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# A STRENGTHENED CAUCHY—SCHWARZ INEQUALITY FOR BIORTHOGONAL WAVELETS

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*Abstract.* A strengthened Cauchy–Schwarz inequality for spaces of biorthogonal wavelets defined on the real line and on the interval is proved. The strengthened Cauchy–Schwarz inequality is a fundamental tool in the analysis of the multilevel methods and, in particular, plays an important role in the a posteriori error estimates for hierarchical methods.

## 1. Introduction

The fundamental tool in the analysis of multilevel finite elements methods is the strengthened Cauchy-Schwarz inequality. The usual Cauchy-Schwarz inequality

$$|(v,w)| \leqslant \sqrt{(v,v)}\sqrt{(w,w)}$$

is refined by the strengthened one in the sense that it states the existence of a constant $\gamma \in [0,1)$ such that

$$|(v,w)| \leqslant \gamma\sqrt{(v,v)}\sqrt{(w,w)};$$

for $v \in V$, $w \in W$, where $V$, $W$ are linear spaces with $V \cap W = \{0\}$, and $\gamma$ depends only on the spaces $V$ and $W$, and not on the choice of the functions $v$ and $w$.

Such inequalities have been widely used in the analysis of hierarchical finite elements methods (see for example [Y], [BDY], [EV]). It is possible to observe that the existence of such inequalities is a natural consequence of the construction of hierarchical basis functions. On the contrary, they have been rarely used in the context of the a posteriori error estimates, where they instead play an important role. Moreover, it has been proved in [De2] that the strengthened Cauchy–Schwarz inequality for biorthogonal wavelets is a necessary hypothesis to obtain a posteriori error estimates for the wavelet–based adaptive finite elements method ([CC], [De1]). We recall that a certain number of forms and proofs of the strengthened Cauchy–Schwarz inequality for finite element spaces, corresponding to different needs, already exist ([Y], [EV], [MT]). We point out that a new approach to the multiscale theory, in which the strengthened Cauchy–Schwarz inequality becomes fundamental, has been proposed in [Dah].

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In this paper a strengthened Cauchy–Schwarz inequality is proved for spaces of biorthogonal wavelets defined on the real line and, afterwards, on the interval. At first, the most important results on wavelet analysis are recalled. The hypotheses made are general and well known to the wavelet specialists.

## 2. Biorthogonal wavelets

We introduce ([CDF], [Dau]) a biorthogonal system of compactly supported wavalets by assigning two functions $m_0$, $\tilde{m}_0$ satisfying the following conditions:

M1. $m_0$, $\tilde{m}_0$ are two $2\pi$–periodic functions. The Fourier expansions of $m_0$ and $\tilde{m}_0$ are of the type

$$m_0(\xi) = \frac{1}{\sqrt{2}} \sum_{n=-\infty}^{+\infty} h_n e^{-in\xi}, \quad \tilde{m}_0(\xi) = \frac{1}{\sqrt{2}} \sum_{n=-\infty}^{+\infty} \tilde{h}_n e^{-in\xi}.$$

M2. $m_0$ and $\tilde{m}_0$ satisfy the identity

$$m_0(\xi)\overline{\tilde{m}_0(\xi)} + m_0(\xi+\pi)\overline{\tilde{m}_0(\xi+\pi)} = 1, \quad \forall \xi \in \mathbf{R}$$

and

$$m_0(0) = \tilde{m}_0(0) = 1, \quad m_0(\pi) = \tilde{m}_0(\pi) = 0.$$

M3. $m_0$ and $\tilde{m}_0$ vanish at $\pi$ with a zero of order $L-1$ and $\tilde{L}-1$ ($\leqslant k$), respectively. In particular such polynomials can be factorized as

$$m_0(\xi) = \left(\frac{1+e^{-i\xi}}{2}\right)^{L} \mathscr{F}(\xi), \quad \tilde{m}_0(\xi) = \left(\frac{1+e^{-i\xi}}{2}\right)^{\tilde{L}} \tilde{\mathscr{F}}(\xi),$$

where $\mathscr{F}$ and $\tilde{\mathscr{F}}$ are $2\pi$–periodic functions.

M4. There exist two integers $p$, $\tilde{p} > 0$ such that, if we set

$$\max_{\xi} |\mathscr{F}(\xi) \ldots \mathscr{F}(2^{p-1}\xi)| = 2^{p\tau},$$

$$\max_{\xi} |\tilde{\mathscr{F}}(\xi) \ldots \tilde{\mathscr{F}}(2^{p-1}\xi)| = 2^{\tilde{p}\tilde{\tau}},$$

then we have $\sigma = L - \frac{1}{2} - \tau > 0$, $\tilde{\sigma} = \tilde{L} - \frac{1}{2} - \tilde{\tau} > 0$, and $\tau$, $\tilde{\tau} \geqslant 0$.

The condition M2 can be translated into a property of the coefficients $h_n$ and $\tilde{h}_n$ (*filters*). In fact we have

$$\sum_n h_n \overline{\tilde{h}}_{n-2k} = \delta_{k0}, \quad \forall k \in \mathbf{Z},$$

$$\sum_n h_n = \sum_n \tilde{h}_n = \sqrt{2},$$

$$\sum_n (-1)^n h_n = \sum_n (-1)^n \tilde{h}_n = 0.$$

The orthogonal case follows when $m_0$ and $\tilde{m}_0$ coincide; it can be proved that in this case $m_0$ generates an orthogonal system.

Now we define the functions

$$\hat{\varphi}(\xi) = \frac{1}{\sqrt{2}} \prod_{j=1}^{\infty} m_0(2^{-j}\xi), \quad \hat{\tilde{\varphi}}(\xi) = \frac{1}{\sqrt{2}} \prod_{j=1}^{\infty} \tilde{m}_0(2^{-j}\xi).$$

From the definition we have the following important relations

$$\hat{\varphi}(2\xi) = \hat{\varphi}(\xi) m_0(\xi), \qquad \hat{\tilde{\varphi}}(2\xi) = \hat{\tilde{\varphi}}(\xi) \tilde{m}_0(\xi), \tag{2.1}$$

and

$$\hat{\varphi}(0) = \frac{1}{\sqrt{2}}, \qquad \hat{\tilde{\varphi}}(0) = \frac{1}{\sqrt{2}}.$$

The anti–transforms $\varphi(x)$ and $\tilde{\varphi}(x)$ are the *scaling functions* in the biorthogonal decomposition of $L^2(\mathbf{R})$. By (2.1) we obtain the *refinement equations*

$$\varphi(x) = \sqrt{2} \sum_n h_n \varphi(2x-n), \quad \tilde{\varphi}(x) = \sqrt{2} \sum_n \tilde{h}_n \tilde{\varphi}(2x-n), \tag{2.2}$$

and the normalization conditions

$$\int_{\mathbf{R}} \varphi(x) dx = 1, \quad \int_{\mathbf{R}} \tilde{\varphi}(x) dx = 1. \tag{2.3}$$

Moreover $\varphi$, $\tilde{\varphi}$ verify the biorthogonality relation

$$(\varphi, \tilde{\varphi}(\cdot - k)) = \delta_{0k}.$$

Next we define the functions

$$\varphi_{0k}(x) = \varphi(x-k), \quad k \in \mathbf{Z},$$

and we set

$$V_0 = \mathrm{span}_{L^2(\mathbf{R})}\{\varphi_{0k} : k \in \mathbf{Z}\}.$$

In order to define the spaces $V_j$ and $\tilde{V}_j$, for $j \neq 0$, we introduce the isometries in $L^2(\mathbf{R})$

$$T_j(v)(x) = 2^{\frac{j}{2}} v(2^j x)$$

and similarly for $\tilde{T}_j$. Then

$$V_j = \{T_j v : v \in V_0\} = \mathrm{span}_{L^2(\mathbf{R})}\{\varphi_{jk} : k \in \mathbf{Z}\}, \tag{2.4}$$

and

$$\tilde{V}_j = \{\tilde{T}_j v : v \in \tilde{V}_0\} = \mathrm{span}_{L^2(\mathbf{R})}\{\tilde{\varphi}_{jk} : k \in \mathbf{Z}\}, \tag{2.5}$$

where, for $j,k \in \mathbf{Z}$,

$$\varphi_{jk}(x) = T_j \varphi_{0k}(x) = 2^{\frac{j}{2}} \varphi(2^j x - k),$$

and

$$\tilde{\varphi}_{jk}(x) = \tilde{T}_j \tilde{\varphi}_{0k}(x) = 2^{\frac{j}{2}} \tilde{\varphi}(2^j x - k).$$

It is possible to prove that, for every $j \in \mathbf{Z}$, we have

$$f(x) \in V_j \Longleftrightarrow f(x - 2^{-j}k) \in V_j, \quad \forall k \in \mathbf{Z},$$
$$f(x) \in V_j \Longleftrightarrow f(2^{-j}x) \in V_0,$$
$$V_j \subset V_{j+1},$$

and moreover

$$\bigcap_{j \in \mathbf{Z}} V_j = \{0\}, \quad \overline{\bigcup_{j \in \mathbf{Z}} V_j} = L^2(\mathbf{R});$$

the same results naturally hold for the $\tilde{V}_j$.

Now, we consider the function (*wavelet mother*)

$$\psi(x) = \sqrt{2} \sum_n g_n \varphi(2x - n),$$

where

$$g_n = (-1)^n \tilde{h}_{1-n}$$

and set

$$\psi_{jk}(x) = 2^{\frac{j}{2}} \psi(2^j x - k). \tag{2.6}$$

We define

$$W_j = \mathrm{span}_{L^2(\mathbf{R})} \{\psi_{jk} : k \in \mathbf{Z}\} \tag{2.7}$$

and

$$\tilde{W}_j = \mathrm{span}_{L^2(\mathbf{R})} \{\tilde{\psi}_{jk} : k \in \mathbf{Z}\}, \tag{2.8}$$

where the $\tilde{\psi}_{jk}$ are defined as the $\psi_{jk}$. Thus we have the orthogonal decompositions

$$V_{j+1} = V_j \oplus W_j, \quad \tilde{V}_{j+1} = \tilde{V}_j \oplus \tilde{W}_j,$$

with

$$W_j \perp \tilde{V}_j, \quad \tilde{W}_j \perp V_j.$$

The Fourier transform of $\psi$ is

$$\hat{\psi}(\xi) = -e^{-i\frac{\xi}{2}} \overline{\tilde{m}_0(\frac{\xi}{2} + \pi)} \hat{\varphi}(\frac{\xi}{2}). \tag{2.9}$$

The wavelets satisfy the relation

$$\left(\psi_{jk}, \tilde{\psi}_{j'l}\right) = \delta_{jj'} \delta_{kl}.$$

The wavelet representation of a generic function $f \in L^2(\mathbf{R})$ is

$$f(x) = \sum_{j,k} (f, \tilde{\psi}_{jk}) \, \psi_{jk}(x),$$

or

$$f(x) = \sum_k (f, \tilde{\varphi}_{j_0,k}) \varphi_{j_0,k}(x) + \sum_{j \geqslant j_0, k} (f, \tilde{\psi}_{jk}) \, \psi_{jk}(x).$$

The following relation is also used

$$\varphi_{j+1,k} = \sum_m (\varphi_{j+1,k}, \tilde{\varphi}_{jm})\varphi_{jm} + \sum_m (\varphi_{j+1,k}, \tilde{\psi}_{jm})\psi_{jm} \tag{2.10}$$

and the analogue for $\tilde{\varphi}_{j+1,k}$.

Now we enunciate some results that will be used in the sequel ([CDF]). They establish regularity, exponential decay to infinity and an inequality verified by $\hat{\varphi}$, $\hat{\tilde{\varphi}}$, respectively.

PROPOSITION 2.1. *The functions $\hat{\varphi}$, $\hat{\tilde{\varphi}}$ belong to the space $\mathscr{C}^k(\mathbf{R})$, where $k$ is the order of regularity of $m_0$ and $\tilde{m}_0$.*

PROPOSITION 2.2. *Let $\sigma$, $\tilde{\sigma}$ be as in M4–hipothesis. There exists a constant $C > 0$ such that $\forall \xi \in \mathbf{R}$*

$$|\hat{\varphi}(\xi)| \leqslant C(1+|\xi|)^{-\frac{1}{2}-\sigma},$$
$$|\hat{\tilde{\varphi}}(\xi)| \leqslant C(1+|\xi|)^{-\frac{1}{2}-\tilde{\sigma}}.$$

PROPOSITION 2.3. *There exist two constants $C_1$, $C_2 > 0$ such that*

$$C_1 \leqslant \sum_m |\hat{\varphi}(\xi + 2m\pi)|^2 \leqslant C_2, \quad \forall \xi \in \mathbf{R},$$
$$C_1 \leqslant \sum_m |\hat{\tilde{\varphi}}(\xi + 2m\pi)|^2 \leqslant C_2, \quad \forall \xi \in \mathbf{R}.$$

The following theorem, which covers the case when $m_0$ and $\tilde{m}_0$ are trigonometric polynomials, is very important. (A first consequence of this hypothesis is that $\varphi$, $\psi$, $\tilde{\varphi}$, $\tilde{\psi}$ have compact support.)

THEOREM 2.4. *The following conditions are equivalent:*
$(i)$

$$\frac{d^l m_0}{d\xi}(\pi) = 0, \quad 0 \leqslant l \leqslant L-1;$$

$(ii)$ $\{\varphi(x-k)\}_{k\in\mathbf{Z}}$ *generate on* $\mathbf{R}$ *the algebraic polynomials of degree* $\leqslant L-1$;
$(iii)$ *we have*

$$\int_{\mathbf{R}} x^l \tilde{\psi}(x)dx = 0, \quad 0 \leqslant l \leqslant L-1. \tag{2.11}$$

Note that the condition $(i)$ is equal to the M3–hypothesis on $m_0$.

Another result, which will be fundamental to prove the strengthened Cauchy–Schwarz inequality for biorthogonal wavelets defined on the interval, is the following one.

PROPOSITION 2.5. *For every $j$ we have*

$$\|\{\alpha_k\}\|_{l^2} \asymp \|\sum_k \alpha_k \varphi_{jk}\|_{L^2(\mathbf{R})}$$

*and thus*

$$V_j = \left\{ \sum_k \alpha_k \varphi_{jk} : \{\alpha_k\} \in l^2 \right\}.$$

Now we consider the wavelets defined on an interval. There are a lot of ways to construct biorthogonal wavelet bases on an interval ([CDV], [DKU]). In this paper the construction presented in [AHJP] is considered. This includes general situations like biorthogonal wavelets and intervals of the real line. Let us consider two scaling functions satisfying the refinement relations

$$\varphi(x) = \sqrt{2} \sum_{k=0}^{2L-1} h_k \varphi(2x-k), \quad \tilde{\varphi}(x) = \sqrt{2} \sum_{k=0}^{2\tilde{L}-1} \tilde{h}_k \tilde{\varphi}(2x-k),$$

where $2L$ and $2\tilde{L}$ are coefficients different to zero. This implies that

$$\operatorname{supp} \varphi = [0, 2L-1], \quad \operatorname{supp} \tilde{\varphi} = [0, 2\tilde{L}-1].$$

At first, we construct $V_j[0,1]$ and $\tilde{V}_j[0,1]$. To make this we consider

$$S_j = \{k : \operatorname{supp} \varphi_{jk} \cap (0,1) \neq \emptyset\} = \{k : (-2L-2) \leqslant k \leqslant 2^j - 1\}.$$

Let $\delta_L$, $\delta_R$, $\tilde{\delta}_L$, $\tilde{\delta}_R$ be fixed non negative integers and define

$$\begin{aligned} S_{j,L} &= \{k : -(2L-2) \leqslant k \leqslant \delta_L - 1\}, \\ S_{j,R} &= \{k : 2^j - (2L-2) - \delta_R \leqslant k \leqslant 2^j - 1\}, \\ S_{j,L} &= \{k : \delta_L \leqslant k \leqslant 2^j - (2L-1) - \delta_R\}. \end{aligned}$$

These three subsets of $S_j$ contain the indices of the basis functions at the left extreme, at the interior and at the right extreme. We assume the parameter $j$ is sufficiently large to guarantee that the sets $S_{j,L}$ and $S_{j,R}$ are disjoint. Thus we have

$$S_j = S_{j,L} \cup S_{j,I} \cup S_{j,R}.$$

The sets $\tilde{S}_j$, $\tilde{S}_{j,L}$, $\tilde{S}_{j,I}$, $\tilde{S}_{j,R}$ are defined in the same way. The integers $\delta_L$, $\delta_R$, $\tilde{\delta}_L$, $\tilde{\delta}_R$ are important since with an appropriate choice it is possible to have a symmetric construction.

By Theorem 2.4 we know that all polynomials $P_{L-1}$ of degree $\leqslant L-1$ can be obtained as linear combinations of the functions $\{\varphi_{jk}\}_{k\in\mathbf{Z}}$. Because this property is strictly related to the approximation property of the wavelets, every construction on the interval must preserve it. This observation is the starting point of the procedure. Every monomial $P_j^\alpha(x) = 2^{\frac{j}{2}}(2^j x)^\alpha$, $\alpha \leqslant L-1$, admits the representation

$$P_j^\alpha(x) = \sum_k (P^\alpha, \tilde{\varphi}_{jk})\, \varphi_{jk}(x).$$

The restriction to the interval $[0,1]$ can be written in the form

$$P_j^{\alpha}(x)|_{[0,1]} = \left( \sum_{k\in S_{j,L}} + \sum_{k\in S_{j,I}} + \sum_{k\in S_{j,R}} \right) \left(P_j^{\alpha}, \tilde{\varphi}_{jk}\right) \varphi_{jk}(x)|_{[0,1]}.$$

Setting

$$\varphi^{\sharp}_{j\alpha,L}(x) = \sum_{k\in S_{j,L}} \left(P_j^{\alpha}, \tilde{\varphi}_{jk}\right) \varphi_{jk}(x)|_{[0,1]},$$
$$\varphi^{\sharp}_{j\alpha,R}(x) = \sum_{k\in S_{j,R}} \left(P_j^{\alpha}, \tilde{\varphi}_{jk}\right) \varphi_{jk}(x)|_{[0,1]},$$

we have

$$P_j^{\alpha}(x)|_{[0,1]} = \varphi^{\sharp}_{j\alpha,L}(x) + \sum_{k\in S_{j,I}} \left(P_j^{\alpha}, \tilde{\varphi}_{jk}\right) \varphi_{jk}(x)|_{[0,1]} + \varphi^{\sharp}_{j\alpha,R}(x).$$

Now we define the space

$$V_j[0,1] = \overline{\{\varphi^{\sharp}_{j\alpha,L}\}_{\alpha\leqslant L-1} \cup \{\varphi_{jk}\}_{k\in S_{j,I}} \cup \{\varphi^{\sharp}_{j\alpha,R}\}_{\alpha\leqslant L-1}}.$$

In the same way we compute the functions $\{\tilde{\varphi}^{\sharp}_{j\alpha,L}\}_{\alpha\leqslant\tilde{L}-1}$, $\{\tilde{\varphi}^{\sharp}_{j\alpha,R}\}_{\alpha\leqslant\tilde{L}-1}$, and we define the spaces $\tilde{V}_j[0,1]$. Now we impose the biorthogonal condition and determine functions, linear combinations of the last defined ones, which satisfy it. It is easy to verify that the spaces defined in this way form an increasing succession.

To have the corresponding wavelets, let $W_j[0,1]$ be the orthogonal complement of $V_j[0,1]$ in $V_{j+1}[0,1]$. The wavelets $\psi_{jk}$, with $k$ such that $L-1 \leqslant k \leqslant 2^j - L$, are in $V_{j+1}[0,1]$. The remaining $2L-2$ wavelets can be found using (2.10). In the same way we define the spaces $\tilde{W}_j[0,1]$ and the wavelets $\tilde{\psi}_{jk}$ belonging to such spaces. Again, it is necessary to impose the biorthogonality condition. Now we have a biorthogonal wavelet basis on the interval $[0,1]$. It is clear that the construction might be generalized to the case of a generic interval $(a,b)$ of the real line.

## 3. Strengthened Cauchy–Schwarz inequality for biorthogonal wavelets on the real line

Since we are interested in derivable wavelets, we consider the case of wavelets satisfying the M4–hypothesis with $\sigma > 1$ (for the spline biorthogonal wavelets this is true if $L \geqslant 2$, because $\tau = 0$).

LEMMA 3.1. *Let* $V_j$, $\tilde{V}_j$ *be defined as in* (2.5), (2.6) *and* $W_j$, $\tilde{W}_j$ *biorthogonal wavelet spaces on* $\mathbf{R}$ *be defined by* (2.8), (2.9). *If we consider the function*

$$\zeta(\xi) = \frac{S(\xi)S(\xi+2\pi)}{A(\xi)S^2(\xi) + B(\xi)S^2(\xi+2\pi) + C(\xi)S(\xi)S(\xi+2\pi)} \tag{3.1}$$

*where*

$$S(\xi) = \sum_l (\xi + 4l\pi)^2 \left|\hat{\varphi}\left(\frac{\xi}{2} + 2l\pi\right)\right|^2,$$

$$A(\xi) = \left|m_0\left(\frac{\xi}{2}\right)\tilde{m}_0\left(\frac{\xi}{2}+\pi\right)\right|^2,$$

$$B(\xi) = \left|m_0\left(\frac{\xi}{2}+\pi\right)\tilde{m}_0\left(\frac{\xi}{2}\right)\right|^2,$$

$$C(\xi) = \left|m_0\left(\frac{\xi}{2}\right)\tilde{m}_0\left(\frac{\xi}{2}\right)\right|^2 + \left|m_0\left(\frac{\xi}{2}+\pi\right)\tilde{m}_0\left(\frac{\xi}{2}+\pi\right)\right|^2,$$

*then there exists* $\gamma < 1$ *such that*

$$|(f', g')| \leqslant \gamma \; \|f'\| \; \|g'\|, \quad \forall f \in V_j, \forall g \in W_j \tag{3.2}$$

*if and only if there exists* $\zeta^\star$ *such that*

$$\zeta(\xi) \geqslant \zeta^\star > 0, \quad \forall \xi \in [0, 2\pi).$$

*Proof.* From the characterization of the spaces $V_0$ and $W_0$

$$\begin{aligned} f \in V_0 \Longleftrightarrow f(x) &= \sum_k \alpha_k \varphi_{0k}(x) \quad \text{with} \quad \{\alpha_k\} \in l^2 \\ \Longleftrightarrow \hat{f}(\xi) &= \sum_k \alpha_k e^{-ik\xi} \hat{\varphi}(\xi) \\ &= F(\xi)\hat{\varphi}(\xi) \quad \text{with} \quad F \in L^2(0, 2\pi), \end{aligned}$$

$$\begin{aligned} g \in W_0 \Longleftrightarrow \hat{g}(\xi) &= e^{-i\frac{\xi}{2}} G(\xi) \tilde{m}_0\left(\frac{\xi}{2}+\pi\right)\hat{\varphi}\left(\frac{\xi}{2}\right) \\ &= G(\xi)\hat{\psi}(\xi), \quad \text{with} \quad G \in L^2(0, 2\pi); \end{aligned}$$

then we have

$$\begin{aligned} f(x) \in V_j &\Longleftrightarrow \hat{f}(\eta) = 2^{-j} F(2^{-j}\eta)\hat{\varphi}(2^{-j}\eta), \\ g(x) \in W_j &\Longleftrightarrow \hat{g}(\eta) = 2^{-j} G(2^{-j}\eta)\hat{\psi}(2^{-j}\eta), \end{aligned}$$

since

$$\begin{aligned} f(x) \in V_j &\Longleftrightarrow f(2^{-j}x) \in V_0 \\ &\Longleftrightarrow \left[f(2^{-j}x)\right]^\wedge(\eta) = F(\eta)\hat{\varphi}(\eta), \\ g(x) \in W_j &\Longleftrightarrow g(2^{-j}x) \in W_0 \\ &\Longleftrightarrow \left[g(2^{-j}x)\right]^\wedge(\eta) = G(\eta)\hat{\psi}(\eta). \end{aligned}$$

From the Parseval formula for the left–hand side of (3.2) we have

$$\begin{aligned}
|(f',g')| &= \left|\int_{\mathbf{R}} f'(x)\overline{g'(x)}dx\right| \\
&= \left|\int_{\mathbf{R}} [f'(x)]^{\wedge}(\eta)\overline{[g'(x)]^{\wedge}(\eta)}d\eta\right| \\
&= \left|\int_{\mathbf{R}} \eta\hat{f}(\eta)\ \overline{\eta\hat{g}(\eta)}d\eta\right| \\
&= 2^{-2j}\left|\int_{\mathbf{R}} \eta^2 F(2^{-j}\eta)\overline{G(2^{-j}\eta)}\hat{\varphi}(2^{-j}\eta)\overline{\hat{\psi}(2^{-j}\eta)}d\eta\right| \\
&= 2^{j}\left|\int_{\mathbf{R}} \xi^2 F(\xi)\overline{G(\xi)}\hat{\varphi}(\xi)\overline{\hat{\psi}(\xi)}d\xi\right|.
\end{aligned}$$

For the right–hand side, in the same way we find

$$\begin{aligned}
(f',f') &= \int_{\mathbf{R}} |f'(x)|^2 dx \\
&= \int_{\mathbf{R}} \eta^2|\hat{f}(\eta)|^2 d\eta \\
&= 2^{-2j}\int_{\mathbf{R}} \eta^2|F(2^{-j}\eta)|^2|\hat{\varphi}(2^{-j}\eta)|^2 d\eta \\
&= 2^{j}\int_{\mathbf{R}} \xi^2|F(\xi)|^2|\hat{\varphi}(\xi)|^2 d\xi,
\end{aligned}$$

and, equivalently,

$$(g',g') = 2^{j}\int_{\mathbf{R}} \xi^2|G(\xi)|^2|\hat{\psi}(\xi)|^2 d\xi.$$

For (3.2) to hold, there must exist a $\gamma < 1$ such that

$$\begin{aligned}
&\left|\int_{\mathbf{R}} \xi^2 F(\xi)\overline{G(\xi)}\hat{\varphi}(\xi)\overline{\hat{\psi}(\xi)}d\xi\right| \\
&\qquad \leqslant \gamma\left(\int_{\mathbf{R}} \xi^2|F(\xi)|^2|\hat{\varphi}(\xi)|^2 d\xi\right)^{\frac{1}{2}} \times \left(\int_{\mathbf{R}} \xi^2|G(\xi)|^2|\hat{\psi}(\xi)|^2 d\xi\right)^{\frac{1}{2}},
\end{aligned} \tag{3.3}$$

with $F, G \in L^2(0,2\pi)$.

Now, (2.1) and (2.9) imply

$$\begin{aligned}
\hat{\varphi}(\xi) &= m_0\left(\frac{\xi}{2}\right)\hat{\varphi}\left(\frac{\xi}{2}\right), \\
\hat{\psi}(\xi) &= -e^{-i\frac{\xi}{2}}\overline{\tilde{m}_0\left(\frac{\xi}{2}+\pi\right)}\hat{\varphi}\left(\frac{\xi}{2}\right).
\end{aligned}$$

Substituing them in (3.3), we have

$$\left| \int_{\mathbf{R}} \xi^2 \; e^{i\frac{\xi}{2}} F(\xi)\overline{G(\xi)} m_0\left(\frac{\xi}{2}\right) \tilde{m}_0\left(\frac{\xi}{2}+\pi\right) \left|\hat{\varphi}\left(\frac{\xi}{2}\right)\right|^2 d\xi \right|$$
$$\leqslant \left( \int_{\mathbf{R}} \xi^2 |F(\xi)|^2 \left|m_0\left(\frac{\xi}{2}\right)\right|^2 \left|\hat{\varphi}\left(\frac{\xi}{2}\right)\right|^2 d\xi \right)^{\frac{1}{2}} \times$$
$$\times \left( \int_{\mathbf{R}} \xi^2 |G(\xi)|^2 \left|\tilde{m}_0\left(\frac{\xi}{2}+\pi\right)\right|^2 \left|\hat{\varphi}\left(\frac{\xi}{2}\right)\right|^2 d\xi \right)^{\frac{1}{2}}$$

and remembering that $F$ and $G$ are $2\pi$–periodic functions, we find

$$\left| \int_0^{2\pi} F(\xi)\overline{G(\xi)} \sum_k (\xi+2k\pi)^2 e^{i(\frac{\xi}{2}+k\pi)} m_0\left(\frac{\xi}{2}+k\pi\right) \times \right.$$
$$\left. \times \tilde{m}_0\left(\frac{\xi}{2}+\pi+k\pi\right) \left|\hat{\varphi}\left(\frac{\xi}{2}+k\pi\right)\right|^2 d\xi \right|$$

$$\leqslant \gamma \left( \int_0^{2\pi} |F(\xi)|^2 \sum_k (\xi+2k\pi)^2 \left|m_0\left(\frac{\xi}{2}+k\pi\right)\right|^2 \left|\hat{\varphi}\left(\frac{\xi}{2}+k\pi\right)\right|^2 d\xi \right)^{\frac{1}{2}} \times$$
$$\times \left( \int_0^{2\pi} |G(\xi)|^2 \sum_k (\xi+2k\pi)^2 \left|\tilde{m}_0\left(\frac{\xi}{2}+\pi+k\pi\right)\right|^2 \left|\hat{\varphi}\left(\frac{\xi}{2}+k\pi\right)\right|^2 d\xi \right)^{\frac{1}{2}}. \tag{3.4}$$

Therefore, if the strengthened Cauchy–Schwarz inequality is proved, there exists $\gamma < 1$ such that

$$\left| \sum_k (\xi+2k\pi)^2 e^{i\left(\frac{\xi}{2}+k\pi\right)} m_0\left(\frac{\xi}{2}+k\pi\right) \tilde{m}_0\left(\frac{\xi}{2}+\pi+k\pi\right) \left|\hat{\varphi}\left(\frac{\xi}{2}+k\pi\right)\right|^2 \right|$$

$$\begin{aligned} &\leqslant \gamma \left( \sum_k (\xi+2k\pi)^2 \left|m_0\left(\frac{\xi}{2}+k\pi\right)\right|^2 \left|\hat{\varphi}\left(\frac{\xi}{2}+k\pi\right)\right|^2 \right)^{\frac{1}{2}} \times \\ &\times \left( \sum_k (\xi+2k\pi)^2 \left|\tilde{m}_0\left(\frac{\xi}{2}+\pi+k\pi\right)\right|^2 \left|\hat{\varphi}\left(\frac{\xi}{2}+k\pi\right)\right|^2 \right)^{\frac{1}{2}}. \end{aligned} \tag{3.5}$$

Now we consider odd $k$ and even $k$, and we divide the sum

$$\left| \sum_l (\xi+4l\pi)^2 e^{i\left(\frac{\xi}{2}+2l\pi\right)} m_0\left(\frac{\xi}{2}+2l\pi\right) \tilde{m}_0\left(\frac{\xi}{2}+\pi+2l\pi\right) \left|\hat{\varphi}\left(\frac{\xi}{2}+2l\pi\right)\right|^2 + \right.$$

$$+\sum_l (\xi+2\pi+4l\pi)^2 e^{i\left(\frac{\xi}{2}+\pi+2l\pi\right)} m_0\left(\frac{\xi}{2}+\pi+2l\pi\right)\times$$
$$\times\tilde{m}_0\left(\frac{\xi}{2}+2l\pi\right)\left|\hat{\varphi}\left(\frac{\xi}{2}+\pi+2l\pi\right)\right|^2\Bigg|$$

$$\leqslant \gamma\left(\sum_l (\xi+4l\pi)^2\left|m_0\left(\frac{\xi}{2}+2l\pi\right)\right|^2\left|\hat{\varphi}\left(\frac{\xi}{2}+2l\pi\right)\right|^2\right.$$
$$\left.+\sum_l (\xi+2\pi+4l\pi)^2\left|m_0\left(\frac{\xi}{2}+\pi+2l\pi\right)\right|^2\left|\hat{\varphi}\left(\frac{\xi}{2}+\pi+2l\pi\right)\right|^2\right)^{\frac{1}{2}}\times$$
$$\times\left(\sum_l (\xi+4l\pi)^2\left|\tilde{m}_0\left(\frac{\xi}{2}+\pi+2l\pi\right)\right|^2\left|\hat{\varphi}\left(\frac{\xi}{2}+2l\pi\right)\right|^2\right.$$
$$\left.+\sum_l (\xi+2\pi+4l\pi)^2\left|\tilde{m}_0\left(\frac{\xi}{2}+2l\pi\right)\right|^2\left|\hat{\varphi}\left(\frac{\xi}{2}+\pi+2l\pi\right)\right|^2\right)^{\frac{1}{2}}.$$

From periodicity of $m_0$ and $\tilde{m}_0$, we obtain

$$\Bigg|e^{i\frac{\xi}{2}} m_0\left(\frac{\xi}{2}\right)\tilde{m}_0\left(\frac{\xi}{2}+\pi\right)\sum_l (\xi+4l\pi)^2\left|\hat{\varphi}\left(\frac{\xi}{2}+2l\pi\right)\right|^2$$
$$-e^{i\frac{\xi}{2}} m_0\left(\frac{\xi}{2}+\pi\right)\tilde{m}_0\left(\frac{\xi}{2}\right)\sum_l (\xi+2\pi+4l\pi)^2\left|\hat{\varphi}\left(\frac{\xi}{2}+\pi+2l\pi\right)\right|^2\Bigg|$$
$$\leqslant \gamma\left(\left|m_0\left(\frac{\xi}{2}\right)\right|^2\sum_l (\xi+4l\pi)^2\left|\hat{\varphi}\left(\frac{\xi}{2}+2l\pi\right)\right|^2\right.$$
$$\left.+\left|m_0\left(\frac{\xi}{2}+\pi\right)\right|^2\sum_l (\xi+2\pi+4l\pi)^2\left|\hat{\varphi}\left(\frac{\xi}{2}+\pi+2l\pi\right)\right|^2\right)^{\frac{1}{2}}\times$$
$$\times\left(\left|\tilde{m}_0\left(\frac{\xi}{2}+\pi\right)\right|^2\sum_l (\xi+4l\pi)^2\left|\hat{\varphi}\left(\frac{\xi}{2}+2l\pi\right)\right|^2\right.$$
$$\left.+\left|\tilde{m}_0\left(\frac{\xi}{2}\right)\right|^2\sum_l (\xi+2\pi+4l\pi)^2\left|\hat{\varphi}\left(\frac{\xi}{2}+\pi+2l\pi\right)\right|^2\right)^{\frac{1}{2}}.$$

Setting

$$S(\xi)=\sum_l (\xi+4l\pi)^2\left|\hat{\varphi}\left(\frac{\xi}{2}+2l\pi\right)\right|^2,$$

we have

$$\left|m_0\left(\frac{\xi}{2}\right)\ \tilde{m}_0\left(\frac{\xi}{2}+\pi\right)S(\xi)-m_0\left(\frac{\xi}{2}+\pi\right)\tilde{m}_0\left(\frac{\xi}{2}\right)S(\xi+2\pi)\right|\leqslant$$

$$\leqslant \gamma \left( \left| m_0\left(\frac{\xi}{2}\right)\right|^2 S(\xi) + \left| m_0\left(\frac{\xi}{2}+\pi\right)\right|^2 S(\xi+2\pi)\right)^{\frac{1}{2}} \times$$
$$\times \left( \left| \tilde{m}_0\left(\frac{\xi}{2}+\pi\right)\right|^2 S(\xi) + \left| \tilde{m}_0\left(\frac{\xi}{2}\right)\right|^2 S(\xi+2\pi)\right)^{\frac{1}{2}}.$$

and then

$$\begin{aligned}
&\left| m_0\left(\frac{\xi}{2}\right)\ \tilde{m}_0\left(\frac{\xi}{2}+\pi\right) S(\xi) - m_0\left(\frac{\xi}{2}+\pi\right)\tilde{m}_0\left(\frac{\xi}{2}\right) S(\xi+2\pi)\right|^2 \\
&\leqslant \gamma^2 \left\{ \left| m_0\left(\frac{\xi}{2}\right)\right|^2 \left| \tilde{m}_0\left(\frac{\xi}{2}+\pi\right)\right|^2 S^2(\xi) \right. \\
&+ \left| m_0\left(\frac{\xi}{2}+\pi\right)\right|^2 \left| \tilde{m}_0\left(\frac{\xi}{2}\right)\right|^2 S^2(\xi+2\pi) \\
&\left. + \left[ \left| m_0\left(\frac{\xi}{2}\right)\right|^2 \left| \tilde{m}_0\left(\frac{\xi}{2}\right)\right|^2 + \left| m_0\left(\frac{\xi}{2}+\pi\right)\right|^2 \left| \tilde{m}_0\left(\frac{\xi}{2}+\pi\right)\right|^2 \right] S(\xi)S(\xi+2\pi) \right\}.
\end{aligned} \tag{3.6}$$

For the left–hand side inequality, we find

$$\begin{aligned}
&\left( m_0\left(\frac{\xi}{2}\right)\ \tilde{m}_0\left(\frac{\xi}{2}+\pi\right) S(\xi) - m_0\left(\frac{\xi}{2}+\pi\right)\tilde{m}_0\left(\frac{\xi}{2}\right) S(\xi+2\pi)\right) \times \\
&\times \left( \overline{m_0\left(\frac{\xi}{2}\right)\tilde{m}_0\left(\frac{\xi}{2}+\pi\right)} S(\xi) - \overline{m_0\left(\frac{\xi}{2}+\pi\right)\tilde{m}_0\left(\frac{\xi}{2}\right)} S(\xi+2\pi)\right) \\
&= \left| m_0\left(\frac{\xi}{2}\right)\tilde{m}_0\left(\frac{\xi}{2}+\pi\right)\right|^2 S^2(\xi) + \left| m_0\left(\frac{\xi}{2}+\pi\right)\tilde{m}_0\left(\frac{\xi}{2}\right)\right|^2 S^2(\xi+2\pi) \qquad (3.7) \\
&- \left\{ m_0\left(\frac{\xi}{2}\right)\overline{\tilde{m}_0\left(\frac{\xi}{2}\right)}\tilde{m}_0\left(\frac{\xi}{2}+\pi\right)\overline{m_0\left(\frac{\xi}{2}+\pi\right)} \right. \\
&\left. + \overline{m_0\left(\frac{\xi}{2}\right)}\tilde{m}_0\left(\frac{\xi}{2}\right) m_0\left(\frac{\xi}{2}+\pi\right)\overline{\tilde{m}_0\left(\frac{\xi}{2}+\pi\right)} \right\} S(\xi)S(\xi+2\pi).
\end{aligned}$$

Moreover, by the M2–hypothesis

$$m_0\left(\frac{\xi}{2}\right)\overline{\tilde{m}_0\left(\frac{\xi}{2}\right)} + m_0\left(\frac{\xi}{2}+\pi\right)\overline{\tilde{m}_0\left(\frac{\xi}{2}+\pi\right)} = 1,$$

we have

$$\begin{aligned}
&\left( m_0\left(\frac{\xi}{2}\right)\ \overline{\tilde{m}_0\left(\frac{\xi}{2}\right)} + m_0\left(\frac{\xi}{2}+\pi\right)\overline{\tilde{m}_0\left(\frac{\xi}{2}+\pi\right)} \right) \times \\
&\times \left( \overline{m_0\left(\frac{\xi}{2}\right)}\tilde{m}_0\left(\frac{\xi}{2}\right) + \overline{m_0\left(\frac{\xi}{2}+\pi\right)}\tilde{m}_0\left(\frac{\xi}{2}+\pi\right) \right)
\end{aligned}$$

$$= \left|m_0\left(\frac{\xi}{2}\right)\tilde{m}_0\left(\frac{\xi}{2}\right)\right|^2 + m_0\left(\frac{\xi}{2}\right)\overline{\tilde{m}_0\left(\frac{\xi}{2}\right)m_0\left(\frac{\xi}{2}+\pi\right)}\tilde{m}_0\left(\frac{\xi}{2}+\pi\right)$$
$$+\left|m_0\left(\frac{\xi}{2}+\pi\right)\tilde{m}_0\left(\frac{\xi}{2}+\pi\right)\right|^2 + \overline{m_0\left(\frac{\xi}{2}\right)}\tilde{m}_0\left(\frac{\xi}{2}\right)m_0\left(\frac{\xi}{2}+\pi\right)\overline{\tilde{m}_0\left(\frac{\xi}{2}+\pi\right)} = 1$$

and thus

$$\begin{aligned}&\left|m_0\left(\frac{\xi}{2}\right)\ \tilde{m}_0\left(\frac{\xi}{2}\right)\right| + \left|m_0\left(\frac{\xi}{2}+\pi\right)\tilde{m}_0\left(\frac{\xi}{2}+\pi\right)\right|^2 - 1\\ &= -\left[\overline{m_0\left(\frac{\xi}{2}\right)}\tilde{m}_0\left(\frac{\xi}{2}\right)m_0\left(\frac{\xi}{2}+\pi\right)\overline{\tilde{m}_0\left(\frac{\xi}{2}+\pi\right)}\right.\\ &\qquad\left. + m_0\left(\frac{\xi}{2}\right)\overline{\tilde{m}_0\left(\frac{\xi}{2}\right)m_0\left(\frac{\xi}{2}+\pi\right)}\tilde{m}_0\left(\frac{\xi}{2}+\pi\right)\right].\end{aligned}$$

Substituing it in (3.7)

$$\begin{aligned}&\left|m_0\left(\frac{\xi}{2}\right)\tilde{m}_0\left(\frac{\xi}{2}+\pi\right)S(\xi) - m_0\left(\frac{\xi}{2}+\pi\right)\tilde{m}_0\left(\frac{\xi}{2}\right)S(\xi+2\pi)\right|^2\\ &= \left|m_0\left(\frac{\xi}{2}\right)\ \tilde{m}_0\left(\frac{\xi}{2}+\pi\right)\right|^2 S^2(\xi) + + \left|m_0\left(\frac{\xi}{2}+\pi\right)\tilde{m}_0\left(\frac{\xi}{2}\right)\right|^2 S^2(\xi+2\pi)\\ &+\left[\left|m_0\left(\frac{\xi}{2}\right)\tilde{m}_0\left(\frac{\xi}{2}\right)\right|^2 + \left|m_0\left(\frac{\xi}{2}+\pi\right)\tilde{m}_0\left(\frac{\xi}{2}+\pi\right)\right|^2 - 1\right] S(\xi)S(\xi+2\pi),\end{aligned}$$

and finally in (3.6), we have

$$\begin{aligned}&\left|m_0\left(\frac{\xi}{2}\right)\ \tilde{m}_0\left(\frac{\xi}{2}+\pi\right)\right|^2 S^2(\xi) + + \left|m_0\left(\frac{\xi}{2}+\pi\right)\tilde{m}_0\left(\frac{\xi}{2}\right)\right|^2 S^2(\xi+2\pi)\\ &+\left[\left|m_0\left(\frac{\xi}{2}\right)\tilde{m}_0\left(\frac{\xi}{2}\right)\right|^2 + \left|m_0\left(\frac{\xi}{2}+\pi\right)\tilde{m}_0\left(\frac{\xi}{2}+\pi\right)\right|^2 - 1\right] S(\xi)S(\xi+2\pi)\\ &\leqslant \gamma^2\left\{\left|m_0\left(\frac{\xi}{2}\right)\tilde{m}_0\left(\frac{\xi}{2}+\pi\right)\right|^2 S^2(\xi) + \left|m_0\left(\frac{\xi}{2}+\pi\right)\tilde{m}_0\left(\frac{\xi}{2}\right)\right|^2 S^2(\xi+2\pi)\right.\\ &\left.+\left[\left|m_0\left(\frac{\xi}{2}\right)\tilde{m}_0\left(\frac{\xi}{2}\right)\right|^2 + \left|m_0\left(\frac{\xi}{2}+\pi\right)\tilde{m}_0\left(\frac{\xi}{2}+\pi\right)\right|^2\right] S(\xi)S(\xi+2\pi)\right\}.\end{aligned}$$

In conclusion, let

$$\gamma^2(\xi) = \frac{A(\xi)S^2(\xi) + B(\xi)S^2(\xi+2\pi) + [C(\xi)-1]S(\xi)S(\xi+2\pi)}{A(\xi)S^2(\xi) + B(\xi)S^2(\xi+2\pi) + C(\xi)S(\xi)S(\xi+2\pi)}$$

where

$$A(\xi) = \left| m_0\left(\frac{\xi}{2}\right)\tilde{m}_0\left(\frac{\xi}{2}+\pi\right)\right|^2,$$

$$B(\xi) = \left| m_0\left(\frac{\xi}{2}+\pi\right)\tilde{m}_0\left(\frac{\xi}{2}\right)\right|^2,$$

$$C(\xi) = \left| m_0\left(\frac{\xi}{2}\right)\tilde{m}_0\left(\frac{\xi}{2}\right)\right|^2 + \left| m_0\left(\frac{\xi}{2}+\pi\right)\tilde{m}_0\left(\frac{\xi}{2}+\pi\right)\right|^2.$$

If the strengthened Cauchy–Schwarz inequality is satisfied, there exists a $\gamma_\star$ such that

$$\gamma^2(\xi) \leqslant \gamma_\star^2 < 1, \quad \forall \xi \in [0,2\pi).$$

To prove that the reverse holds, it is sufficient to verify that, if there exists a point where (3.5) is not satisfied, then there are no functions $F$, $G \in L^2(0,2\pi)$ such that (3.4) holds with $\gamma < 1$.

Let $\xi^\star \in [0,2\pi)$ be such that (3.5) holds with $\gamma = 1$. This means that, if we denote by $\chi(\xi)$ the left–hand side of (3.5), and by $\Gamma(\xi)$ and $\Lambda(\xi)$ the factors in the right–hand side, then we have $\forall \varepsilon > 0$

$$(1-\varepsilon)\Gamma(\xi)\Lambda(\xi) \leqslant \chi(\xi) \leqslant (1+\varepsilon)\Gamma(\xi)\Lambda(\xi), \quad \forall \xi \in B = B(\xi^\star, \delta_\varepsilon).$$

Now let $F_\varepsilon$, $G_\varepsilon$ be two functions in $L^2(0,2\pi)$ such that supp $F_\varepsilon$, supp $G_\varepsilon \subset B$. Then

$$\begin{aligned}(1-\varepsilon)\int_B &\left|F_\varepsilon(\xi)\overline{G_\varepsilon(\xi)}\right|\Gamma(\xi)\Lambda(\xi)d\xi \\ &\leqslant \int_B \left|F_\varepsilon(\xi)\overline{G_\varepsilon(\xi)}\right|\chi(\xi)d\xi \\ &\leqslant (1+\varepsilon)\int_B \left|F_\varepsilon(\xi)\overline{G_\varepsilon(\xi)}\right|\Gamma(\xi)\Lambda(\xi)d\xi, \quad \forall \xi \in B.\end{aligned}$$

Set

$$F_\varepsilon(\xi) = H_\varepsilon(\xi)\Gamma(\xi),$$
$$G_\varepsilon(\xi) = H_\varepsilon(\xi)\Lambda(\xi),$$

where $H_\varepsilon \in L^2(0,2\pi)$, supp $H_\varepsilon \subset B$, then we have

$$\begin{aligned}(1-\varepsilon)\int_B &|H_\varepsilon(\xi)\Gamma(\xi)\Lambda(\xi)|^2 d\xi \\ &\leqslant \int_B \left|F_\varepsilon(\xi)\overline{G_\varepsilon(\xi)}\right|\chi(\xi)d\xi \\ &\leqslant (1+\varepsilon)\int_B |H_\varepsilon(\xi)\Gamma(\xi)\Lambda(\xi)|^2 d\xi,\end{aligned}$$

$$\begin{aligned}(1-\varepsilon)&\left(\int_B |H_\varepsilon(\xi)\Gamma(\xi)\Lambda(\xi)|^2 d\xi\right)^{\frac12}\left(\int_B |H_\varepsilon(\xi)\Gamma(\xi)\Lambda(\xi)|^2 d\xi\right)^{\frac12} \\ &\leqslant \int_B \left|F_\varepsilon(\xi)\overline{G_\varepsilon(\xi)}\right|\chi(\xi)d\xi \\ &\leqslant (1+\varepsilon)\left(\int_B |H_\varepsilon(\xi)\Gamma(\xi)\Lambda(\xi)|^2 d\xi\right)^{\frac12}\left(\int_B |H_\varepsilon(\xi)\Gamma(\xi)\Lambda(\xi)|^2 d\xi\right)^{\frac12}\end{aligned}$$

and

$$\begin{aligned}(1-\varepsilon)&\left(\int_B |F_\varepsilon(\xi)\Lambda(\xi)|^2\,d\xi\right)^{\frac12}\left(\int_B |G_\varepsilon(\xi)\Gamma(\xi)|^2\,d\xi\right)^{\frac12}\\ &\leqslant \int_B \left|F_\varepsilon(\xi)\overline{G_\varepsilon(\xi)}\right|\chi(\xi)d\xi\\ &\leqslant (1+\varepsilon)\left(\int_B |F_\varepsilon(\xi)\Lambda(\xi)|^2\,d\xi\right)^{\frac12}\left(\int_B |G_\varepsilon(\xi)\Gamma(\xi)|^2\,d\xi\right)^{\frac12}.\end{aligned}$$

Thus

$$\left|\int_B F_\varepsilon(\xi)\overline{G_\varepsilon(\xi)}\chi(\xi)d\xi\right| \leqslant \left(\int_B |F_\varepsilon(\xi)\Lambda(\xi)|^2\,d\xi\right)^{\frac12}\left(\int_B |G_\varepsilon(\xi)\Gamma(\xi)|^2\,d\xi\right)^{\frac12},$$

and

$$\left|\int_0^{2\pi} F_\varepsilon(\xi)\overline{G_\varepsilon(\xi)}\chi(\xi)d\xi\right| \leqslant \left(\int_0^{2\pi} |F_\varepsilon(\xi)\Lambda(\xi)|^2\,d\xi\right)^{\frac12}\left(\int_0^{2\pi} |G_\varepsilon(\xi)\Gamma(\xi)|^2\,d\xi\right)^{\frac12},$$

which implies the assertion. □

LEMMA 3.2. *The function*

$$S(\xi) = \sum_l (\xi+4l\pi)^2\left|\hat\varphi\left(\frac{\xi}{2}+2l\pi\right)\right|^2$$

*is continuous on the interval* $[0,4\pi]$.

*Proof.* For $\xi\in[0,4\pi]$ fixed, let $l_0\in\mathbf{Z}$ satisfy both conditions

$$|l_0|>1, \quad \left|\frac{\xi}{2}+2(l_0+1)\pi\right|\geqslant 1.$$

Then

$$\begin{aligned}S(\xi) &= S_1(\xi)+S_2(\xi)\\ &= \sum_{|l|\leqslant l_0}(\xi+4l\pi)^2\left|\hat\varphi\left(\frac{\xi}{2}+2l\pi\right)\right|^2+\sum_{|l|>l_0}(\xi+4l\pi)^2\left|\hat\varphi\left(\frac{\xi}{2}+2l\pi\right)\right|^2.\end{aligned}$$

The finite sum $S_1$ is continuous due to Proposition 2.1. Moreover, the exponential decay to infinity of $\hat{\varphi}$ in Proposition 2.2, implies

$$\begin{aligned}
S_2(\xi) &= \sum_{|l|>l_0} (\xi + 4l\pi)^2 \left|\hat{\varphi}\left(\frac{\xi}{2} + 2l\pi\right)\right|^2 \\
&\leqslant C \sum_{|l|>l_0} (\xi + 4l\pi)^2 \frac{1}{\left(1 + \left|\frac{\xi}{2} + 2l\pi\right|\right)^{1+2\sigma}} \\
&\leqslant C \sum_{|l|>l_0} \frac{(\xi + 4l\pi)^2}{\left|\frac{\xi}{2} + 2l\pi\right|^{1+2\sigma}} \\
&= K \sum_{|l|>l_0} \frac{1}{|\xi + 4l\pi|^{2\sigma-1}} \\
&\leqslant K \sum_{|l|>l_0} \frac{1}{|4l\pi|^{2\sigma-1}}
\end{aligned}$$

which is convergent because $\sigma > 1$. □

LEMMA 3.3. *There exists* $\zeta^\star$ *such that*

$$\zeta(\xi) \geqslant \zeta^\star > 0, \quad \forall \xi \in [0, 2\pi),$$

*where* $\zeta(\xi)$ *is defined by* (3.1).

*Proof.* Let be $\xi \in (0, 2\pi)$. In this case $A(\xi)$, $B(\xi) \geqslant 0$, and, by M2–hypothesis, $C(\xi) > 0$. Continuity of $A$, $B$, $C$ implies that these functions have maximum in $[0, 2\pi]$. Moreover, if $S(\xi) = 0$, with $\xi \in (0, 4\pi)$, this would mean $\hat{\varphi}(\xi + 2l\pi) = 0$ for every $l \in \mathbf{Z}$, and then we have

$$\sum_l \hat{\varphi}(\xi + 2l\pi) = 0$$

contrary to Proposition 2.3. Note that for $\xi = 0$, $\zeta(\xi)$ is an indeterminate form, since

$$A(0) = B(0) = S(0) = S(2\pi) = 0, \quad C(0) = 1.$$

In this case we have (by M3–hypothesis about $m_0$, $\tilde{m}_0$), for $\xi \to 0$,

$$\begin{aligned}
A(\xi) &\sim \xi^{2\tilde{L}}, \\
B(\xi) &\sim \xi^{2L}, \\
C(\xi) &\sim \xi^2.
\end{aligned}$$

With regard to $S(\xi + 2\pi)$, in a sufficiently small neighbourhood of $0$, we have (using

(2.1) and dividing the sum)

$$\begin{aligned} S(\xi+2\pi) &= \sum_l (\xi+2\pi+4l\pi)^2 \left|\hat{\varphi}\left(\frac{\xi}{2}+\pi+2l\pi\right)\right|^2 \\ &= \sum_l (\xi+2\pi+4l\pi)^2 \left|m_0\left(\frac{\xi}{4}+\frac{\pi}{2}+l\pi\right)\right|^2 \left|\hat{\varphi}\left(\frac{\xi}{4}+\frac{\pi}{2}+l\pi\right)\right|^2 \\ &= \sum_k (\xi+2\pi+8k\pi)^2 \left|m_0\left(\frac{\xi}{4}+\frac{\pi}{2}+2k\pi\right)\right|^2 \left|\hat{\varphi}\left(\frac{\xi}{4}+\frac{\pi}{2}+2k\pi\right)\right|^2 \\ &\quad + \sum_k (\xi+6\pi+8k\pi)^2 \left|m_0\left(\frac{\xi}{4}+\frac{3\pi}{2}+2k\pi\right)\right|^2 \left|\hat{\varphi}\left(\frac{\xi}{4}+\frac{3\pi}{2}+2k\pi\right)\right|^2, \end{aligned}$$

and using the periodicity of $m_0$ and $\tilde{m}_0$, we find

$$\begin{aligned} S(\xi+2\pi) &= \left|m_0\left(\frac{\xi}{4}+\frac{\pi}{2}\right)\right|^2 \sum_k (\xi+2\pi+8k\pi)^2 \left|\hat{\varphi}\left(\frac{\xi}{4}+\frac{\pi}{2}+2k\pi\right)\right|^2 \\ &\quad + \left|m_0\left(\frac{\xi}{4}+\frac{3\pi}{2}\right)\right|^2 \sum_k (\xi+6\pi+8k\pi)^2 \left|\hat{\varphi}\left(\frac{\xi}{4}+\frac{3\pi}{2}+2k\pi\right)\right|^2 \\ &\geqslant (2\pi)^2 \left|m_0\left(\frac{\xi}{4}+\frac{\pi}{2}\right)\right|^2 \sum_k \left|\hat{\varphi}\left(\frac{\xi}{4}+\frac{\pi}{2}+2k\pi\right)\right|^2 \\ &\quad + \pi^2 \left|m_0\left(\frac{\xi}{4}+\frac{3\pi}{2}\right)\right|^2 \sum_k \left|\hat{\varphi}\left(\frac{\xi}{4}+\frac{3\pi}{2}+2k\pi\right)\right|^2. \end{aligned}$$

From the M2–hypothesis, we have

$$m_0\left(\frac{\xi}{4}+\frac{\pi}{2}\right) \neq 0 \quad \text{or} \quad m_0\left(\frac{\xi}{4}+\frac{3\pi}{2}\right) \neq 0,$$

and then we obtain (by Proposition 2.3)

$$S(\xi+2\pi) \geqslant c_1 \left|m_0\left(\frac{\xi}{4}+\frac{\pi}{2}\right)\right|^2 + c_2 \left|m_0\left(\frac{\xi}{4}+\frac{3\pi}{2}\right)\right|^2 \geqslant c_3 > 0.$$

Thus

$$\zeta(\xi) \to 1 \qquad \text{for} \quad \xi \to 0.$$

In conclusion, by Lemma 3.2, in all cases $\zeta(\xi)$ is greater than a positive constant. □

Now it is possible to formulate the fundamental result.

THEOREM 3.4. *The strengthened Cauchy-Schwarz inequality holds for biorthogonal wavelet spaces $V_j(\mathbf{R})$ and $W_j(\mathbf{R})$ satisfying the hypothesis M1–M2–M3–M4.*

## 4. A strengthened Cauchy-Schwarz inequality for biorthogonal wavelets on the interval

In this paragraph a strengthened Cauchy–Schwarz inequality for biorthogonal wavelets defined on an interval is proved. In the proof we use the previous result for biorthogonal wavelets defined on the real line.

THEOREM 4.1. *Let* $V_j([0,1])$ *and* $W_j([0,1])$ *be respectively the spaces of scaling functions and biorthogonal wavelets, defined on the interval* $[0,1]$. *There exists* $\gamma < 1$ *such that*

$$|((v,w))| \leqslant \gamma \; \|v\| \; \|w\|, \qquad \forall u \in V_j[0,1], \quad \forall w \in W_j[0,1],$$

*where* $\|\cdot\|$ *is the semi–norm in* $H^1([0,1])$ *and* $((u,u)) = \|u\|^2$.

*Proof.* First we note that we can consider the interval $[0,+\infty)$. Moreover, $T_j$ is an isometry for all $j$, and this implies that it is sufficient to prove inequality at level $j=0$. Let

$$\gamma = \sup_{\substack{v \in V_0, \; \|v\|=1 \\ w \in W_0, \; \|w\|=1}} ((v,w)),$$

where $V_0 = V_0[0,+\infty)$, $W_0 = W_0[0,+\infty)$. By usual Cauchy–Schwarz inequality we have that $\gamma \leqslant 1$. Now we set $\gamma = 1$. Then there exist sequences

$$\{v_n\} \subset V_0[0,+\infty),$$
$$\{w_n\} \subset W_0[0,+\infty),$$

such that

$$\|v_n\| = 1, \qquad \|w_n\| = 1,$$

and

$$((v_n,w_n)) \to 1.$$

Moreover, we observe that

$$\begin{aligned} \|v_n - w_n\|^2 &= \|v_n\|^2 + \|w_n\|^2 - 2((v_n,w_n)) \\ &= 2\,[1 - ((v_n,w_n))] \end{aligned}$$

and thus

$$\|v_n - w_n\| \to 0. \tag{4.1}$$

We note that

$$V_0[0,+\infty), \; W_0[0,+\infty) \subset V_1[0,+\infty)$$

and thus we have

$$v_n(x) = \sum_{k \geqslant 0} \alpha_{1k}^n \varphi_{1k}(x),$$
$$w_n(x) = \sum_{k \geqslant 0} \beta_{1k}^n \varphi_{1k}(x).$$

Moreover, from Proposition 2.5,

$$\|v_n - w_n\|^2 \asymp \sum_{k\geqslant 0} c_k|\alpha_{1k}^n - \beta_{1k}^n|^2.$$

For all $k$, it follows from $(4.1)$ that

$$|\alpha_{1k}^n - \beta_{1k}^n| \to 0.$$

Now we fix a $K \in N$ and define the finite dimensional spaces

$$V_0^K[0,+\infty) = V_0[0,+\infty) \cap \operatorname{span}\{\varphi_{1k}; 0 \leqslant k \leqslant K\},$$
$$W_0^K[0,+\infty) = W_0[0,+\infty) \cap \operatorname{span}\{\varphi_{1k}; 0 \leqslant k \leqslant K\}.$$

The disjointness of the spaces $V_0[0,+\infty)$ and $W_0[0,+\infty)$ implies

$$V_0^K[0,+\infty) \cap W_0^K[0,+\infty) = \{0\}.$$

Let $V_0^I[0,+\infty)$ and $W_0^I[0,+\infty)$ be infinite dimensional spaces such that

$$V_0[0,+\infty) = V_0^K[0,+\infty) \oplus V_0^I[0,+\infty),$$
$$W_0[0,+\infty) = W_0^K[0,+\infty) \oplus W_0^I[0,+\infty).$$

Hence, for every $n$, it is possible to consider the decompositions of $v_n$ and $w_n$

$$v_n = v_n^K + v_n^I,$$
$$w_n = w_n^K + w_n^I.$$

We notice that

$$\|v_n^K - w_n^K\| \to 0,$$

and thus

$$\|v_n^K\| \to 0, \qquad \|w_n^K\| \to 0,$$

$$\|v_n^I\| \to 1, \qquad \|w_n^I\| \to 1.$$

But $v_n^I$ and $w_n^I$ could be considered as elements of $V_0(\mathbf{R})$ and $W_0(\mathbf{R})$, respectively, and hence they have coefficients not equal to zero only from a certain point on. This implies that for $v_n^I \in V_0(\mathbf{R})$ and $w_n^I \in W_0(\mathbf{R})$ we have

$$\frac{((v_n^I, w_n^I))}{\|v_n^I\|\ \|w_n^I\|} \to 1$$

contrary to Theorem 3.4 about the strengthened Cauchy–Schwarz inequality for biorthogonal wavelets on the real line. $\square$

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