^\mu + t^\mu} \left\{ M(p, t) - m(p, K) \right\}. \quad (3.2)$$

Now for $0 < r \leqslant R \leqslant K$ and $0 \leqslant \theta < 2\pi$, we have

$$\left|p\left(Re^{i\theta}\right)\right| \leqslant \left|p\left(re^{i\theta}\right)\right| + \int_r^R |p'(te^{i\theta}|dt$$

which implies

$$M(p,R) \leqslant M(p,r) + \int_r^R M(p',t)dt.$$

Combining the above inequality with (3.2), we get

$$M(p,R) \leqslant M(p,r) + \int_r^R \frac{nt^{\mu-1}}{K^\mu + t^\mu}\Big\{M(p,t) - m(p,K)\Big\}dt$$

or

$$M(p,R) \leqslant M(p,r) + \Big[\int_r^R \frac{nt^{\mu-1}}{K^\mu + t^\mu} M(p,t)dt - \int_r^R \frac{nt^{\mu-1}}{K^\mu + t^\mu} m(p,K)dt\Big]. \tag{3.3}$$

Now using the arguments similar to those used in Dewan [3], the theorem follows.

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