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INEQUALITIES FOR THE ZEROS OF THE ASSOCIATED ULTRASPHERICAL POLYNOMIALS

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Abstract. Using a functional analytic method based on the three-term recurrence relation of orthogonal polynomials, we study the monotonicity of the zeros of the associated ultraspherical (or Gegenbauer) polynomials, and we give some inequalities for the largest zeros.

1. Introduction

The orthogonal polynomials $P_n(x)$, $n \geqslant 0$, of degree n, with respect to a positive Borel measure on the real line with infinite mass points, can be defined by the recurrence relation:

$$\begin{gathered} \alpha_n P_{n+1}(x) + \alpha_{n-1} P_{n-1}(x) + b_n P_n(x) = x P_n(x) \\ P_{-1}(x) = 0, \quad P_0(x) = 1, \end{gathered} \tag{1.1}$$

where $\alpha_n > 0$ and b_n real sequences.

The associated polynomials $P_n(x;c)$ of the above polynomials are obtained when we replace n by $n+c$ in the coefficients α_n and b_n of (1.1), i.e.,

$$\begin{gathered} \alpha_{n+c} P_{n+1}(x;c) + \alpha_{n+c-1} P_{n-1}(x;c) + b_{n+c} P_n(x;c) = x P_n(x;c) \\ P_{-1}(x;c) = 0, \quad P_0(x;c) = 1 \end{gathered}$$

for arbitrary real $c \geqslant 0$ or $c > -1$.

The associated polynomials are called associated of order c, if c is an integer greater than 1 and numerator polynomials, if $c = 1$. One of the most important results for the last case was given by A. Elbert and A. Laforgia in [5]. Results for particular cases of associated polynomials concerning explicit forms, orthogonality measures, monotonicity properties and differential inequalities for their zeros were given in [1], [3], [7], [8], [9], [16], [24], [26], [30]. We point out that the above associated polynomials do not satisfy a second order differential equation of Sturm-Liouville type. Therefore the effective methods which are based on properties of differential equations of Sturm-Liouville type cannot be applied. They satisfy a 4th-order differential equation. For particular cases of associated polynomials, many authors [28], [29], [31, and the references therein] gave explicitly the corresponding 4th-order differential equations.

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In this paper we study monotonicity of the zeros of the associated ultraspherical (or Gegenbauer) polynomials and we present some inequalities for the largest zeros. In section 2 we present the method we shall use, and in section 3 we give the main results.

2. The method

We consider the general recurrence relation

$$\alpha_n P_{n+1}(x) + \alpha_{n-1} P_{n-1}(x) + b_n P_n(x) = x P_n(x) \tag{2.1}$$

$$P_{-1}(x) = 0, \quad P_0(x) = 1, \tag{2.2}$$

where α_n, b_n real sequences, with $\alpha_n > 0$, $n = 0, 1, \dots$ and present briefly the method we shall use.

Let e_k, $k = 0, 1, 2, \dots, n-1$ be an orthonormal basis in a finite dimensional Hilbert space H_N and let V be the truncated shift operator

$$V e_k = e_{k+1},\ k = 0, 1, \dots .n-2, \quad V e_{n-1} = 0. \tag{2.3}$$

The adjoint of V satisfies

$$V^* e_k = k e_{k-1},\ k = 0, 1, \dots, n-2, \quad V^* e_0 = 0. \tag{2.4}$$

Let A, B be the diagonal operators

$$A e_k = \alpha_k e_k, \quad B e_k = b_k e_k,\ k = 0, 1, \dots, n-1. \tag{2.5}$$

Then we can see easily the following [13].

THEOREM 2.1. *The zeros of the polynomial $P_n(x)$ of degree n defined by*

$$\alpha_n P_{n+1}(x) + \alpha_{n-1} P_{n-1}(x) + b_n P_n(x) = x P_n(x)$$
$$P_{-1}(x) = 0, \quad P_0(x) = 1$$

are the eigenvalues of the operator

$$T = AV^* + VA + B$$

i.e.,

$$(AV^* + VA + B) x_k = \lambda_k x_k$$

or

$$\lambda_k = ((AV^* + VA + B) x_k, x_k), \quad \|x_k\| = 1, \quad k = 0, 1, \dots, n-1$$

and vice versa.

REMARK 2.1. The identification of zeros of orthogonal polynomials as eigenvalues of a tridiagonal matrix is a very old result. What is new in our approach is the separation of this tridiagonal matrix as a sum of products of simple matrices in the operator form

$$T = AV^* + VA + B.$$

Some results have been found very easily, due to this separation. We think that these results do not follow easily from the usual tridiagonal matrix form of T.

Upper bounds for the zeros $\lambda_{n,k}$ of the polynomial $P_n(x)$.

Using the relation

$$\lambda_{n,k} = ((AV^* + VA + B))x_{n,k}, x_{n,k}), \quad \|x_{n,k}\| = 1,\ k = 0, 1, 2, \dots, n-1 \tag{2.7}$$

we can easily obtain *upper bounds for all the zeros* $\lambda_{n,k}$.

If fact from (2.7) we have:

$$\lambda_{n,k} \leqslant \|A\| \cdot \|V^*\| + \|V\| \cdot \|A\| + \|B\| = 2\|A\| + \|B\| = 2 \max_k |\alpha_k| + \max_k |b_k|,$$

since $\|V\| = \|V^*\| = 1$.

One can find more stringent upper bounds using chain sequences [9].

Lower bounds for the largest zero $\lambda_{\max}$.

Since the operator

$$AV^* + VA + B$$

is self-adjoint, we have that:

$$\lambda_{\max}=\|AV^*+VA+B\| = \sup_{f\in H_n} |((AV^*+VA+B)f,f)| \geqslant ((AV^*+VA+B)f,f), \ \|f\| = 1. \tag{2.8}$$

Choosing $f = 1/\sqrt{2}\,(e_{n-1} + e_{n-2})$, we find from (2.8)

$$\lambda_{\max} \geqslant \frac{1}{2}[2\alpha_{n-2} + b_{n-2} + b_{n-1}], \quad n \geqslant 2. \tag{2.9}$$

We can find sharper lower bounds if we use the relation

$$\begin{aligned}\lambda_{\max}^{(i)}&=\|AV^*+VA+B\|^{(i)}=\|(AV^*+VA+B)^i\|= \sup_{f\in H_n} |((AV^*+VA+B)^i f,f)|\\ &\geqslant ((AV^* + VA + B)^i f, f), \quad \|f\| = 1,\ i = 1, 2, 3, \dots\end{aligned} \tag{2.10}$$

So, from (2.9) for $i = 2$, $f = 1/\sqrt{2}\,(e_{n-1} + e_{n-2})$ we obtain

$$\lambda_{\max}^2 \geqslant \frac{1}{2}[2\alpha_{n-3}^2 + 2\alpha_{n-2}^2 + b_{n-2}^2 + b_{n-1}^2]. \tag{2.11}$$

Differentiability of the Eigenvalues.

If the operators A and B of the operator $T = AV^* + VA + B$ depend on a real parameter v and $A(v)$, $B(v)$ are uniformley bounded for v in some interval and differentiable in the operator norm, then the eigenvectors $x_k(v)$ of the self-adjoint operator

$$T(v) = A(v)V^* + VA(v) + B(v)$$

are strongly differentiable and the derivative of the corresponding eigenvalues $\lambda_k(v)$ is given by

$$\frac{d\lambda_k(v)}{dv} \equiv \lambda_k'(v) = ((A'(v)V^* + VA'(v) + B'(v))x_k(v), x_k(v)), \quad \|x_k(v)\| = 1$$

where $A'(v)e_k = \alpha_k' e_k$, $B'(v)e_k = b_k' e_k$, $k = 0, 1, \dots, n-1$ and primes mean differentiation with respect to v. For details see [10].

When we want to examine the sign of $\frac{d\lambda_k(v)}{dv}$ a difficulty appears in the scalar product $((A'(v)V^* + VA'(v) + B'(v))x_k(v), x_k(v))$, because the operator $(A'(v)V^* + VA'(v))$ is not positive although in many interesting cases the sequence $\alpha_n'(v)$, is strictly positive.

From a more general result proved in [12], we know that the components of the eigenvector $x_{\max}$ which corresponds to the largest eigenvalue $\lambda_{\max}$ of the operator $AV^* + VA + B$ are strictly positive numbers. Therefore the scalar product in

$$\frac{d\lambda_k(v)}{dv} = ((A'(v)V^* + VA'(v) + B'(v))x_k(v), x_k(v))$$

is greater than zero provided that $a'(v) > 0$ and $b'(v) > 0$. For the largest eigenvalue $\lambda_{\max}(v)$ the following relation also holds

$$\frac{d^2\lambda_{\max}(v)}{dv^2} \geqslant ((A''(v)V^* + VA''(v) + B''(v))x_{\max}(v), x_{\max}(v))$$

provided that $A(v)$ is twice differentiable [11].

REMARK. Formally, the differentiability of the eigenvalues with respect to a parameter is an old result, known as the Hellmann-Feyman theorem [R. P. Feyman, Phys. Rev 56(1939) 340–343, H. Hellmann, Einfuhrung in die Quantenchemie, (Deuticke, Vienna 1937)]. The first application of this theorem to the zeros of orthogonal polynomlals, to the best of our knowledge, is that of M. E. H. Ismail, [22]. Previously, this theorem has been applied by J. T. Lewis and M. E. Muldoon, [27], for the zeros of Bessel functions of the first kind and of order $v > -1$. Rigorous proofs of the Hellmann-Feyman theorem were given with different assumptions, in [10], [11], [22].

Finally we point out that the above method is general and is valid not only for the particular case considered in this paper. It has been used successfully by E. K. Ifantis and the author in previous work [10, 13–19].

3. Main Results

The associated Ultraspherical (or Gegenbauer) Polynomials

$$P_n^{\lambda}(x;c),\ c \geqslant 0,\ \lambda \geqslant -1/2,\ n = 0, 1, 2, \ldots$$

are defined as follows

$$\begin{gathered}(n+c+1)P_{n+1}^{\lambda}(x;c)+(n+c+2\lambda-1)P_{n-1}^{\lambda}(x;c)=2x(n+c+\lambda)P_n^{\lambda}(x;c)\\ P_{-1}^{\lambda}(x;c) = 0, \quad P_0^{\lambda}(x;c) = 1.\end{gathered} \tag{3.1}$$

The associated Legendre polynomials obtained with $c = \frac{1}{2}$ were studied in [2]. Setting in (3.1)

$$U_{-1} = 0,\ U_0 = 1,\ U_n = \frac{n + c + 2\lambda - 1}{n + c} U_{n-1},\ n \geqslant 1$$

so that $U_n > 0$, and

$$P_n^\lambda(x;c) = \Big(\frac{U_n}{n+c+\lambda}\Big)^{1/2} Q_n^\lambda(x;c)$$

we find that

$$\frac{1}{2}\Big[\frac{(n+c+1)(n+c+2\lambda)}{(n+c+1+\lambda)(n+c+\lambda)}\Big]^{1/2} Q_{n+1}^\lambda(x;c) + \frac{1}{2}\Big[\frac{(n+c)(n+c+2\lambda-1)}{(n+c+\lambda)(n+c+\lambda-1)}\Big]^{1/2} Q_{n-1}^\lambda(x;c) = xQ_n^\lambda(x;c)$$

$$Q_{-1}^\lambda(x;c) = 0, \quad Q_0^\lambda(x;c) = 1 \tag{3.2}$$

The polynomials $Q_n^\lambda(x;c)$ and the associated ultraspherical polynomials $P_n^\lambda(x;c)$, have the same zeros, say $x_{nk}^\lambda(c)$, $k = l, 2, \ldots [n/2]$, in decreasing order. According to our aproach the zeros, $x_{nk}^\lambda(c)$ are eigenvalues of the operator:

$$A(\lambda,c)V^* + VA(\lambda,c)$$

i.e.

$$\frac{1}{2}(A(\lambda,c)V^* + VA(\lambda,c))f_{nk}^\lambda(c) = x_{nk}^\lambda(c) f_{nk}^\lambda(c), \quad \|f_{nk}^\lambda(c)\| = 1$$

or

$$x_{nk}^\lambda(c) = \frac{1}{2}((A(\lambda,c)V^* + VA(\lambda,c))f_{nk}^\lambda(c), f_{nk}^\lambda(c)),\ k = 1,2,\ldots,[n/2], \|f_{nk}^\lambda(c)\| = 1 \tag{3.3}$$

where

$$A(\lambda,c)e_j = a_j(\lambda,c)e_j, j = 0,1,2,\ldots,n-1$$

and

$$\alpha_j(\lambda,c) = \frac{1}{2}\Big[\frac{(j+c+1)(j+c+2\lambda)}{(j+c+1+\lambda)(j+c+\lambda)}\Big]^{1/2} \tag{3.4}$$

Upper and lower bounds for the largest zero $x_{n1}^\lambda(c)$.

From (3.3) we obtain the upper bounds:

$$x_{nk}^\lambda(c) \leqslant \|A\| = \max_j |a_j(\lambda,c)| = \frac{1}{2}\Big[\frac{(n+c)(n+c+2\lambda-1)}{(n+c+\lambda)(n+c+\lambda-1)}\Big]^{1/2}$$

for all zeros $x_{nk}^\lambda(c)$, $k = 1, 2 \ldots$.

For $c = 0$ we obtain upper bounds for all zeros x_{nk}^λ of the classical ultraspherical polynomials $P_n^\lambda(x)$.

Applying the relations (2.9) and (2.11) for $b_n = 0$ and $\alpha_j(\lambda,c)$, $j = 1,2,\ldots,n-1$ given by (3.4), we obtain the following lower bounds for the largest zero $x_{n1}^\lambda(c)$.

$$x_{n1}^\lambda(c) \geqslant \alpha_{n-2} = \frac{1}{2}\Big[\frac{(n+c-1)(n+c+2\lambda-2)}{(n+c+\lambda-1)(n+c+\lambda-2)}\Big]^{1/2}, \quad \lambda \geqslant 1$$

$$(x_{n1}^\lambda(c))^2 \geqslant \alpha_{n-2}^2 + \alpha_{n-3}^2 =$$

$$= \frac{1}{4(n+c+\lambda-2)}\Big[\frac{(n+c-1)(n+c+2\lambda-2)}{(n+c+\lambda-1)} + \frac{(n+c-2)(n+c+2\lambda-3)}{(n+c+\lambda-3)}\Big]$$

$$= \frac{(n+c-1)(n+c+2\lambda-2)(n+c+\lambda-3)+(n+c-2)(n+c+2\lambda-3)(n+c+\lambda-1)}{4(n+c+\lambda-1)(n+c+\lambda-2)(n+c+\lambda-3)}, \lambda \geqslant 2.$$

We can obtain more stringent lower bounds for the largest zero $x_{n1}^{\lambda}(c)$ if we use relation (2.10) for $i = 3, 4, \dots$

For $c = 0$, in the above inequalities, we obtain the corresponding lower bounds for the largest zero x_{n1}^{λ}, of the classical Ultraspherical polynomials.

Monotonicity of the largest zero $x_{n1}^{\lambda}(c)$.

The differential equation for the largest zero $x_{n1}^{\lambda}(c)$ with respect to λ, according to our approach, is the following

$$\frac{d}{d\lambda}x_{n1}^{\lambda}(c) = \frac{1}{2}\left(\left(\frac{d}{d\lambda}A(\lambda,c)V^* + V\frac{d}{d\lambda}A(\lambda,c)f_{n1}^{\lambda}(c)\right), f_{n1}^{\lambda}(c)\right)$$

or, since $\left(\frac{d}{d\lambda}A(\lambda,c)V^*f_{n1}^{\lambda}(c), f_{n1}^{\lambda}(c)\right)$ is real

$$\frac{d}{d\lambda}x_{n1}^{\lambda}(c) = \left(\frac{d}{d\lambda}A(\lambda,c)V^*f_{n1}^{\lambda}(c), f_{n1}^{\lambda}(c)\right) \tag{3.5}$$

where

$$\frac{d}{d\lambda}A(\lambda,c)e_j = \frac{d}{d\lambda}a_j(\lambda,c)e_j, \quad j = 0,1,2,\dots,n-1$$
$$\frac{d}{d\lambda}A(\lambda,c) = -A(\lambda,c)\Gamma(\lambda,c)$$

and

$$\Gamma(\lambda,c)e_j = \gamma_j(\lambda,c)e_j, \quad \gamma_j(\lambda,c) = \frac{\lambda^2 + (\lambda - 1/2)(j+c)}{(j+c+2\lambda)(j+c+\lambda+1)(j+c+\lambda)}. \tag{3.6}$$

The differential equation (3.5) now is written as

$$-\frac{d}{d\lambda}x_{n1}^{\lambda}(c) = (\Gamma(\lambda,c)A(\lambda,c)V^*f_{n1}^{\lambda}(c), f_{n1}^{\lambda}(c)). \tag{3.7}$$

From (3.7) it follows immediately that

$$\frac{d}{d\lambda}x_{n1}^{\lambda}(c) < 0, \quad \text{for } \lambda \geqslant 1/2, \quad \text{since } (f_{n1}^{\lambda}(c), e_n) > 0.$$

Also, since

$$\gamma_1(\lambda,c) < \frac{1}{2(j+c+\lambda+1)}, \quad j = 0,1,2,\dots,n-1$$

it follows from the differential equation (3.7) that

$$-\frac{d}{d\lambda}x_{n1}^{\lambda}(c) < \frac{1}{c+\lambda+1}(A(\lambda,c)V^*f_{n1}^{\lambda}(c), f_{n1}^{\lambda}(c)) = \frac{1}{c+\lambda+1}x_{n1}^{\lambda}, \quad \lambda \geqslant 1/2$$

or

$$\frac{d}{d\lambda}[(c+\lambda+1)^{1/2}x_{n1}^{\lambda}(c)] > 0, \quad \lambda \geqslant 1/2.$$

From the above we obtain the following:

THEOREM 3.1. *For any fixed* n, c *and* $\lambda \geqslant 1/2$ *we have*
(i) the largest zero $x_{n1}^{\lambda}(c)$ *decreases with respect to* λ.
(ii) The function $(c+\lambda+1)^{1/2}x_{n1}^{\lambda}(c)$, *increases with respect to* λ.

REMARKS. (i) For $c = 0$, we have the corresponding result for the classical ultraspherical polynomials $P_n^{\lambda}(x)$ [14], [4].

(ii) Recently it is proved [6] that the function $\left[\lambda + \dfrac{2n^2+1}{4n+2}\right]^{1/2} x_{nk}^{\lambda}$, $k = 1, 2, \ldots$ where $x_{n,k}^{\lambda}$ are the zeros of the classical Ultraspherical polynomials increases as λ increases for $\lambda > -1/2$. This result answers and improves the following conjectures

LAFORGIA'S CONJECTURE [25]. For $n \geqslant 2$ and $1 \leqslant k \leqslant \left[\dfrac{n}{2}\right]$ the function λx_{nk}^{λ}, $k = 1, 2, \ldots$ increases as λ increases for $\lambda > 0$.

ISMAIL-LETTESSIER-ASKEY CONJECTURE [21]. For $n \geqslant 2$ and $1 \leqslant k \leqslant [n/2]$ the function $(\lambda+1)^{1/2}x_{nk}^{\lambda}$, $k = 1, 2, \ldots$ increases as λ increases for $\lambda > -1/2$.

Similarly, in the same way as for the differential equation (37) we can find the *differential equation for the largest zero* $x_{n1}^{\lambda}(c)$, *with respect to* c.

$$\frac{d}{dc}x_{n1}^{\lambda}(c) = (\Gamma(\lambda,c)A(\lambda,c)V^{*}f_{n1}^{\lambda}(c), f_{n1}^{\lambda}(c)) \tag{3.8}$$

where:

$$\Gamma(\lambda,c)e_j = \gamma_j(\lambda,c)e_j,\ \gamma_j(\lambda,c) = \frac{[2(j+c+\lambda)+1]\lambda(\lambda-1)}{(j+c+1)(j+c+2\lambda)(j+c+\lambda+1)(j+c+\lambda)} \tag{3.9}$$

and $\alpha_j(\lambda,c) = \dfrac{1}{2}\left[\dfrac{(j+c+1)(j+c+2\lambda)}{(j+c+1+\lambda)(j+c+\lambda)}\right]^{1/2} > 0$.

From (3.8) and (3.9) we obtain the following:

THEOREM 3.2. *The largest zero* $x_{n1}^{\lambda}(c)$ *of* $P_n^{\lambda}(x,c)$, *satisfy the differential inequalities:*

(i) $$0 < \frac{2\lambda(\lambda-1)}{(n+c+2\lambda-1)^3}x_{n1}^{\lambda}(c) < \frac{d}{dc}x_{n1}^{\lambda}(c) < \frac{2\lambda(\lambda-1)}{(n+c)^3}x_{n1}^{\lambda}(c),\ \lambda>1,\ -\frac{1}{2}<\lambda<0 \tag{3.10}$$

(ii) $$\frac{2\lambda(\lambda-1)}{(n+c+2\lambda-1)^3}x_{n1}^{\lambda}(c) < \frac{d}{dc}x_{n1}^{\lambda}(c) < \frac{2\lambda(\lambda-1)}{(n+c+\lambda)^3}x_{n1}^{\lambda}(c) < 0,\ 0<\lambda<1. \tag{3.11}$$

Proof. Since $(f_{n1}^{\lambda}(c), e_n) > 0$ and $\gamma_j(\lambda,c) > 0$, for $\lambda > 1$ $\gamma_j(\lambda,c) < 0$, for $0 < \lambda < 1$ it follows from (3.8) that

$$\frac{dx_{n1}^{\lambda}(c)}{dc} > 0 \quad \text{for } \lambda > 1 \quad \text{and} \quad \frac{dx_{n,1}^{\lambda}(c)}{dc} < 0 \quad \text{for } 0 < \lambda < 1.$$

Also since for $\lambda > 1$

$$\gamma_j(\lambda,c) < \frac{2(j+c+\lambda+1)\lambda(\lambda-1)}{(j+c+1)(j+c+2\lambda)(j+c+\lambda+1)(j+c+\lambda)} < \frac{2\lambda(\lambda-1)}{(j+c+1)^3}$$

and

$$\gamma_j(\lambda,c) > \frac{2(j+c+\lambda)\lambda(\lambda-1)}{(j+c+1)(j+c+2\lambda)(j+c+\lambda+1)(j+c+\lambda)} > \frac{2\lambda(\lambda-1)}{(j+c+2\lambda)^3}$$

it follows from (3.8) that

$$0 < \frac{2\lambda(\lambda-1)}{(n+c+2\lambda-1)^3} x_{n1}^{\lambda}(c) < \frac{d}{dc} x_{n1}^{\lambda}(c) < \frac{2\lambda(\lambda-1)}{(n+c)^3} x_{n1}^{\lambda}(c), \quad \lambda > 1.$$

Similarly we find that

$$\frac{2\lambda(\lambda-1)}{(n+c+2\lambda-1)^3} x_{n1}^{\lambda}(c) < \frac{d}{dc} x_{n1}^{\lambda}(c) < \frac{2\lambda(\lambda-1)}{(n+c+\lambda)^3} x_{n1}^{\lambda}(c) < 0, \quad 0 < \lambda < 1.$$

COROLLARY. *For the largest zeros* $x_{n1}^{\lambda}(c)$ *of the associated Ultraspherical polynomials the following inequalities hold.*

$$(i) \qquad e^{\lambda(\lambda-1)\left[\frac{1}{(n+2\lambda-1)^2} - \frac{1}{(n+2\lambda+c-1)^2}\right]} x_{n1}^{\lambda} < x_{n1}^{\lambda}(c) < e^{\lambda(\lambda-1)\left[\frac{1}{n^2} - \frac{1}{(n+c)^2}\right]} x_{n1}^{\lambda},$$

$$-\frac{1}{2} < \lambda < 0, \quad \lambda > 1$$

$$(ii) \qquad e^{\lambda(\lambda-1)\left[\frac{1}{(n+\lambda-1)^2} - \frac{1}{(n+\lambda+c-1)^2}\right]} x_{n1}^{\lambda} < x_{n1}^{\lambda}(c) < e^{\lambda(\lambda-1)\left[\frac{1}{(n+\lambda)^2} - \frac{1}{(n+c+\lambda)^2}\right]} x_{n1}^{\lambda},$$

$$0 < \lambda < 1.$$

Proof. It follows by integration with respect to c, from 0 to c, from the inequalities (3.10), (3.11). The above inequalities become equalities when $c = 0$, and therefore are stringent when c is near to zero in the interval $(-1, +\infty)$.

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