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DETERMINANTAL INEQUALITIES FOR THE PSI FUNCTION

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Abstract. Some inequalities concerning determinants which involve the psi function (logarithmic derivative of the gamma function) are established.

1. Introduction and background

In this paper some determinantal inequalities for the psi function are studied. We recall that the function ψ is defined as the logarithmic derivative of the eulerian function Γ, that is $\psi(x) = \dfrac{\Gamma'(x)}{\Gamma(x)}$. In what follows we shall only consider these functions on the interval $(0, +\infty)$, but it is well-known that they may be defined for every complex z, except for the non-positive integer numbers (indeeed, in the points $z = 0, -1, -2, \ldots$ gamma and psi functions have a pole of the first order).

First we shall deal with the determinants

$$\begin{vmatrix} \psi'(x) & \psi''(x) \\ \psi''(x) & \psi'''(x) \end{vmatrix}, \quad \begin{vmatrix} \psi''(x) & \psi'''(x) \\ \psi'''(x) & \psi^{(4)}(x) \end{vmatrix}, \quad \begin{vmatrix} \psi'''(x) & \psi^{(4)}(x) \\ \psi^{(4)}(x) & \psi^{(5)}(x) \end{vmatrix}, \ldots \tag{1.1}$$

and, more generally, with the determinant $\begin{vmatrix} \psi^{(k+2a)}(x) & \psi^{(k+a+b)}(x) \\ \psi^{(k+a+b)}(x) & \psi^{(k+2b)}(x) \end{vmatrix}$, where k is a positive integer and a and b are non-negative integers.

Finally, we shall consider the derterminant

$$\begin{vmatrix} \psi^{(m)}(x) & \psi^{(m)}(x+h) \\ \psi^{(m)}(x+k) & \psi^{(m)}(x+h+k) \end{vmatrix}, \tag{1.2}$$

where h and k are two real positive number and m is a non-negative integer.

We recall that a function f defined on an interval $I \subseteq \mathbf{R}$ is said *absolutely monotonic* on I if it is continuous on I and it has non-negative derivatives of all orders on $\mathring{I}$, i.e. $f^{(k)}(x) \geqslant 0$ for $x \in \mathring{I}$ and $k = 0, 1, 2, \ldots$.

A function f defined on an interval $I \subseteq \mathbf{R}$ is said *completely monotonic* in I if it is continuous on I and its derivatives are alternatively non-negative and non-positive

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on $\overset{\circ}{I}$, i.e. $(-1)^k f^{(k)}(x) \geqslant 0$ for $x \in \overset{\circ}{I}$ and $k = 0, 1, 2, \ldots$; it is clear that $f(x)$ is completely monotonic in I if and only if $f(-x)$ is absolutely monotonic on the interval $-I$, that is the interval whose elements are the opposites of the numbers of I.

It is also to be noted that if f is absolutely monotonic in I, then all its derivatives $f^{(k)}$ are absolutely monotonic in I as well, while if f is completely monotonic in I, then all the functions $(-1)^k f^{(k)}(x)$ are completely monotonic in I.

Now, for the absolutely and completely monotonic functions the well-known *Hankel determinantal inequalities* hold, see [3 p. 167]: if f is absolutely monotonic in $(-\infty, 0)$, then for any negative x we have the inequalities:

$$f(x) \geqslant 0, \qquad \begin{vmatrix} f(x) & f'(x) \\ f'(x) & f''(x) \end{vmatrix} \geqslant 0, \qquad \begin{vmatrix} f(x) & f'(x) & f''(x) \\ f'(x) & f''(x) & f'''(x) \\ f''(x) & f'''(x) & f^{IV}(x) \end{vmatrix} \geqslant 0, \ldots$$

$$f'(x) \geqslant 0, \qquad \begin{vmatrix} f'(x) & f''(x) \\ f''(x) & f'''(x) \end{vmatrix} \geqslant 0, \qquad \begin{vmatrix} f'(x) & f''(x) & f'''(x) \\ f''(x) & f'''(x) & f^{IV}(x) \\ f'''(x) & f^{IV}(x) & f^{V}(x) \end{vmatrix} \geqslant 0, \ldots \tag{1.3}$$

and so on. Taking into account the relation between absolutely and completely monotonic functions, we also may say that inequalities (2.1) are valid for $x > 0$ if f is completely monotonic in $(0, +\infty)$ and if in the determinants $f^{(k)}(x)$ is replaced by $(-1)^k f^{(k)}(x)$; in particular, if f is completely monotonic in $(0, +\infty)$ we have for $x > 0$:

$$\begin{vmatrix} f(x) & -f'(x) \\ -f'(x) & f''(x) \end{vmatrix} = \begin{vmatrix} f(x) & f'(x) \\ f'(x) & f''(x) \end{vmatrix} \geqslant 0,$$

$$\begin{vmatrix} -f'(x) & f''(x) \\ f''(x) & -f'''(x) \end{vmatrix} = \begin{vmatrix} f'(x) & f''(x) \\ f''(x) & f'''(x) \end{vmatrix} \geqslant 0,$$

and so on.

2. Hankel type inequalities for the psi function

The result of this section is obtained as an immediate consequence of the known results recalled in the introduction.

THEOREM 1. *For any fixed positive integer m the inequality*

$$\begin{vmatrix} \psi^{(m)}(x) & \psi^{(m+1)}(x) \\ \psi^{(m+1)}(x) & \psi^{(m+2)}(x) \end{vmatrix} \geqslant 0 \tag{2.1}$$

holds for any $x > 0$. More generally, for any fixed non-negative integers a and b, and for any fixed positive integer k, the inequality

$$\begin{vmatrix} \psi^{(k+2a)}(x) & \psi^{(k+a+b)}(x) \\ \psi^{(k+a+b)}(x) & \psi^{(k+2b)}(x) \end{vmatrix} \geqslant 0 \tag{2.2}$$

holds for any $x > 0$.

Proof. To prove formula (2.1) it is sufficient to show that the function $\psi'(x)$ is completely monotonic in $(0,+\infty)$. For this, we consider the well-known integral representation $\psi(x) = \int_0^\infty \Big(\frac{e^{-t}}{t} - \frac{e^{-xt}}{1-e^{-t}}\Big)dt$ holds for any $x > 0$ [2, p. 259, formula (6.3.21)]. taking derivatives in this formula m times with respect to x we have:

$$\psi^{(m)}(x) = (-1)^{m+1}\int_0^\infty t^m \frac{e^{-xt}}{1-e^{-t}}dt, \tag{2.3}$$

which proves that derivatives of even order of the function $\psi'(x)$ are positive in $(0,+\infty)$, while the derivatives of odd order are negative in the same interval. Due to the inequalities recalled above, we may state that inequality (2.1) is valid for every fixed positive integer m and for $x > 0$.

For what concerns inequality (2.2), we recall that in [2, p. 283, formula (3.3)] Hankel determinantal inequalities have been extended as follows: let $k, a_1, a_2, \ldots, a_n$ be non negative integers; if f is absolutely monotonic in $I = (-\infty, 0)$ or $I = (-\infty, 0]$, then the determinant $|f^{(a_i+a_j+k)}(x_i + x_j)|_n$ is non-negative for each pair (x_i, x_j) such that $x_i + x_j \in I$, and the same is true for the determinant $|(-1)^k f^{(a_i+a_j+k)}(x_i + x_j)|_n$; instead, if f is completely monotonic in $I = (0,+\infty)$ or $I = [0,+\infty)$, then we have $|(-1)^k f^{(a_i+a_j+k)}(x_i + x_j)|_n \geqslant 0$ and also $|(-1)^{a_i+a_j+k} f^{(a_i+a_j+k)}(x_i + x_j)|_n \geqslant 0$ whenever $x_i + x_j \in I$. In the particular case $n = 2$, $x_i = x_j = \frac{x}{2}$, $a_1 = a$, $a_2 = b$, the last two inequalities become $\begin{vmatrix} f^{(k+2a)}(x) & f^{(k+a+b)}(x) \\ f^{(k+a+b)}(x) & f^{(k+2b)}(x) \end{vmatrix} \geqslant 0$. Hence formula (2.2) for a positive integer k follows from the fact that $\psi'(x)$ is completely monotonic in $(0,+\infty)$. □

3. Further determinal inequalities for the psi function

In the previous section we considered determinants in which different derivatives of the psi function are calculated at the same x. Now, we shall consider a determinant such as $\begin{vmatrix} \psi^{(m)}(x) & \psi^{(m)}(x+h) \\ \psi^{(m)}(x+k) & \psi^{(m)}(x+h+k) \end{vmatrix}$, in which the same derivative of $\psi(x)$ is calculated at different points.

It is well-known that a function f defined on an open interval I is said *convex* (*strictly convex*) on I if for every pair $(x,y) \in I \times I$ and for any $\alpha \in (0,1)$ we get $f(\alpha x + (1-\alpha)y) \leqslant \alpha f(x) + (1-\alpha)f(y)$ ($<$ for strict convexity) or, which is the same, if the function $\Phi(u,v,w) = \dfrac{(w-v)f(u) + (u-w)f(v) + (v-u)f(w)}{(u-v)(v-w)(w-u)}$, symmetric with respect to the three variables u, v, w, is non-negative (positive) for every triplet (u,v,w) of pairwise distinct number of I. Instead, f is *concave* (*strictly concave*) on I if $f(\alpha x + (1-\alpha)y \geqslant$ (or$>$) $\alpha f(x) + (1-\alpha)f(y)$, i.e. if $\Phi(u,v,w)$ is non-positive (negative).

Another equivalent condition is the following: for a fixed $h > 0$, let us define the function $g(x) = f(x+h) - f(x)$, which is defined for every $x \in I$ such that $x + h \in I$

too; we may say that f is convex (strictly convex) on I if and only if g is non-decreasing (increasing), that is if for $x<y$ (with $x,y\in I$ such that $x+h,y+h\in I$) we have $f(x+h)-f(x)\leqslant f(y+h)-f(y)$ ($<$ for strict convexity). Similarly, f is concave (strictly concave) on I if and only if g is non-increasing (decreasing) in I.

Let f be positive on the open interval I; the function f is said *log-convex* on I (strictly log-convex) if $\log f$ is (strictly) convex on I, and it is said (strictly) *log-concave* on I if $\log f$ is (strictly) concave on I. Due to the characterization above, we may say that $f(x)>0$ is (strictly) log-convex on I if and only if the function $\log f(x+h)-\log f(x)$ is non-decreasing (increasing) on I, i.e., if $G(x)=\dfrac{f(x+h)}{f(x)}$ is non-decreasing (increasing) on I, while f is (strictly) log-concave on I if and only if $G(x)$ is non-increasing (decreasing) on I.

If f is negative on I, we may say again that $-f$ is log-convex if G is non-decreasing; in fact we have $G(x)=\dfrac{-f(x+h)}{-f(x)}=\dfrac{f(x+h)}{f(x)}$.

It is well-known that a function f twice differentiable on I is there convex if and only if $f''(x)\geqslant 0$ on I; hence if f is positive in I and twice differentiable and log-convex on I, it is also convex on I, and if f is concave and positive it is log-concave too.

It follows that the function ψ' is log-convex in $(0,+\infty)$, since $\begin{vmatrix}\psi'(x) & \psi''(x)\\ \psi''(x) & \psi'''(x)\end{vmatrix}\geqslant 0$ for the (2.1). Hence, for a fixed $h>0$, the function $\dfrac{\psi'(x+h)}{\psi'(x)}$ is non-decreasing, therefore for $k>0$ we have $\dfrac{\psi'(x+h)}{\psi'(x)}\leqslant\dfrac{\psi'(x+h+k)}{\psi'(x+k)}$, that is

$$\begin{vmatrix}\psi'(x) & \psi'(x+h)\\ \psi''(x+k) & \psi'(x+h+k)\end{vmatrix}\geqslant 0.$$

A similar result may be obtained for $\psi^{(m)}(x)$ (m is positive integer): in fact for odd m the function $\psi^{(m)}(x)$ is log-convex, since

$$D^2\big(\log\psi^{(m)}\big)=\frac{\psi^{(m)}(x)\psi^{(m+2)}(x)-[\psi^{(m+1)}(x)]^2}{[\psi^{(m)}(x)]^2},$$

which is non-negative for the formula (2.1). For even m we have again

$$D^2\big(\log(-\psi^{(m)}(x))\big)=\frac{\psi^{(m)}(x)\psi^{(m+2)}(x)-[\psi^{(m+1)}(x)]^2}{[\psi^{(m)}(x)]^2},$$

which is non-negative for the (2.1).

So we have proved the following

THEOREM 2. *For any fixed positive integer m, and for any $h,k>0$, the inequality*

$$\begin{vmatrix}\psi^{(m)}(x) & \psi^{(m)}(x+h)\\ \psi^{(m)}(x+k) & \psi^{(m)}(x+h+k)\end{vmatrix}\geqslant 0 \tag{3.1}$$

holds for any $x>0$.

Some particular cases of (3.1) are the following: for $k = h$ it becomes

$$\begin{vmatrix} \psi^{(m)}(x) & \psi^{(m)}(x+h) \\ \psi^{(m)}(x+h) & \psi^{(m)}(x+2h) \end{vmatrix} \geqslant 0; \tag{3.2}$$

this formula for $h = 1$ gives

$$\begin{vmatrix} \psi^{(m)}(x) & \psi^{(m)}(x+1) \\ \psi^{(m)}(x+1) & \psi^{(m)}(x+2) \end{vmatrix} \geqslant 0. \tag{3.3}$$

Formula (3.1) with $k = 2h$ becomes

$$\begin{vmatrix} \psi^{(m)}(x) & \psi^{(m)}(x+h) \\ \psi^{(m)}(x+2h) & \psi^{(m)}(x+3h) \end{vmatrix} \geqslant 0, \tag{3.4}$$

and (3.4) for $h = 1$ gives

$$\begin{vmatrix} \psi^{(m)}(x) & \psi^{(m)}(x+1) \\ \psi^{(m)}(x+2) & \psi^{(m)}(x+3) \end{vmatrix} \geqslant 0. \tag{3.5}$$

Similar results for the ψ function seem more difficult to obtain, since $\psi(x)$ has not constant sign on $(0,\infty)$: indeed, ψ is concave and strictly monotone on $(0,+\infty)$, and it is zero only in $\alpha \cong 1.46163$. So, Theorem 1 is false for $m = 0$, since the function $A(x) = \begin{vmatrix} \psi(x) & \psi'(x) \\ \psi'(x) & \psi''(x) \end{vmatrix}$ has not constant sign (it is $A(\alpha) = -(\psi'(\alpha))^2$ and $A\left(\frac{1}{2}\right) \cong 8.69$).

On the other hand, formula (3.1) is true, with the inequality reversed, in the interval $[\alpha,+\infty)$: in fact, in the interval $(\alpha,+\infty)$ ψ is concave and positive, then it is log-concave, and the inequality also holds for $x = \alpha$; according to the values of the parameters h and k it may hold also in $(0,+\infty)$. First we give independent proof of two particular cases, then we shall prove a more general theorem.

THEOREM 3. *The inequalities*

$$\begin{vmatrix} \psi(x) & \psi(x+1) \\ \psi(x+1) & \psi(x+2) \end{vmatrix} < 0 \tag{3.6}$$

and

$$\begin{vmatrix} \psi(x) & \psi(x+1) \\ \psi(x+2) & \psi(x+3) \end{vmatrix} < 0 \tag{3.7}$$

hold for any $x > 0$.

Proof. First we prove some asymptotic formulas which we shall use later:

$$\psi(x) = -\frac{1}{x} - \gamma + o(1); \tag{3.8}$$

$$\psi'(x) = \frac{1}{x^2} + \frac{\pi^2}{6} + o(1); \tag{3.9}$$

$$\psi''(x) = -\frac{2}{x^3} - 2\zeta(3) + o(1), \tag{3.10}$$

where the o-symbols are intended for $x \to 0^+$, the symbol γ denotes the well-known Euler-Mascheroni constant, defined as $\gamma = \lim_{n\to\infty}\left(1 + \frac{1}{2} + \cdots + \frac{1}{n} - \log n\right)$, and $\zeta(s)$ is of course the Riemann's zeta function, defined as $\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$ for every $s \in \mathbf{C}$ with $\Re e(s) > 1$.

To prove (3.8), first we write $\psi(1+x) = \psi(1) + o(1) = -\gamma + o(1)$ for $x \to 0^+$ (for the formula $\psi(1) = -\gamma$, see [1, p. 258, formula (6.3.2)]). Now, by remembering that $\psi(1+x) = \psi(x) + \frac{1}{x}$, see [1, p. 258, formula (6.3.5)], we have at once the (3.8).

Likewise, taking into account that $\psi^{(m)}(1) = (-1)^{m+1} m! \zeta(m+1)$ for any positive integer m [2, p. 260, formula (6.4.2)], in particular $\psi'(1) = \zeta(2) = \frac{\pi^2}{6}$, we have $\psi'(1+x) = \frac{\pi^2}{6} + o(1)$; since $\psi'(1+x) = \psi'(x) - \frac{1}{x^2}$ ([2, p. 260, (6.4.6)]), we obtain the (3.9). Formula (3.10) is obtained in a similar way, using $\psi''(1+x) = \psi''(x) + \frac{2}{x^3}$.

Now, by calculating explicitly the determinant $\begin{vmatrix} \psi(x) & \psi(x+1) \\ \psi(x+1) & \psi(x+2) \end{vmatrix}$, we find $-\frac{g(x)}{x^2(x+1)}$, where $g(x) = x\psi(x) + x + 1$; then we note that the first two derivatives of g are

$$\begin{cases} g'(x) = \psi(x) + x\psi'(x) + 1 \\ g(x) = 2\psi'(x) + x\psi''(x) \end{cases}$$

and determine the sign of $g''(x)$. by using (2.3) we find $g''(x) = \int_0^\infty \frac{(2t - t^2 x)e^{-xt}}{1 - e^{-t}} dt$; but we may note that $\frac{d}{dt}(t^2 e^{-xt}) = (2t - t^2 x)e^{-xt}$. So, an integration by parts gives

$$g''(x) = \int_0^\infty \frac{t^2 e^{-(x+1)t}}{(1 - e^{-t})^2} dt,$$

which is positive for any $x > 0$. Hence $g'(x)$ is increasing on $(0, +\infty)$.

Using formulas (3.8-9) we may easily calculate the limit of $g'(x)$ as $x \to 0^+$:

$$\begin{aligned} \lim_{x\to 0^+} g'(x) &= \lim_{x\to 0^+} (\psi(x) + x\psi'(x) + 1) \\ &= \lim_{x\to 0^+}\left(-\frac{1}{x} - \gamma + o(1) + \frac{1}{x} + \frac{\pi^2}{6}x = o(x) + 1\right) = 1 - \gamma, \end{aligned}$$

which is positive. Therefore $g'(x)$ is positive for any $x > 0$.

Now, we calculate in the same way the limit of $g(x)$ as $x \to 0^+$:

$$\lim_{x\to 0^+} g(x) = \lim_{x\to 0^+} (x\psi(x) + x + 1) = \lim_{x\to 0^+} (-1 - \gamma x + o(x) + x + 1) = 0.$$

This proves that $g(x)$ is positive for any $x > 0$, hence $\begin{vmatrix} \psi(x) & \psi(x+1) \\ \psi(x+1) & \psi(x+2) \end{vmatrix}$ is negative for any $x > 0$.

For what concerns formula (3.7), we have, proceeding as above,

$$\begin{vmatrix} \psi(x) & \psi(x+1) \\ \psi(x+2) & \psi(x+3) \end{vmatrix} = -\frac{h(x)}{x^2(x+1)(x+2)},$$

where

$$h(x) = (2x^2+2x)\psi(x) + 2x^2 + 5x + 2,$$

and

$$\begin{aligned} h'(x) &= (4x+2)\psi(x) + (2x^2+2x)\psi'(x) + 4x + 5, \\ h''(x) &= 4\psi(x) + (8x+4)\psi'(x) + (2x^2+2x)\psi''(x) + 4, \\ h'''(x) &= 12\psi'(x) + (12x+6)\psi''(x) + (2x^2+2x)\psi'''(x). \end{aligned}$$

Using (2.3) again, we find

$$h'''(x) = 2\int_0^{+\infty} \frac{(6t - 6t^2x - 3t^2 + t^3x^2 + t^3x)e^{-tx}}{1-e^{-t}}dt.$$

Since

$$\frac{d}{dt}(t^2e^{-tx}) = (2t - t^2x)e^{-tx}, \quad \frac{d}{dt}(t^3e^{-tx}) = (3t^2 - t^3x)e^{-tx},$$

we have

$$h'''(x) = 6\int_0^{+\infty} \frac{\frac{d}{dt}(t^2e^{-tx})}{1-e^{-t}}dt - 2(x+1)\int_0^{+\infty} \frac{\frac{d}{dt}\left(t^3e^{-tx}\right)}{1-e^{-t}}dt,$$

which, integrating by parts, becomes

$$h'''(x) = 2\int_0^{+\infty} \frac{\left(3t^2 - (x+1)t^3\right)e^{-t(x+1)}}{(1-e^{-t})^2}dt.$$

Again, we have $\frac{d}{dt}\left(t^3e^{-t(x+1)}\right) = \left(3t^2 - t^3(x+1)\right)e^{-t(x+1)}$, so we may integrate by parts again and obtain $h'''(x) = 4\int_0^{+\infty} \frac{t^3e^{-t(x+2)}}{(1-e^{-t})^3}dt$, which is positive for every $x > 0$. Therfore, h'' is increasing.

Using (3.8–10), we have $\lim_{x\to 0^+} = -4\gamma + \frac{2}{3}\pi^2 > 0$, so h' is also increasing on $(0,+\infty)$; then, $\lim_{x\to 0^+} h'(x) = 3 - 2\gamma > 0$, so h is also increasing on $(0,+\infty)$; finally, $\lim_{x\to 0^+} h(x) = 0$, and h is positive on $(0,+\infty)$. □

Finally, we have the more general result

THEOREM 4. *The inequality*

$$\begin{vmatrix} \psi(x) & \psi(x+h) \\ \psi(x+k) & \psi(x+h+k) \end{vmatrix} < 0 \tag{3.11}$$

holds for any $x > 0$ *if and only if the positive parameters* h *and* k *are such that* $h+k \geqslant \alpha$*, where* α *is the unique zero of* $\psi(x)$*.*

Proof. We already know that (3.11) is true, independently of h and k, on $[\alpha, +\infty)$; so we have only to consider $0 < x < \alpha$.

First, we put

$$B(x) = \begin{vmatrix} \psi(x) & \psi(x+h) \\ \psi(x+k) & \psi(x+h+k) \end{vmatrix} = \psi(x)\psi(x+h+k) - \psi(x+h)\psi(x+k)$$

and calculate $\lim_{x\to 0^+} B(x)$; if $h+k < \alpha$, then $\psi(h+k) < 0$, so $\lim_{x\to 0^+} B(x) = +\infty$, and similarly in the case $h+k > \alpha$ we have $\lim_{x\to 0^+} B(x) = -\infty$; for $h+k = \alpha$ we we may note that $\psi(x+\alpha) = \psi'(\alpha)x + o(x)$ as $x \to 0^+$; this equality, together with (3.8), gives $\lim_{x\to 0^+} B(x) = -\psi'(\alpha) - \psi(h)\psi(k) < 0$. This proves that (3.11) cannot hold for any $x > 0$ if $h+k < \alpha$.

Now, let us consider the case $h+k = \alpha$, with $h \leqslant k$; we have to analyze the following subcases:

A1) $0 < x < \alpha < x+h \leqslant x+k < x+\alpha$;
A2) $0 < x < \alpha = x+h \leqslant x+k < x+\alpha$;
A3) $0 < x < x+h < \alpha < x+k < x+\alpha$;
A4) $0 < x < x+h \leqslant x+k = \alpha < x+\alpha$;
A5) $0 < x < x+h \leqslant x+k < \alpha < x+\alpha$.

The product $\psi(x)\psi(x+\alpha)$ is negative, because $\psi(x) < 0$ and $\psi(x+\alpha) > 0$; in the subcases A1 and A5 $\psi(x+h)\psi(x+k)$ is positive (product of two factors with the same sign), therefore $B(x) < 0$, and the same is true in the subcases A2 and A4, since $\psi(x+h)\psi(x+k) = 0$. In the case A3 the product $\psi(x+h)\psi(x+k)$ is also negative, but $|\psi(x+h)\psi(x+k)|$ is lesser than $|\psi(x)\psi(x+\alpha)|$, so we have $B(x) < 0$ again.

In the case $h+k > \alpha$, with $h \leqslant k$, the subcases to be considered are:

B1) $\alpha \leqslant h \leqslant k,\ 0 < x < \alpha < x+h \leqslant x+k < x+h+k$;

C1) $h < \alpha \leqslant k,\ 0 < x < \alpha < x+h < x+k < x+h+k$;
C2) $h < \alpha \leqslant k,\ 0 < x < \alpha = x+h < x+k < x+h+k$;
C3) $h < \alpha \leqslant k,\ 0 < x < x+h < \alpha < x+k < x+h+k$;

D1) $h \leqslant k < \alpha,\ 0 < x < \alpha < x+h \leqslant x+k < x+h+k$;
D2) $h \leqslant k < \alpha,\ 0 < x < \alpha = x+h \leqslant x+k < x+h+k$;

D3) $h \leqslant k < \alpha,\ 0 < x < x+h < \alpha < x+k < x+h+k;$

D4) $h \leqslant k < \alpha,\ 0 < x < x+h \leqslant x+k = \alpha < x+h+k;$

D5) $h \leqslant k < \alpha,\ 0 < x < x+h \leqslant x+k < \alpha < x+h+k.$

In the subcase B1 we have $\psi(x) < 0$, $\psi(x+h) > 0$, $\psi(x+k) > 0$ and $\psi(x+h+k) > 0$, so $B(x)$ is negative; subcases C1–C3 are similar to A1–A3, and finaly subcases D1–D5 are similar to A1–A5; therfore we have $B(x) < 0$ in every case. □

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