Mathematical
Inequalities
& Applications
Volume 2, Number 2 (1999), 207–221

INEQUALITIES FOR SOME COEFFICIENTS OF UNIVALENT FUNCTIONS

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(*communicated by J. Pečarić*)

Abstract. Let $\mathscr{S}$ be the usual class of normalized analytic and univalent functions in the open unit disk. We write

$$\log \frac{f(z)}{z} = 2\sum_{n=1}^{\infty} \gamma_n z^n \quad (f \in \mathscr{S}).$$

The well-known de Branges' theorem shows that

$$I_n = \sum_{k=1}^{n} (n-k+1)\left(k|\gamma_k|^2 - \frac{1}{k}\right) \leqslant 0 \quad (n \in \mathbb{N} := \{1,2,3,\cdots\};\ f \in \mathscr{S}).$$

In this paper we use the properties of I_n to obtain some coefficient inequalities for univalent functions. The results obtained here extend and unify several known results.

1. Introduction

Let $\mathscr{S}$ be the class of functions of the form:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1.1}$$

which are analytic and univalent in the open unit disk

$$\mathscr{U} := \{z : z \in \mathbb{C} \quad \text{and} \quad |z| < 1\}.$$

We write

$$\log \frac{f(z)}{z} = 2\sum_{n=1}^{\infty} \gamma_n z^n \tag{1.2}$$

and

$$\left(\frac{f(z)}{z}\right)^{\lambda} = \sum_{n=0}^{\infty} a_n(\lambda) z^n \qquad (0 < \lambda < \infty). \tag{1.3}$$

Mathematics subject classification (1991): 30C45, 30C50, 30A10.

Key words and phrases: Analytic functions; univalent functions; coefficient inequalities; Bieberbach, Robertson, and Milin conjectures; de Branges' theorem; convex hull; Bazilević conjecture; fractional derivatives; Abel transformation.

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Paper MIA-02-19

The Koebe function $K(z) = z(1-z)^{-2}$ has the logarithmic coefficients

$$\gamma_n = \frac{1}{n} \qquad (n \in \mathbb{N}) \tag{1.4}$$

and

$$\left(\frac{K(z)}{z}\right)^{\lambda} = \sum_{n=0}^{\infty} d_n(2\lambda)\, z^n. \tag{1.5}$$

In 1984 de Branges proved the Milin conjecture that, for $f \in \mathscr{S}$ and $n \in \mathbb{N}$, the following sharp inequality holds true:

$$I_n = \sum_{k=1}^{n} (n-k+1)\left(k\,|\gamma_k|^2 - \frac{1}{k}\right) \leqslant 0. \tag{1.6}$$

This was extended to the form:

$$I_n(\alpha,\beta) = \sum_{k=1}^{n} d_{k-1}(\alpha)\, d_{n-k}(\beta)\left(|\gamma_k|^2 - \frac{1}{k^2}\right) \leqslant 0 \quad (n \in \mathbb{N};\ 0 < \alpha \leqslant 2;\ \beta \geqslant 2), \tag{1.7}$$

by showing that these inequalities are in the convex hull of the inequalities (1.6) (see [3, p. 27]). Recently, Dong [5] proved that

$$-I_n \geqslant M_n = \begin{cases} \dfrac{3m}{4m^2-1}\,(1-|\gamma_1|^2) & (n = 2m-1;\ m \in \mathbb{N}) \\[2ex] \dfrac{15m(m+1)}{(4m^2-1)(2m+3)}\left(\dfrac{5}{4} - |\gamma_1|^2 - |\gamma_2|^2\right) & (n = 2m;\ m \in \mathbb{N}). \end{cases} \tag{1.8}$$

In this paper we use the properties of I_n to obtain some inequalities for the coefficients γ_n and $a_n(\lambda)$ defined as above. The results obtained here extend and unify several known results.

2. Inequalities for the Coefficients $a_n(\lambda)$

It is well-known that de Branges' result (1.6) implies Robertson's conjecture:

$$\sum_{n=0}^{N} \left|a_n\left(\tfrac{1}{2}\right)\right|^2 \leqslant N+1 \qquad (N \in \mathbb{N} \setminus \{1\}), \tag{2.1}$$

and hence also the celebrated Bieberbach conjecture:

$$|a_n| \leqslant n \qquad (n \in \mathbb{N} \setminus \{1\}). \tag{2.2}$$

Equality in each of the results (1.6), (2.1), and (2.2) holds true only if

$$f(z) = K_\chi(z) := \frac{z}{(1-\chi z)^2} \qquad (|\chi| = 1).$$

The inequalities (2.1) and (2.2) have been extended by several authors (*cf.* [1], [3], [8], and [14]). In particular, Milin and Grinshpan [14] proved the following results:

THEOREM A. *Let $f \in \mathscr{S}$ be defined by* (1.1). *Then the following sharp inequalities hold true:*

$$|a_N| \leqslant \sum_{k=0}^{N-1} \left|a_k\left(\tfrac{1}{2}\right)\right|^2 \leqslant \xi N \qquad (N \in \mathbb{N} \setminus \{1\}); \tag{2.3}$$

$$\sum_{k=n}^{N} \left|a_k\left(\tfrac{1}{2}\right)\right|^2 \leqslant \xi \left[(N+1)\,\exp\left(\frac{I_N}{N+1}\right) - n\,\exp\left(\frac{I_{n-1}}{n}\right)\right] \tag{2.4}$$

$$\big(n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\};\ \ N = n, n+1, n+2, \cdots;\ \ N+n \neq 0;\ \ I_{-1} = I_0 = 0\big),$$

where

$$\xi = \alpha \exp(1-\alpha) \leqslant 1 \qquad \left(\alpha := \frac{4+|a_2|^2}{8}\right). \tag{2.5}$$

THEOREM B. *Let $f \in \mathscr{S}$ be defined by* (1.1). *Then*

$$|a_N(\lambda)| \leqslant \{\theta(\lambda)\}^{\frac{1}{2}}\, d_N(2\lambda) \qquad (\lambda \geqslant 1;\ \ N \in \mathbb{N}), \tag{2.6}$$

where

$$\theta(\lambda) = \alpha(\lambda)\,\exp\left(1-\alpha(\lambda)\right) \leqslant 1 \qquad \left(\alpha(\lambda) = \frac{2+\lambda|a_2|^2}{2(2\lambda+1)}\right). \tag{2.7}$$

Equality in (2.5) *is attained by the function* $K_\chi(z)$.

Aharonov [1] also obtained (2.3). Hayman and Hummel [8] discussed as to what extent the analogue of (2.2) holds true for the coefficients $a_n(\lambda)$ defined by (1.3). They proved that

$$\sum_{n=0}^{N} \frac{|a_n(\lambda)|^2}{d_n(2\lambda)} \leqslant d_N(2\lambda+1) \qquad (\lambda \geqslant \tfrac{1}{2};\ \ N \in \mathbb{N}_0).$$

The following theorem will provide a unification and extension of the above results.

THEOREM 1. *Let $f \in \mathscr{S}$ be defined by* (1.1). *Then*

$$\sum_{k=0}^{n} \frac{|a_k(\lambda)|^2}{d_k(2\lambda)} \leqslant \theta(\lambda)\delta_n(2\lambda+1,1)\, d_n(2\lambda+1) \qquad (\lambda \geqslant \tfrac{1}{2};\ \ n \in \mathbb{N} \setminus \{1\}) \tag{2.8}$$

and

$$|a_n(\lambda)| \leqslant \{\theta(\lambda)\delta_n(2\lambda,0)\}^{\frac{1}{2}}\, d_n(2\lambda) \qquad (\lambda \geqslant 1;\ \ n \in \mathbb{N} \setminus \{1\}), \tag{2.9}$$

where

$$\delta_n(\beta,\rho) = \exp\left(-\frac{(\beta-1)(\beta-2)(\beta-\rho)n}{(n+\beta-1)(n+\beta-2)(n+\beta-3)}\left(1-\left|\frac{a_2}{2}\right|^2\right) - \frac{\beta-\rho}{d_n(\beta)}\,M_n\right), \tag{2.10}$$

and $\theta(\lambda)$ *and* M_n *are defined by* (2.7) *and* (1.8), *respectively. Equality in* (2.8) *and* (2.9) *holds true only if* $f(z) = K_\chi(z)$.

Proof. We note from (1.2) and (1.3) that

$$\sum_{k=0}^{\infty} a_k(\lambda)\, z^k = \exp\left(\sum_{k=1}^{\infty} 2\lambda\, \gamma_k\, z^k\right). \tag{2.11}$$

Applying Milin's inequalities [11, pp. 33–37], we find from (2.11) that

$$\sum_{k=0}^{n} \frac{|a_k(\lambda)|^2}{d_k(2\lambda)} = \theta_n(2\lambda)\, d_n(2\lambda+1)\, \exp\left(\frac{2\lambda}{d_n(2\lambda+1)}\, I_n(2\lambda+1)\right) \tag{2.12}$$

and

$$|a_n(\lambda)| \leqslant \{\theta_n(2\lambda)\}^{\frac{1}{2}}\, d_n(2\lambda)\, \exp\left(\frac{\lambda}{d_n(2\lambda)}\, I_n(2\lambda)\right), \tag{2.13}$$

where $\{\theta_n(2\lambda)\}$ is a non-increasing sequence of positive numbers:

$$\theta_n(2\lambda) \leqslant \theta_{n-1}(2\lambda) \leqslant \cdots \leqslant \theta_1(2\lambda) \leqslant \theta_0(2\lambda) := 1 \tag{2.14}$$

and $I_n(\beta) := I_n(2,\beta)$ is defined by (1.7). The equality $\theta_n(2\lambda) = 1$ holds true for some $n \in \mathbb{N}$ only if

$$\gamma_k = \frac{\chi^k}{k} \qquad (k \in \mathbb{N};\ |\chi| = 1).$$

We now apply the Abel transformation to the quantity $I_n(\beta)$. This yields

$$\begin{aligned} I_n(\beta) &= \sum_{k=1}^{n} d_{n-k}(\beta)\left(k|\gamma_k|^2 - \frac{1}{k}\right) \\ &= \sum_{k=1}^{n} \{d_{n-k}(\beta) - 2d_{n-k-1}(\beta) + d_{n-k-2}(\beta)\}\, I_k \\ &= \sum_{k=1}^{n-2} d_{n-k}(\beta)\, \frac{(\beta-1)(\beta-2)}{(n-k+\beta-1)(n-k+\beta-2)}\, I_k + (\beta-2)\, I_{n-1} + I_n, \end{aligned} \tag{2.15}$$

where $d_{-2}(\beta) = d_{-1}(\beta) = 0$.

It follows from (1.6) that

$$I_n(\beta) \leqslant d_{n-1}(\beta)\, \frac{(\beta-1)(\beta-2)}{(n+\beta-2)(n+\beta-3)}\left(\left|\frac{a_2}{2}\right|^2 - 1\right) \leqslant 0 \quad (\beta \geqslant 2;\ n \in \mathbb{N}\setminus\{1\}). \tag{2.16}$$

Furthermore, by applying (1.8), we have (for $\beta \geqslant 2$ and $n \in \mathbb{N}$)

$$I_n(\beta) \leqslant M_n(\beta) := \begin{cases} -M_n & (\beta = 2) \\ -\displaystyle\sum_{k=1}^{n} d_{n-k}(\beta)\, \frac{(\beta-1)(\beta-2)}{(n-k+\beta-1)(n-k+\beta-2)}\, M_k & (\beta > 2). \end{cases} \tag{2.17}$$

The equality in (2.16) and (2.17) holds true only for the function $K_\chi(z)$.

Since

$$\theta_n(2\lambda) \leqslant \theta_1(2\lambda) = \theta(\lambda) \qquad (n \in \mathbb{N}),$$

we get from (2.12) and (2.17) that

$$\begin{aligned}
\sum_{k=0}^{n} \frac{|a_k(\lambda)|^2}{d_k(2\lambda)} &\leqslant \left\{\theta(\lambda)\, \exp\left(\frac{2\lambda}{d_n(2\lambda+1)} M_n(2\lambda+1)\right)\right\} d_n(2\lambda+1) \\
&\leqslant \left\{\theta(\lambda)\, \exp\left(\frac{-2\lambda}{d_n(2\lambda+1)} \cdot \frac{2\lambda(2\lambda-1)\, d_{n-1}(2\lambda+1)}{(n+2\lambda-1)(n+2\lambda-2)} \cdot M_1 \right.\right. \\
&\qquad \left.\left. - \frac{2\lambda}{d_n(2\lambda+1)} M_n\right)\right\} d_n(2\lambda+1) \\
&= \{\theta(\lambda)\delta_n(2\lambda+1,1)\}\, d_n(2\lambda+1) \qquad \left(\lambda \geqslant \tfrac{1}{2};\ n \in \mathbb{N}\setminus\{1\}\right),
\end{aligned} \tag{2.18}$$

which yields (2.8).

Similarly, from (2.13) and (2.17), we have

$$\begin{aligned}
|a_n(\lambda)| &\leqslant \left\{\{\theta(\lambda)\}^{\frac{1}{2}}\, \exp\left(\frac{\lambda}{d_n(2\lambda)} M_n(2\lambda)\right)\right\} d_n(2\lambda) \\
&\leqslant \left\{\{\theta(\lambda)\}^{\frac{1}{2}}\, \exp\left(\frac{-\lambda}{d_n(2\lambda)} \cdot \frac{(2\lambda-1)(2\lambda-2)\, d_{n-1}(2\lambda)}{(n+2\lambda-2)(n+2\lambda-3)} \cdot M_1 \right.\right. \\
&\qquad \left.\left. - \frac{\lambda}{d_n(2\lambda)} M_n\right)\right\} d_n(2\lambda) \\
&= \{\theta(\lambda)\delta_n(2\lambda,0)\}^{\frac{1}{2}}\, d_n(2\lambda) \qquad (\lambda \geqslant 1;\ n \in \mathbb{N}\setminus\{1\}),
\end{aligned} \tag{2.19}$$

which gives (2.9). The assertion concerning the attainment of equality in (2.8) and (2.9) is obvious. This completes the proof of Theorem 1.

REMARK 1. It follows from (2.12), (2.14), and (2.17) that the analogue of (2.4) holds true for the coefficients $a_n(\lambda)$ $\left(\lambda \geqslant \frac{1}{2}\right)$.

3. Inequalities for the Coefficients γ_n

Milin and Grinshpan [14] noted that de Branges' result (1.6) implies the Bazilević conjecture on the estimation of the logarithmic area, namely,

$$\sum_{k=1}^{\infty} k\, |\gamma_k|^2\, r^k \leqslant \log \frac{1}{1-r} \qquad (0 < r < 1). \tag{3.1}$$

Zemyan [18], Andreev and Duren [2], and Milin [12] also gave different proofs of (3.1). Recently, Li [10] and Nikitin [15] obtained the stronger inequality:

$$\sum_{k=1}^{\infty} k\,|\gamma_k|^2\,r^k \leqslant \log\frac{1}{1-r} - r(1-r)^2\left(1-|\gamma_1|^2\right) \qquad (0<r<1). \tag{3.2}$$

The following theorem will improve this inequality further.

THEOREM 2. *Let* $f \in \mathscr{S}$ *and* $0<r<1$. *Then*

$$\begin{aligned}\sum_{k=1}^{\infty} k\,|\gamma_k|^2\,r^k \leqslant{}& \log\frac{1}{1-r} - \frac{3}{8}\,(1-r)^2\left(\frac{1+r^2}{r^2}\,\log\frac{1+r}{1-r} - \frac{2}{r}\right)\left(1-|\gamma_1|^2\right)\\ &- \frac{15}{32}(1-r)^2\left\{\frac{3+2r^2+3r^4}{2r^3}\,\log\frac{1+r}{1-r} - \frac{3(1+r^2)}{r^2}\right\}\left(\frac{5}{4}-|\gamma_1|^2-|\gamma_2|^2\right).\end{aligned} \tag{3.3}$$

Proof. For $\beta>0$, we observe that

$$\begin{aligned}\sum_{k=1}^{\infty} k\,|\gamma_k|^2\,r^k &= (1-r)^{\beta}\sum_{k=1}^{\infty} k\,|\gamma_k|^2\,r^k\sum_{k=0}^{\infty} d_k(\beta)\,r^k\\ &= (1-r)^{\beta}\left\{\sum_{n=1}^{\infty} I_n(\beta)\,r^n + \sum_{n=1}^{\infty}\left(\sum_{k=1}^{n}\frac{d_{n-k}(\beta)}{k}\right)r^n\right\}\\ &= \log\frac{1}{1-r} + (1-r)^{\beta}\sum_{n=1}^{\infty} I_n(\beta)\,r^n.\end{aligned} \tag{3.4}$$

When $\beta=2$, we conclude from (2.17) that

$$\sum_{k=1}^{\infty} k\,|\gamma_k|^2\,r^k \leqslant \log\frac{1}{1-r} - (1-r)^2\sum_{k=1}^{\infty} M_k\,r^k. \tag{3.5}$$

Since

$$\begin{aligned}\sum_{k=1}^{\infty} M_k\,r^k = {}& \left(1-|\gamma_1|^2\right)\sum_{k=1}^{\infty}\frac{3k}{4k^2-1}\,r^{2k-1}\\ &+\left(\frac{5}{4}-|\gamma_1|^2-|\gamma_2|^2\right)\sum_{k=1}^{\infty}\frac{15k(k+1)}{(4k^2-1)(2k+3)}\,r^{2k},\end{aligned}$$

and since

$$\sum_{k=1}^{\infty} \frac{3k}{4k^2-1} r^{2k-1} = \frac{3}{4} \sum_{k=1}^{\infty} \left(\frac{1}{2k-1} + \frac{1}{2k+1} \right) r^{2k-1}$$
$$= \frac{3}{8} \left(\frac{1+r^2}{r^2} \log \frac{1+r}{1-r} - \frac{2}{r} \right)$$

and

$$\sum_{k=1}^{\infty} \frac{15k(k+1)}{(4k^2-1)(2k+3)} = \frac{15}{32} \sum_{k=1}^{\infty} \left(\frac{3}{2k-1} + \frac{2}{2k+1} + \frac{3}{2k+3} \right) r^{2k}$$
$$= \frac{15}{32} \left(\frac{3+2r^2+3r^4}{2r^3} \log \frac{1+r}{1-r} - \frac{3(1+r^2)}{r^2} \right),$$

it is easily seen that the inequality (3.5) reduces to (3.3). This proves Theorem 2.

Duren and Leung [6] showed that

$$\sum_{k=1}^{\infty} |\gamma_k|^2 \leqslant \sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6} \tag{3.6}$$

for all $f \in \mathscr{S}$. One question that arises naturally is whether the partial-sum inequality:

$$\sum_{k=1}^{n} |\gamma_k|^2 \leqslant \sum_{k=1}^{n} \frac{1}{k^2} \tag{3.7}$$

holds true for $n \in \mathbb{N} \setminus \{1,2\}$. Both Milin and Grinshpan [14] and Andreev and Duren [2] provide some supportive evidence that (3.7) should hold true for $n \geqslant 3$, although neither work provides a proof. Recently, Zemyan [18] gave an estimate of the left-hand side of (3.7). Li [10] improved Zemyan's result to the following form:

$$\sum_{k=1}^{n} |\gamma_k|^2 \leqslant \sum_{k=1}^{n} \frac{1}{k^2} + \frac{\delta}{n} - \frac{1}{3} \left(1 - |\gamma_1|^2\right) \qquad (n \in \mathbb{N} \setminus \{1,2\}),$$

where $\delta < 0.312$ is the Milin constant. We shall now prove

THEOREM 3. *Let $f \in \mathscr{S}$. Then*

$$\sum_{k=1}^{n} |\gamma_k|^2 \leqslant \sum_{k=1}^{n} \frac{1}{k^2} + \frac{\delta}{n+1} - L(n)\left(1 - |\gamma_1|^2\right) - H(n)\left(\frac{5}{4} - |\gamma_1|^2 - |\gamma_2|^2\right) \tag{3.8}$$
$$(n \in \mathbb{N} \setminus \{1,2\}),$$

where $\delta < 0.312$, $L(n)$ $\left(\geqslant \frac{1}{3}\right)$ *and* $H(n)$ $\left(\geqslant \frac{1}{6}\right)$ *are defined by*

$$L(n) = \begin{cases} \dfrac{3}{2}\left(\dfrac{\pi^2}{8} - 1 - \dfrac{1}{4}\psi'\left(\dfrac{1}{2}+m\right) + \dfrac{m}{4m^2-1}\right) & (n = 2m-1;\ m \in \mathbb{N}) \\ \dfrac{3}{2}\left(\dfrac{\pi^2}{8} - 1 - \dfrac{1}{4}\psi'\left(\dfrac{1}{2}+m\right) + \dfrac{m+1}{(2m+1)^2}\right) & (n = 2m;\ m \in \mathbb{N}) \end{cases} \tag{3.9}$$

and

$$H(n) = \begin{cases} \dfrac{15}{8}\left[\dfrac{4}{3} - \dfrac{\pi^2}{8} + \dfrac{1}{4}\psi'\left(\dfrac{1}{2}+m\right) - \dfrac{m}{4m^2-1}\right] & (n = 2m-1;\ m \in \mathbb{N}) \\ \dfrac{15}{8}\left[\dfrac{4}{3} - \dfrac{\pi^2}{8} + \dfrac{1}{4}\psi'\left(\dfrac{1}{2}+m\right)\right. & \\ \qquad\qquad \left. + \dfrac{2-(m+2)(4m^2-1)}{(2m-1)(2m+1)^2(2m+3)}\right] & (n = 2m;\ m \in \mathbb{N}), \end{cases} \tag{3.10}$$

and $\psi(z) := \Gamma'(z)/\Gamma(z)$ *denotes the Psi or Digamma function.*

Proof. By using summation by parts twice, we have

$$\sum_{k=1}^{n} \frac{1}{k}\left(k|\gamma_k|^2 - \frac{1}{k}\right) = \sum_{k=1}^{n} \frac{2}{k(k+1)(k+2)} I_k + \frac{1}{(n+1)(n+2)} I_n + \frac{1}{n+1}\sum_{k=1}^{n}\left(k|\gamma_k|^2 - \frac{1}{k}\right). \tag{3.11}$$

From (2.17) and Milin's Lemma, we obtain

$$\sum_{k=1}^{n} |\gamma_k|^2 \leqslant \sum_{k=1}^{n} \frac{1}{k^2} + \frac{\delta}{n+1} - R_n, \tag{3.12}$$

where

$$R_n = \sum_{k=1}^{n} \frac{2}{k(k+1)(k+2)} M_k + \frac{1}{(n+1)(n+2)} M_n. \tag{3.13}$$

Now

$$\sum_{k=1}^{2m} \frac{2}{k(k+1)(k+2)} M_k = 3\left(1 - |\gamma_1|^2\right) \sum_{k=1}^{m} \frac{1}{(4k^2-1)^2} + \frac{15}{2}\left(\frac{5}{4} - |\gamma_1|^2 - |\gamma_2|^2\right) \sum_{k=1}^{m} \frac{1}{(2k-1)(2k+1)^2(2k+3)},$$

$$\sum_{k=1}^{m} \frac{1}{(4k^2-1)^2} = \frac{1}{4}\left\{\sum_{k=1}^{m} \frac{1}{(2k-1)^2} + \sum_{k=1}^{m} \frac{1}{(2k+1)^2} - \sum_{k=1}^{m} \frac{2}{(2k-1)(2k+1)}\right\}$$

$$= \frac{1}{2}\left\{\frac{\pi^2}{8} - 1 - \frac{1}{4}\,\psi'\left(\tfrac{1}{2}+m\right) + \frac{m+1}{(2m+1)^2}\right\},$$

and

$$\sum_{k=1}^{m} \frac{1}{(2k-1)(2k+1)^2(2k+3)} = \frac{1}{8}\left\{\sum_{k=1}^{m} \frac{1}{(2k-1)(2k+1)} + \sum_{k=1}^{m} \frac{1}{(2k+1)(2k+3)} - 2\sum_{k=1}^{m} \frac{1}{(2k+1)^2}\right\}$$

$$= \frac{1}{4}\left\{\frac{4}{3} - \frac{\pi^2}{8} + \frac{1}{4}\,\psi'\left(\tfrac{1}{2}+m\right) - \frac{2+(2m+1)(m+2)}{(2m+1)^2(2m+3)}\right\},$$

where we have used the fact that

$$\sum_{k=1}^{m} \frac{1}{(2k-1)^2} = \frac{\pi^2}{8} - \frac{1}{4}\,\psi'\left(\tfrac{1}{2}+m\right). \tag{3.14}$$

Thus (3.12) gives (3.8) when $n = 2m$. Since

$$R_{2m-1} = R_{2m} + \frac{1}{2m(2m+1)}\,M_{2m-1} - \frac{1}{2m(2m+1)}\,M_{2m},$$

the inequality (3.12) also gives (3.8) in the case when $n = 2m-1$. This completes the proof of Theorem 3.

In view of the fact that $\psi'\left(\frac{1}{2}+m\right) \to 0 \quad (m \to \infty)$, Theorem 3 yields the following

COROLLARY. *Let $f \in \mathscr{S}$. Then*

$$\sum_{k=1}^{\infty} |\gamma_k|^2 \leqslant \frac{\pi^2}{6} - \frac{3(\pi^2-8)}{16}\left(1-|\gamma_1|^2\right) - \frac{5(32-3\pi^2)}{64}\left(\frac{5}{4} - |\gamma_1|^2 - |\gamma_2|^2\right). \tag{3.15}$$

4. Remarks on the Estimation of $\left|f^{(n)}(z)/f'(z)\right|$

The well-known Landau's theorem shows that, if $f \in \mathscr{S}$ is defined by (1.1), then

$$|a_n| \leqslant n \Leftrightarrow \left|f^{(n)}(z)\right| \leqslant \frac{n!\,(n+|z|)}{(1-|z|)^{n+2}} \qquad \left(z \in \mathscr{U};\ n \in \mathbb{N}\setminus\{1\}\right). \tag{4.1}$$

Gong [7] extended the inequality (4.1) to the following sharp forms:

$$|a_n| \leqslant n \Leftrightarrow \left|\frac{f^{(n)}(z)}{f'(z)}\right| \leqslant \frac{n!\,(n+|z|)}{(1-|z|)^{n-1}(1+|z|)} \qquad \big(z \in \mathscr{U};\ \ n \in \mathbb{N}\setminus\{1\}\big); \tag{4.2}$$

$$|a_n| \leqslant 1 \Leftrightarrow \left|\frac{f^{(n)}(z)}{f'(z)}\right| \leqslant \frac{n!}{(1-|z|)^{n-1}} \qquad \big(z \in \mathscr{U};\ \ n \in \mathbb{N}\setminus\{1\}\big). \tag{4.3}$$

Therefore, de Branges' result (2.2) also improves several results on the estimation of $|f^{(n)}(z)/f'(z)|$. For example, one may cite the results obtained by Jakubowski [9] and Todorov [17]. (See also a recent work of Srivastava [16] for various extensions of inequalities like (4.1) to *fractional* derivatives.)

Recently, Chua [4] generalized de Branges' theorem (2.2) to arbitrary simply-connected domains. The following two theorems were established by him [4].

THEOREM C. *Let*

$$w = g(z) = z + \sum_{n=2}^{\infty} a_n z^n \in \mathscr{S}.$$

For $n = k, k+1, k+2, \cdots \ \ (k \in \mathbb{N})$, *let* $B_{n,k}(a_2, a_3, \cdots, a_n)$ *be the coefficient of* w^n *in the expansion:*

$$G_k(w) = \big[g^{-1}(w)\big]^k = \sum_{n=k}^{\infty} B_{n,k}\, w^n,$$

where g^{-1} *is the inverse of* g. *Then the sharp inequality:*

$$|B_{n,k}(a_2, a_3, \cdots, a_n)| \leqslant \frac{k}{n}\binom{2n}{n-k} \tag{4.4}$$

holds true for $n = k, k+1, k+2, \cdots \ \ (k \in \mathbb{N})$, *with equality precisely for the function* $K_\chi(z)$.

THEOREM D. *Let* $f(w)$ *be an analytic and univalent function on a convex domain* $\mathscr{E}$. *Then*

$$\left|\frac{f^{(n)}(w)}{f'(w)}\right| \leqslant (n+1)!\ 2^{n-2}\{\lambda_{\mathscr{E}}(w)\}^{n-1} \qquad (w \in \mathscr{E};\ \ n = 2, 3, 4), \tag{4.5}$$

where $\lambda_{\mathscr{E}}(w)$ *denotes the hyperbolic metric on* $\mathscr{E}$. *Furthermore, if* $f(w)$ *is also convex, then*

$$\left|\frac{f^{(n)}(w)}{f'(w)}\right| \leqslant n!\ 2^{n-1}\,\{\lambda_{\mathscr{E}}(w)\}^{n-1} \qquad (w \in \mathscr{E};\ \ n = 2, 3, 4). \tag{4.6}$$

In this section, we first give a simple proof of Theorem C. We say that

$$f(z) = \sum_{n=0}^{\infty} a_n z^n$$

is dominated by

$$F(z) = \sum_{n=0}^{\infty} A_n z^n,$$

and we write $f(z) \ll F(z)$, if

$$|a_n| \leqslant A_n \qquad (n \in \mathbb{N}_0).$$

Obviously, if $f(z) \ll F(z)$, then

$$[f(z)]^k \ll [F(z)]^k \qquad (k \in \mathbb{N}_0).$$

Löwner's result shows that, if

$$w = g(z) = z + \sum_{n=2}^{\infty} a_n z^n \in \mathscr{S} \quad \text{and} \quad z = g^{-1}(w) = \sum_{n=1}^{\infty} B_{n,1}\, w^n,$$

then

$$|B_{n,1}| \leqslant \frac{1}{n}\binom{2n}{n-1} \qquad (n \in \mathbb{N}), \tag{4.7}$$

with equality for a given n only if

$$g^{-1}(w) = \bar{\chi} K^{-1}(\chi w) \quad \text{and} \quad |\chi| = 1.$$

For the Koebe function $w = K_1(z) = z/(1-z)^2 =: K(z)$, we have

$$z = K_1^{-1}(w) = \frac{1 - 2w - \sqrt{1-4w}}{2w} = w + \sum_{n=2}^{\infty} \frac{1}{n}\binom{2n}{n-1} w^n. \tag{4.8}$$

Thus Löwner's result can be rewritten as follows:

$$g^{-1}(w) \ll K_1^{-1}(w).$$

This gives

$$\left[g^{-1}(w)\right]^k \ll \left[K_1^{-1}(w)\right]^k \qquad (k \in \mathbb{N}_0). \tag{4.9}$$

Since

$$\left[K_1^{-1}(w)\right]^k = \frac{\left(1 - \sqrt{1-4w}\,\right)^{2k}}{(4w)^k} = \sum_{n=k}^{\infty} \frac{k}{n}\binom{2n}{n-k} w^n, \tag{4.10}$$

(4.9) is equivalent to (4.4).

Next we prove an extension of Theorem D from $2 \leqslant n \leqslant 4$ to $2 \leqslant n \leqslant 7$. For each $w_0 \in \mathscr{E}$, let $w = g(z)$ be a conformal mapping of $\mathscr{U}$ onto $\mathscr{E}$ with $g(0) = w_0$. Then the function

$$h(z) = \frac{g(z) - g(0)}{g'(0)}$$

is a normalized convex univalent function in $\mathscr{U}$ and

$$\lambda_{\mathscr{E}}(w_0) = \frac{1}{|g'(0)|}.$$

Let

$$\left[h^{-1}(w)\right]^k = \sum_{n=k}^{\infty} B_{n,k}\, w^n \qquad (k \in \mathbb{N}).$$

Then, from

$$(f \circ g)(z) = \sum_{j=0}^{\infty} \frac{(f \circ g)^{(j)}(0)}{j!}\, z^j \qquad (z \in \mathscr{U}),$$

we have

$$\begin{aligned} f(w) &= \sum_{j=0}^{\infty} \frac{(f \circ g)^{(j)}(0)}{j!} \left[g^{-1}(w)\right]^j \\ &= \sum_{j=0}^{\infty} \frac{(f \circ g)^{(j)}(0)}{j!} \sum_{n=j}^{\infty} B_{n,j} \left(\frac{w - w_0}{g'(0)}\right)^n \\ &= \sum_{n=0}^{\infty} \left\{ \sum_{j=1}^{n} \frac{(f \circ g)^{(j)}(0)}{j!} \cdot \frac{B_{n,j}}{[g'(0)]^n} \right\} (w - w_0)^n, \end{aligned}$$

which yields

$$\frac{f^{(n)}(w_0)}{n!} = \frac{1}{[g'(0)]^n} \sum_{j=1}^{n} \frac{(f \circ g)^{(j)}(0)}{j!}\, B_{n,j} \qquad (n \in \mathbb{N}). \tag{4.11}$$

Since

$$\frac{[(f \circ g)(z) - (f \circ g)(0)]}{f'(w_0)\, g'(0)} \in \mathscr{S},$$

we find from (4.11) and (2.2) that

$$\begin{aligned} \left|\frac{f^{(n)}(w_0)}{n!}\right| &\leqslant \frac{1}{|g'(0)|^n} \sum_{j=1}^{n} j|f'(w_0)\, g'(0)|\ |B_{n,j}| \\ &= |f'(w_0)|\ \{\lambda_{\mathscr{E}}(w_0)\}^{n-1} \sum_{j=1}^{n} j|B_{n,j}|. \end{aligned} \tag{4.12}$$

Let $B_{n,1} = B_n \ \ (n \in \mathbb{N})$. Then

$$\left[h^{-1}(w)\right]^k = w^k \sum_{j=0}^{k} \binom{k}{j} \left(\sum_{n=1}^{\infty} B_n\, w^{n-1}\right)^j = \sum_{n=k}^{\infty} B_{n,k}\, w^n, \tag{4.13}$$

which yields

$$B_{k,k}=1,\quad B_{k+1,k}=\binom{k}{1}B_2,\quad B_{k+2,k}=\binom{k}{1}B_3+\binom{k}{2}B_2^2,$$

$$B_{k+3,k}=\binom{k}{1}B_4+2\binom{k}{2}B_2B_3+\binom{k}{3}B_2^3,$$

$$B_{k+4,k}=\binom{k}{1}B_5+\binom{k}{2}\left(B_3^2+2B_2B_4\right)+3\binom{k}{3}B_2^2B_3+\binom{k}{4}B_2^4,$$

$$\begin{aligned}B_{k+5,k}=&\binom{k}{1}B_6+2\binom{k}{2}\left(B_2B_5+B_3B_4\right)+3\binom{k}{3}\left(B_2^2B_4+B_3^2B_2\right)\\&+4\binom{k}{4}B_2^3B_3+\binom{k}{5}B_2^5,\end{aligned}$$

$$\begin{aligned}B_{k+6,k}=&\binom{k}{1}B_7+\binom{k}{2}\left(B_4^2+2B_2B_6+2B_3B_5\right)+\binom{k}{3}\left(B_3^3+3B_2^2B_5+6B_2B_3B_4\right)\\&+\binom{k}{4}\left(4B_2^3B_4+6B_2^2B_3^2\right)+5\binom{k}{5}B_2^4B_3+\binom{k}{6}B_2^6.\end{aligned}\tag{4.14}$$

For the convex function $h(z)$, Libera and Zlotkiewicz [11] proved that

$$|B_n|\leqslant 1\qquad(n=2,3,4,5,6,7),\tag{4.15}$$

and also that these bounds are sharp.

In view of (4.15), we deduce from (4.14) that

$$|B_{k,k}|=1,\quad |B_{k+1,k}|\leqslant\binom{k}{1}=k,\quad |B_{k+2,k}|\leqslant\binom{k}{1}+\binom{k}{2}=\binom{k+1}{k-1},$$

$$|B_{k+3,k}|\leqslant\binom{k}{1}+2\binom{k}{2}+\binom{k}{3}=\binom{k+2}{k-1},$$

$$|B_{k+4,k}|\leqslant\binom{k}{1}+3\binom{k}{2}+3\binom{k}{3}+\binom{k}{4}=\binom{k+3}{k-1},$$

$$|B_{k+5,k}|\leqslant\binom{k}{1}+4\binom{k}{2}+6\binom{k}{3}+4\binom{k}{4}+\binom{k}{5}=\binom{k+4}{k-1},$$

$$|B_{k+6,k}|\leqslant\binom{k}{1}+5\binom{k}{2}+10\binom{k}{3}+10\binom{k}{4}+5\binom{k}{5}+\binom{k}{6}=\binom{k+5}{k-1},$$

that is,

$$|B_{n,k}|\leqslant\binom{n-1}{k-1}\qquad(n=k,k+1,\cdots,k+6).\tag{4.16}$$

It follows from (4.12) and (4.16) that, for $2 \leqslant n \leqslant 7$,

$$\begin{aligned} \frac{|f^{(n)}(w_0)|}{n!} &\leqslant |f'(w_0)| \, \{\lambda_{\mathscr{E}}(w_0)\}^{n-1} \sum_{j=1}^{n} j \binom{n-1}{j-1} \\ &= |f'(w_0)| \, \{\lambda_{\mathscr{E}}(w_0)\}^{n-1} \, (n+1) \, 2^{n-2}, \end{aligned} \tag{4.17}$$

which is (4.5).

If $f(w)$ is also convex, then

$$\frac{[(f \circ g)(z) - (f \circ g)(0)]}{f'(w_0) \, g'(0)}$$

is a normalized convex univalent function in $\mathscr{U}$. Hence, in the case when $2 \leqslant n \leqslant 7$, (4.11) and (4.16) yield

$$\begin{aligned} \frac{|f^{(n)}(w_0)|}{n!} &\leqslant \frac{1}{|g'(0)|^n} \sum_{j=1}^{n} |f'(w_0) \, g'(0)| \, |B_{n,j}| \\ &\leqslant |f'(w_0)| \, \{\lambda_{\mathscr{E}}(w_0)\}^{n-1} \sum_{j=1}^{n} \binom{n-1}{j-1} \\ &= |f'(w_0)| \, \{\lambda_{\mathscr{E}}(w_0)\}^{n-1} \, 2^{n-1}, \end{aligned} \tag{4.18}$$

which is (4.6).

This completes the proof of our extension of Theorem D.

Acknowledgments. The present investigation was supported, in part, by the *Natural Sciences and Engineering Research Council of Canada* under Grant OGP0007353.

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(Received November 10, 1997)

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