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SHARP INEQUALITIES CONNECTED TO THE HOMOGENIZED p–POISSON EQUATION

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Abstract. In this paper we study inequalities for the effective energy density associated with the homogenized p-Poisson equation. We prove that the inequalities are sharpest possible and we even find all cases of equality. Our results implies uniqueness of rank 1 laminates within the class of multi-phase structures.

1. Introduction

We study the p-Poisson equation

$$-\mathrm{div}\Big(\lambda(\frac{x}{\varepsilon})\,|Du|^{p-2}\,Du\Big) = f \ \text{ on } \Omega \subset R^N,\ p > 1,$$

with Dirichlet boundary data where λ is periodic relative to a cell Y and bounded between to positive constants. Considering $\varepsilon = \varepsilon_h$, $h = 1, 2, \ldots,$ as a sequence such that $\varepsilon_h \to 0$ as $h \to \infty$ we get a family of p-Poisson equations. Homogenization results for monotone operators then yield the existence of a corresponding homogenized problem on the form

$$-\mathrm{div}(b(Du)) = f \ \text{ on } \Omega.$$

In this paper we consider the following sharp inequalities for the *effective energy density* $(b(e_i), e_i)$

$$q_i \leqslant (b(e_i), e_i) \leqslant q^i. \tag{1.1}$$

Here e_i is the usual basis-vector in the i'th direction and q_i and q^i are real numbers obtained as compositions of power-means (see Section 2). E.g. in the case when Y is the unit cube $]0,1[^N$ these numbers can be written as

$$q_i = \int_0^1 \cdots \int_0^1 \left(\int_0^1 \lambda^{\frac{1}{1-p}}\,dx_i\right)^{1-p} dx_1 \cdots dx_{i-1}dx_{i+1} \cdots dx_N,$$

$$q^i = \left(\int_0^1 \left(\int_0^1 \cdots \int_0^1 \lambda\,dx_1 \cdots dx_{i-1}dx_{i+1} \cdots dx_N\right)^{\frac{1}{1-p}} dx_i\right)^{1-p}.$$

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The first proof of (1.1) was presented in [15]. This result represents a generalization of the inequality

$$q_i \leqslant q^i$$

which has been known in the theory of inequalities for quite a long time (see e.g. [1], [3, p. 170] and [7, p. 148]). Variants and generalizations of (1.1) has been developed later (see [10, 11, 12, 13, 14]).

These inequalities represent a generalization of the non-linear Wiener bounds for the p-Poisson equation. Moreover, it seems to be possible to interpret the actual inequalities as some type of refinements of some version of the Minkowski inequality. In many cases the upper and lower bounds q_i and q^i are sufficiently close to give a better estimate of the effective energy density than the non-linear Hashin-Shtrikman bounds. In [9] a theorem was given (without proof) stating the precise conditions for which $q_i = q^i$ (see Theorem 1). In this paper we present the detailed proof of this result. We also point out a new application of this theorem concerning the proof of the uniqueness of *rank 1 laminates* (see Theorem 2). Moreover, we describe how (1.1) can be used to estimate shear stiffness in sandwich plates consisting of composite power-law material in the core (see Section 5).

2. Preliminaries

In the space R^N, we consider a fixed parallelepiped $Y = \prod_{j=1}^{N} \left]0, x_j^0\right[$ (a Y-cell). Using general formulae for monotonic problems (see e.g. [6]) we get that the *homogenized operator* $b : R^N \to R^N$ for the p-Poisson problem takes the following form:

$$b(\xi) = \frac{1}{|Y|} \int_Y \lambda(x) \left|\xi + Dw^{\xi}(x)\right|^{p-2} (\xi + Dw^{\xi}(x))dx,$$

where w^{ξ} is the unique solution (up to an additive constant) to the *local problem*: Find w^{ξ} in $W^{1,p}_{per}(Y)$ (= the space of all Y-periodic $\psi \in W^{1,p}(Y)$) such that,

$$\int_Y (\lambda(x) \left|\xi + Dw^{\xi}(x)\right|^{p-2} (\xi + Dw^{\xi}(x)), Dv)dx = 0 \tag{2.1}$$

for all $v \in W^{1,p}_{per}(Y)$.

It is possible to prove (see [4] and the references therein) that w^{ξ} is the unique minimum point of the problem:

$$f_\lambda(\xi) = \min_{u \in W^{1,p}_{per}(Y)} \frac{1}{|Y|} \int_Y \lambda(x) \left|\xi + Du(x)\right|^p dx \tag{2.2}$$

Observe also that by putting $v = w^{\xi}$ in (2.1) we get

$$(b(\xi), \xi) = f_\lambda(\xi). \tag{2.3}$$

Now, let i be a fixed positive integer less then or equal to N. The following notations will be used: If $s \in R^1$ and $t \in R^{N-1}$, then $x = (s,t)$ denotes the element in R^N with coordinates: $x_i = s$, $x_j = t_j$ whenever $j < i$, and $x_j = t_{j-1}$ whenever $i < j$.

The spaces T and S are defined by $T = \prod_{j\neq i} \left]0, x_j^0\right[$ and $S = \left]0, x_i^0\right[$, equipped with the corresponding Lebesgue measures η and μ, respectively. Finally we define the numbers q_i and q^i as follows:

$$q_i = |Y|^{-1} \int_Y \lambda_i dx,$$

where

$$\lambda_i(s,t) = \left(|S|^{-1} \int_S (\lambda(\cdot,t))^{\frac{1}{1-p}} d\mu \right)^{1-p}$$

for all $s \in S$, and almost all $t \in T$.

$$q^i = \left(|Y|^{-1} \int_Y (\lambda^i)^{\frac{1}{1-p}} dx \right)^{1-p},$$

where

$$\lambda^i(s,t) = |T|^{-1} \int_T \lambda(s,\cdot) d\eta$$

for almost all $s \in S$, and all $t \in T$.

3. Sharpness conditions

In this section we will prove the following result which states necessary and sufficient conditions such that $q_i = (b(e_i), e_i) = q^i$.

THEOREM 1. *The following statements are equivalent:*

1. $q_i = (b(e_i), e_i)$
2. $(b(e_i), e_i) = q^i$
3. $\lambda = k_{i1}k_{i2}$, *where* k_{i1} *is dependent of only the* i *'th coordinate and* k_{i2} *is independent of the* i *'th coordinate.*

In order to prove the theorem we need to following result, which is of independent interest.

LEMMA 1. *Suppose* $f \in L^p(Y)$ *where* $p \geqslant 1$ *and let* U_1 *and* U_2 *be subsets of* Y *on the form* $I_{i1} \times \prod_{j\neq i} I_j$ *and* $I_{i2} \times \prod_{j\neq i} I_j$, *respectively, where* $\prod_{j\neq i} I_j \subseteq T$, $\{I_j\}$ *are segments and* I_{i1} *and* I_{i2} *are disjoint segments of* S. *Furthermore, assume that for every such pair (* U_1, U_2*),*

$$|U_1| \int_{U_2} f\,dx = |U_2| \int_{U_1} f\,dx.$$

Then f *is independent a.e. of the* i *'th coordinate.*

Proof. We start by proving the lemma for the case $N = 1$, i.e. we prove that: Suppose $f \in L^p(Y)$ and that the following holds for every disjoint segments I_1 and I_2 of Y : $|I_2| \int_{I_1} f\,dx = |I_1| \int_{I_2} f\,dx$. Then f is constant a.e. We state that

$$\int_V f\,dx = \int_V k dx \tag{3.1}$$

for every open or closed set $V \subseteq Y$, where $k = |Y|^{-1} \int_Y f\, dx$. This can be seen by observing that (3.1) holds for every segment, hence also for open sets since they are unions of countable collections of disjoint segments. Finally (3.1) holds for closed sets since they are complements of open sets.

Fix $\epsilon > 0$. Since Y is bounded, $f \in L^1(Y)$ and there is a continuous function g such that $\|f - g\|_1 < \epsilon/2$. Let V be the open set $\{x \in Y : g(x) > k\}$. By (3.1) it yields that

$$\int_Y |g - k|\, dx = \left| \int_V g - f\, dx \right| + \left| \int_{Y \setminus V} g - f\, dx \right| \leqslant \int_Y |g - f|\, dx < \frac{\epsilon}{2}.$$

The triangle inequality gives

$$\int_Y |f - k|\, dx \leqslant \int_Y |f - g|\, dx + \int_Y |g - k|\, dx < \epsilon,$$

and since ϵ was arbitrarily chosen, $\|f - k\|_1 = 0$ which, in its turn, implies that $f = k$ a.e.

The proof of the general case is similar. In fact, we get that (3.1) holds for $f \in L^p(Y)$ if we replace "k" by "$|S|^{-1} \int_S f(\cdot, t) d\mu$" and "$V$" first by a subset of the type "U_1" and next by a closed or open subset of Y. The rest of the proof follows exactly as above, i.e. we get that $f = k$ a.e., and, since k is independent of the i'th coordinate, this completes the proof.

Proof of Theorem 1. It is easily seen by inspection that 3 implies that $q_i = q^i$. Hence 3 ⇒ 1 and 3 ⇒ 2.

Proof of 1 ⇒ 3: Let u denote the solution of (2.1) for $\xi = e_i$. Since u is Y-periodic we have that $\int_S D_i u(\cdot, t) d\mu = 0$. Accordingly,

$$|S| \leqslant \left| \int_S (e_i + Du)(\cdot, t) d\mu \right| \leqslant \int_S |(e_i + Du)|\, (\cdot, t) d\mu. \tag{3.2}$$

Moreover, by the Hölder inequality,

$$\int_S \lambda^{-\frac{1}{p}} \lambda^{\frac{1}{p}} (|e_i + Du|)(\cdot, t) d\mu \leqslant$$

$$\leqslant \left(\int_S \lambda(\cdot, t)^{\frac{1}{1-p}} d\mu \right)^{\frac{p-1}{p}} \left(\int_S \lambda (|e_i + Du|^p)(\cdot, t) d\mu \right)^{\frac{1}{p}} \tag{3.3}$$

Hence,

$$|S|\, \lambda_i \leqslant \int_S \lambda (|e_i + Du|^p)(\cdot, t) d\mu$$

(for almost all $t \in T$). Now, by integrating with respect to the Lebesgue measure in R^N, by using Fubini's theorem, (2.3) and the assumption that $(b(e_i), e_i) = q_i$ this implies that we have equality in (3.2) and (3.3) for almost all $t \in T$. Equality in (3.2) implies that $Du_j = 0$ for $j \neq i$, that is, $|e_i + Du|$ is only dependent of the i'th coordinate. Hence, equality in (3.3) implies that λ is a product $k_{i1}k_{i2}$ a.e., where k_{i1} is dependent of only the i'th coordinate and k_{i2} is independent of the i'th coordinate (cf. [17, p. 65]).

Proof of $2 \Rightarrow 3$: Let $(b(e_i), e_i) = q^i$ and v be the solution of (2.1) for $\xi = e_i$ with λ replaced by λ^i. We observe that v is only dependent of the i'th coordinate. This fact can be seen by extending v to a Y-periodic function in R^N. Due to the independence of λ^i in the j-direction $(j \neq i)$ any translation of v in this direction restricted to Y will be a solution of the cell problem. But this solution is unique up to an additive constant, so v have to be independent of the j'th coordinate. We also have that v is a minimum point of (2.2) with λ replaced by λ^i. According to the independence of λ^i in the j-direction $(j \neq i)$ (2.1) yields that the minimum value $f_{\lambda^i}(e_i)$ of (2.2) is equal to q^i . Moreover, it is easily seen that

$$f_{\lambda^i}(e_i) = |Y|^{-1} \int_Y \lambda^i \, |\xi + Dv|^p \, dx = |Y|^{-1} \int_Y \lambda \, |\xi + Dv|^p \, dx.$$

Hence, since $q^i = (b(e_i), e_i) = f_\lambda(e_i)$ we get that both u and v are minimum points of (2.2), that is, $u = v$ by the uniqueness of the local problem. Let U_1 and U_2 be as in Lemma 1. We construct the function ϕ as follows. ϕ is defined equal to 1 on the set K between U_1 and U_2. On $Y \backslash (U_1 \cup U_2 \cup K)$, ϕ is defined to be equal to 0. On U_1 and U_2, respectively, ϕ is defined by linear interpolation between the function-values on the two opposite traces which are normal to e_i. This definition implies that we can find functions $\omega_h \in W^{1,p}_{per}(Y)$ such that $\partial\omega_h/\partial x_i \to \partial\phi/\partial x_i$ in $L^p(Y)$. Hence, according to (2.1) and the properties of u we have that

$$0 = \int_Y \lambda \frac{\partial \omega_h}{\partial x_i} \left| \frac{\partial u}{\partial x_i} + 1 \right|^{p-2} (\frac{\partial u}{\partial x_i} + 1) dx =$$

$$\int_Y \lambda \frac{\partial \phi}{\partial x_i} \left| \frac{\partial u}{\partial x_i} + 1 \right|^{p-2} \Big(\frac{\partial u}{\partial x_i} + 1\Big) dx = \int_{U_1 \cup U_2} \lambda \frac{\partial \phi}{\partial x_i} \left| \frac{\partial u}{\partial x_i} + 1 \right|^{p-2} \Big(\frac{\partial u}{\partial x_i} + 1\Big) dx.$$

Consequently,

$$|U_1| \int_{U_2} \lambda \left| \frac{\partial u}{\partial x_i} + 1 \right|^{p-2} \Big(\frac{\partial u}{\partial x_i} + 1\Big) dx = |U_2| \int_{U_1} \lambda \left| \frac{\partial u}{\partial x_i} + 1 \right|^{p-2} \Big(\frac{\partial u}{\partial x_i} + 1\Big) dx,$$

and, by Lemma 1,

$$\lambda \, |\partial u/\partial x_i + 1|^{p-2} \, (\partial u/\partial x_i + 1)$$

is thereby independent of the i'th coordinate a.e. Accordingly, if $\partial u/\partial x_i + 1 = 0$ for some point in Y, the same holds for

$$\lambda \, |\partial u/\partial x_i + 1|^{p-2} \, (\partial u/\partial x_i + 1)$$

on the line parallel to e_i containing this point. But since $\lambda > 0$ and u is Y-periodic, this can only occur on a set of measure zero, i.e. $\partial u/\partial x_i + 1 \neq 0$ a.e. and we conclude that λ is a product $k_{i1}k_{i2}$ a.e., where k_{i1} is dependent of only the i'th coordinate and k_{i2} is independent of the i'th coordinate. This completes the proof.

4. Applications

In this section we point out new consequences of (1.1) and Theorem 1. We start by recalling the Wiener inequalities associated with the homogenized p-Poisson equation:

$$P_{1/(1-p)}(\lambda) \leqslant (b(e_i), e_i) \leqslant P_1(\lambda),$$

where $P_k(\lambda)$ denotes the k*'th power mean* of λ, i.e.

$$P_k(\lambda) = \left(|Y|^{-1}\int_Y \lambda^k dx\right)^{\frac{1}{k}}.$$

For the proof, see e.g. [15]. We will now prove the following result which states necessary and sufficient conditions such that $(b(e_i), e_i) = P_{1/(1-p)}(\lambda)$ or $(b(e_i), e_i) = P_1(\lambda)$

THEOREM 2. *The following statements are equivalent:*
1. $(b(e_i), e_i) = P_{1/(1-p)}(\lambda)$
2. $(b(e_j), e_j) = P_1(\lambda)$ *for all* $j \neq i$
3. λ *is dependent of only the* i*'th coordinate.*

REMARK. Functions λ which is dependent of only the i'th coordinate are often referred to as rank 1 laminates. The above theorem gives that these functions are unique in the sense that no other functions can induce property 1 and 2. This is interesting since if

$$P_{1/(1-p)}(\lambda) < (b(e_i), e_i) < P_1(\lambda),$$

then an infinite number of types of functions will lead to the same effective energy density $(b(e_i), e_i)$, at least for the case when $p = 2$ (see e.g. [2, p. 91] and the references given there)

REMARK. Theorem 2 represents a generalization of a statement given in [8, p. 193] (without proof) for the special case $p = 2$.

Proof of Theorem 2. It is easily seen by inspection that 3 implies that $q_i = q^i = P_{1/(1-p)}(\lambda)$ and $q_j = q^j = P_1(\lambda)$ for all $j \neq i$. Hence (1.1) gives that $3 \Rightarrow 1$ and $3 \Rightarrow 2$.

Proof of $1 \Rightarrow 3$: Since $P_{1/(1-p)}(\lambda) \leqslant q_j$ and $q^j \leqslant P_1(\lambda)$, (1.1) gives that

$$P_{1/(1-p)}(\lambda) \leqslant q_j \leqslant (b(e_j), e_j) \leqslant q^j \leqslant P_1(\lambda)$$

for $j = 1, 2, \ldots, N$ Hence, assuming that 1 holds we obtain that $q_i = (b(e_i), e_i)$, and by Theorem 1 this gives that $\lambda = k_{i1}k_{i2}$, where k_{i1} is dependent of only the i'th coordinate and k_{i2} is independent of the i'th coordinate. Thus,

$$q_i = \left(|S|^{-1}\int_S (k_{i1})^{\frac{1}{1-p}} d\mu\right)^{1-p} |T|^{-1}\int_T k_{i2} d\eta$$

and

$$P_{(1-p)^{-1}}(\lambda) = \left(|S|^{-1}\int_S (k_{i1})^{\frac{1}{1-p}}d\mu\right)^{1-p}\left(|T|^{-1}\int_T (k_{i2})^{\frac{1}{1-p}}d\eta\right)^{1-p}.$$

Therefore, since

$$q_i = P_{1/(1-p)}(\lambda)$$

we find that

$$\left(|T|^{-1}\int_T (k_{i2})^{\frac{1}{1-p}}d\eta\right)^{1-p} = |T|^{-1}\int_T k_{i2}d\eta.$$

Thus $k_{i2} = \text{const}$ a.e. [using the fact that $P_k(f) < P_t(f)$ for $k < t$ unless $f = \text{const}$, see e.g. [3]] and we conclude that λ is dependent of only the i'th coordinate.

Proof of $2 \Rightarrow 3$: We argue similarly as above and obtain that $\lambda = k_{j1}k_{j2}$ where $k_{j1} = \text{const}$ a.e. and k_{j2} is independent of the j'th coordinate for $j = 1, 2, \ldots i-1, i+1, \ldots, N$; in other words λ is dependent of only the i'th coordinate. This completes the proof.

5. Final discussion and concluding remarks

The p-Poisson equation serves as a model for many physical problems. Depending of the type of problem considered the function λ can be the electric conductivity, the thermal conductivity or the magnetic permeability.

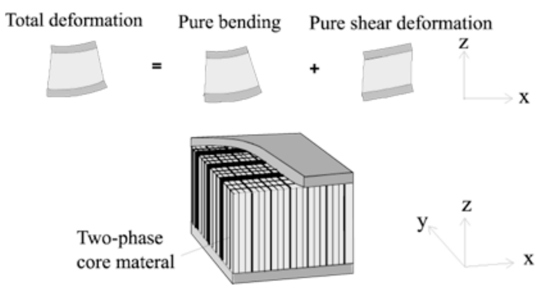

Figure 5.1. Illustration of contributions to the bending of a sandwich beam

Moreover, for 3 dimensional problems where λ and u are only dependent of two coordinates, say x, y, it is also possible to interpret λ as the *shear module* and u as the *displacement* of an elastic body. This fact seems to be useful e.g. in connection with the determination of the effective behavior of core materials in sandwich beams when the core is an unidirectional mixture of several *power law materials*. In such constructions the stiffness is provided partly by membrane action in the thin facings but mostly by the transverse shear strain resistance in the core material (see Figure 5.1). The relation between the effective transverse shear strain in the core γ_{xz} and the effective transverse shear stress τ_{xz} is as follows:

$$\tau_{xz} = b(e_x, e_x)\,|\gamma_{xz}|^{p-2}\,\gamma_{xz}.$$

Thus, as the transverse shear stress often is known we can use (1.1) to give estimates on the transverse shear strain.

We note also that for $p = 2$ the p-Poisson equation reduces to the linear heat conduction equation.

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