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GENERALIZED FURUTA INEQUALITY IN BANACH $*$–ALGEBRAS AND ITS APPLICATIONS

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Dedicated to Professor Zirô Takeda
on his 77^{th} birthday
with respect and affection

(*communicated by F. Hansen*)

Abstract. Okayasu [12] proved the useful Löwner-Heinz inequality in Banach $*$-algebra as follows. Let A be a unital hermitian Banach $*$-algebra with continuous involution and $a, b \in A$. If $a \geqslant b > 0$, then $a^p \geqslant b^p$ for $p \in (0,1]$. For $a > 0$, a^α =exp($\alpha \log a$),where log is the principal branch of the complex logarithm. As a nice application of this result, K.Tanahashi and M.Uchiyama [15] proved the following very interesting inequality. Let $a, b \in A$. Let $R \ni p, q, r \geqslant 0$ satisfy $(1+r)q \geqslant p+r$ and $q \geqslant 1$.

$$(b^{\frac{r}{2}} a^p b^{\frac{r}{2}})^{\frac{1}{q}} \geqslant (b^{\frac{r}{2}} b^p b^{\frac{r}{2}})^{\frac{1}{q}} \quad \text{if } a \geqslant b > 0.$$

This inequality may be called to be "Banach $*$-algebra version" of Furuta inequality. By using this result and Löwner-Heinz inequality in Banach $*$-algebra in Okayasu [12], we show the following generalized Furuta inequality. Let $a, b \in A$. If $a \geqslant b > 0$, then for each $1 \geqslant q \geqslant t \geqslant 0$ and $p \geqslant q$

$$a^{q-t+r} \geqslant \{a^{\frac{r}{2}} (a^{\frac{-t}{2}} b^p a^{\frac{-t}{2}})^s a^{\frac{r}{2}}\}^{\frac{q-t+r}{(p-t)s+r}}$$

holds for $s \geqslant 1$ and $r \geqslant t$. Moreover as an application of this inequality, we show that if $a \geqslant b > 0$, for each $t \in [0,1]$, $q \geqslant 0$ and $p \geqslant t$,

$$G_{p,q,t}(a,b,r,s) = a^{\frac{-r}{2}} \{a^{\frac{r}{2}} (a^{\frac{-t}{2}} b^p a^{\frac{-t}{2}})^s a^{\frac{r}{2}}\}^{\frac{q-t+r}{(p-t)s+r}} a^{\frac{-r}{2}}$$

is decreasing for $r \geqslant t$ and $s \geqslant 1$ such that $(p-t)s \geqslant q-t$.

1. Introduction

Let A and B be bounded linear operators on a Hilbert space H. We have obtained [4] the following order preserving operator inequalities as an extension of Löwner-Heinz inequality [9] and [11].

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THEOREM F (Furuta inequality).

If $A \geqslant B \geqslant 0$, *then for each* $r \geqslant 0$,

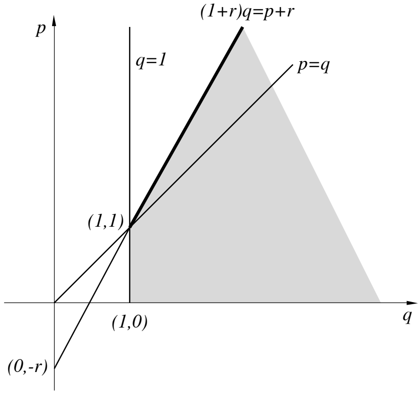
$$\text{(i)} \quad (B^{\frac{r}{2}}A^pB^{\frac{r}{2}})^{\frac{1}{q}} \geqslant (B^{\frac{r}{2}}B^pB^{\frac{r}{2}})^{\frac{1}{q}}$$

and

$$\text{(ii)} \quad (A^{\frac{r}{2}}A^pA^{\frac{r}{2}})^{\frac{1}{q}} \geqslant (A^{\frac{r}{2}}B^pA^{\frac{r}{2}})^{\frac{1}{q}}$$

hold for $p \geqslant 0$ *and* $q \geqslant 1$ *with*

$$(1+r)q \geqslant p+r.$$



Figure

Alternative proofs of Theorem F are given in [2] and [10] and also an elementary one page proof in [5]. The domain drawn for p, q and r in the Figure is the best possible one for Theorem F in [13]. In [6, Theorem 1.1] we established the following Theorem G as extensions of Theorem F and Ando-Hiai [1, Theorem 3.5].

THEOREM G [6]. *If* $A \geqslant B \geqslant 0$ *with* $A > 0$, *then for each* $t \in [0,1]$ *and* $p \geqslant 1$,

$$A^{1-t+r} \geqslant \{A^{\frac{r}{2}}(A^{\frac{-t}{2}}B^pA^{\frac{-t}{2}})^sA^{\frac{r}{2}}\}^{\frac{1-t+r}{(p-t)s+r}} \tag{1.1}$$

holds for any $s \geqslant 1$ *and* r *such that* $r \geqslant t$.

Another mean theoretic proof of Theorem G is given in [3]. The best possibility of (1.1) is proved in [14] by using skillful technique.

Let A be a unital Banach $*$-algebra with unit e and $a, b \in A$. A is said to be hermitian if $a^* = a$ then the spectrum $\sigma(a) \subset R$. $a \geqslant 0$ means that $a = a^*$ and $\sigma(a) \subset [0,\infty)$. $a > 0$ means $a \geqslant 0$ and $0 \notin \sigma(a)$. For $a > 0$, $a^\alpha = \exp(\alpha \log a)$, where log is the principal branch of the complex logarithm. Recently Okayasu [12] proved the following useful Löwner-Heinz inequality in Banach $*$-algebra.

THEOREM A [12]. *Let* A *be a unital hermitian Banach* $*$*-algebra with continuous involution. Let* $a, b \in A$ *and* $p \in (0,1]$. *Then* $a^p > b^p$ *if* $a > b > 0$, *and* $a^p \geqslant b^p$ *if* $a \geqslant b > 0$.

Using Theorem A, Tanahashi and Uchiyama [15] proved the following interesting inequality.

THEOREM B [15]. *Let* A *be a unital hermitian Banach* $*$*-algebra with continuous involution and* $a, b \in A$. *Let* $R \ni p, q, r \geqslant 0$ *satisfy* $(1+r)q \geqslant p+r$ *and* $q \geqslant 1$. *Then*

$$\text{(i)} \quad (b^{\frac{r}{2}}a^pb^{\frac{r}{2}})^{\frac{1}{q}} \geqslant (b^{\frac{r}{2}}b^pb^{\frac{r}{2}})^{\frac{1}{q}} \quad \text{if } a \geqslant b > 0$$

and

$$\text{(ii)} \quad (b^{\frac{r}{2}}a^pb^{\frac{r}{2}})^{\frac{1}{q}} > (b^{\frac{r}{2}}b^pb^{\frac{r}{2}})^{\frac{1}{q}} \quad \text{if } a > b > 0.$$

In this paper we shall give an extension of Theorem B by applying Theorem A and Theorem B and also we shall show an application of this extension.

2. Results

THEOREM 1. *Let A be a unital hermitian Banach $*$-algebra with continuous involution. Let $a,b \in A$ and $R \ni p,q,r,s,t \geqslant 0$ satisfy $1 \geqslant q \geqslant t \geqslant 0$, $p \geqslant q$, $s \geqslant 1$ and $r \geqslant t$. Then*

$$(i) \qquad a^{q-t+r} \geqslant \{a^{\frac{r}{2}}(a^{\frac{-t}{2}}b^p a^{\frac{-t}{2}})^s a^{\frac{r}{2}}\}^{\frac{q-t+r}{(p-t)s+r}} \quad \text{if } a \geqslant b > 0$$

and

$$(ii) \qquad a^{q-t+r} > \{a^{\frac{r}{2}}(a^{\frac{-t}{2}}b^p a^{\frac{-t}{2}})^s a^{\frac{r}{2}}\}^{\frac{q-t+r}{(p-t)s+r}} \quad \text{if } a > b > 0.$$

Theorem 1 implies the following Corollary 2 which is essentially equivalent to Theorem B (see Remark 3).

COROLLARY 2. *Let A be a unital hermitian Banach $*$-algebra with continuous involution. Let $a,b \in A$, $p \geqslant 1$ and $r \geqslant 0$. Then the following (i), (ii), (iii) and (iv) hold.*

$$(i) \qquad (a^{\frac{r}{2}}a^p a^{\frac{r}{2}})^{\frac{1+r}{p+r}} \geqslant (a^{\frac{r}{2}}b^p a^{\frac{r}{2}})^{\frac{1+r}{p+r}} \quad \text{if } a \geqslant b > 0$$

$$(ii) \qquad (b^{\frac{r}{2}}a^p b^{\frac{r}{2}})^{\frac{1+r}{p+r}} \geqslant (b^{\frac{r}{2}}b^p b^{\frac{r}{2}})^{\frac{1+r}{p+r}} \quad \text{if } a \geqslant b > 0$$

$$(iii) \qquad (a^{\frac{r}{2}}a^p a^{\frac{r}{2}})^{\frac{1+r}{p+r}} > (a^{\frac{r}{2}}b^p a^{\frac{r}{2}})^{\frac{1+r}{p+r}} \quad \text{if } a > b > 0$$

and

$$(iv) \qquad (b^{\frac{r}{2}}a^p b^{\frac{r}{2}})^{\frac{1+r}{p+r}} > (b^{\frac{r}{2}}b^p b^{\frac{r}{2}})^{\frac{1+r}{p+r}} \quad \text{if } a > b > 0.$$

As an application of Theorem 1, we show the following Theorem 3 associated with functions implying Theorem 1 (see Remark 5).

THEOREM 3. *Let A be a unital hermitian Banach $*$-algebra with continuous involution and $a,b \in A$. If $a \geqslant b > 0$, then for each $t \in [0,1]$, $q \geqslant 0$ and $p \geqslant t$,*

$$G_{p,q,t}(a,b,r,s) = a^{\frac{-r}{2}}\{a^{\frac{r}{2}}(a^{\frac{-t}{2}}b^p a^{\frac{-t}{2}})^s a^{\frac{r}{2}}\}^{\frac{q-t+r}{(p-t)s+r}} a^{\frac{-r}{2}}$$

is decreasing for $r \geqslant t$ and $s \geqslant 1$ such that $(p-t)s \geqslant q-t$.

COROLLARY 4. *Let A be a unital hermitian Banach $*$-algebra with continuous involution and $a,b \in A$. If $a \geqslant b > 0$, then for each $t \in [0,1]$ and $p \geqslant 1$,*

$$a^{1-t+r} \geqslant \{a^{\frac{r}{2}}(a^{\frac{-t}{2}}b^p a^{\frac{-t}{2}})^s a^{\frac{r}{2}}\}^{\frac{1-t+r}{(p-t)s+r}}$$

holds for $s \geqslant 1$ and $r \geqslant t$. Moreover, if $a \geqslant b > 0$, then for each $t \in [0,1]$ and $p \geqslant 1$,

$$G_{p,q,t}(a,b,r,s) = a^{\frac{-r}{2}}\{a^{\frac{r}{2}}(a^{\frac{-t}{2}}b^p a^{\frac{-t}{2}})^s a^{\frac{r}{2}}\}^{\frac{1-t+r}{(p-t)s+r}} a^{\frac{-r}{2}}$$

is decreasing for $r \geqslant t$ and $s \geqslant 1$.

REMARK 1. In case for bounded linear operators on a Hilbert space, (i) of Theorem 1 is shown in [8], Theorem 3 is obtained in [7] and Corollary 4 is also shown in [6] and [3].

3. Proofs of results

In what follows, a small letter means an element in Banach $*$-algebra A. We need the following usefull lemmas to give proofs of the results in 2.

LEMMA C [15]. *Let* $a, b > 0$. *For any real number* λ

$$(bab)^{\lambda} = ba^{\frac{1}{2}}(a^{\frac{1}{2}}b^2a^{\frac{1}{2}})^{\lambda-1}a^{\frac{1}{2}}b.$$

LEMMA D [15]. *If* $a > b > 0$, *then* $b^{-1} > a^{-1} > 0$. *Also if* $a \geqslant b > 0$, *then* $b^{-1} \geqslant a^{-1} > 0$.

LEMMA E [15]. *If* $a > b > 0$ *and* $c > 0$, *then* $cac > cbc$. *Also if* $a \geqslant b > 0$ *and* $c > 0$, *then* $cac \geqslant cbc$.

LEMMA F [12]. *If* $a, b \in A$, *then either* $a \geqslant b > 0$ *or* $a > b \geqslant 0$ *implies* $a > 0$.

Proof of Theorem 1. We shall show (i). Let $a \geqslant b > 0$. Then $a > 0$ by Lemma F. We have only to consider the case $q \neq 0$ since the result is trivial in case $q = 0$. First of all we remark the following (3.0);

$$d = a^{\frac{-t}{2}}b^pa^{\frac{-t}{2}} > 0 \tag{3.0}$$

since $d = a^{\frac{-t}{2}}b^pa^{\frac{-t}{2}} = (b^{\frac{p}{2}}a^{\frac{-t}{2}})^*(b^{\frac{p}{2}}a^{\frac{-t}{2}})$. Next we prove that if $a \geqslant b > 0$, then

$$a^q \geqslant \{a^{\frac{t}{2}}(a^{\frac{-t}{2}}b^pa^{\frac{-t}{2}})^sa^{\frac{t}{2}}\}^{\frac{q}{(p-t)s+t}} \quad \text{for } 1 \geqslant q \geqslant t \geqslant 0,\ p \geqslant q \text{ and } s \geqslant 1. \tag{3.1}$$

In case $2 \geqslant s \geqslant 1$. We recall $a^t \geqslant b^t > 0$ by Theorem A since $t \in [0, 1]$, so $b^{-t} \geqslant a^{-t} > 0$ by Lemma D and we have

$$\begin{aligned} 0 < b_1 &= \{a^{\frac{t}{2}}(a^{\frac{-t}{2}}b^pa^{\frac{-t}{2}})^sa^{\frac{t}{2}}\}^{\frac{q}{(p-t)s+t}} \qquad (3.2)\\ &= \{b^{\frac{p}{2}}(b^{\frac{p}{2}}a^{-t}b^{\frac{p}{2}})^{s-1}b^{\frac{p}{2}}\}^{\frac{q}{(p-t)s+t}} \quad \text{by Lemma C}\\ &\leqslant \{b^{\frac{p}{2}}(b^{\frac{p}{2}}b^{-t}b^{\frac{p}{2}})^{s-1}b^{\frac{p}{2}}\}^{\frac{q}{(p-t)s+t}}\\ &= b^q\\ &\leqslant a^q = a_1 \quad \text{for } 1 \geqslant q \geqslant t \geqslant 0,\ p \geqslant q \text{ and } 2 \geqslant s \geqslant 1 \end{aligned}$$

because the first inequality follows by (3.0) and Theorem A since $\frac{q}{(p-t)s+t} \in (0, 1]$ and the second one follows by Lemma E and Theorem A since $s-1, \frac{q}{(p-t)s+t} \in (0, 1]$, and the last one follows by Theorem A since $1 \geqslant q \geqslant 0$.

Repeating (3.2) for $a_1 \geqslant b_1 > 0$, then we have

$$a_1^{q_1} \geqslant \{a_1^{\frac{t_1}{2}}(a_1^{\frac{-t_1}{2}}b_1^{p_1}a_1^{\frac{-t_1}{2}})^{s_1}a_1^{\frac{t_1}{2}}\}^{\frac{q_1}{(p_1-t_1)s_1+t_1}} > 0 \tag{3.3}$$

for $1 \geqslant q_1 \geqslant t_1 \geqslant 0$, $p_1 \geqslant q_1$ and $2 \geqslant s_1 \geqslant 1$. Put $1 = q_1 \geqslant t_1 = \frac{t}{q} \geqslant 0$ and $p_1 = \frac{(p-t)s+t}{q} \geqslant q_1 = 1$ in (3.3). Then

$$\begin{aligned} a^q &\geqslant \{a^{\frac{t}{2}}[a^{\frac{-t}{2}}a^{\frac{t}{2}}(a^{\frac{-t}{2}}b^pa^{\frac{-t}{2}})^sa^{\frac{t}{2}}a^{\frac{-t}{2}}]^{s_1}a^{\frac{t}{2}}\}^{\frac{q}{(p-t)ss_1+t}} \qquad (3.4)\\ &= \{a^{\frac{t}{2}}(a^{\frac{-t}{2}}b^pa^{\frac{-t}{2}})^{ss_1}a^{\frac{t}{2}}\}^{\frac{q}{(p-t)ss_1+t}} > 0 \end{aligned}$$

for $1 \geqslant q \geqslant t \geqslant 0$, $p \geqslant q$ and $4 \geqslant ss_1 \geqslant 1$. Repeating this process from (3.2) to (3.4), consequently we obtain (3.1) for $1 \geqslant q \geqslant t \geqslant 0$, $p \geqslant q$ and *any* $s \geqslant 1$.

Put $a_2 = a^q$ and $b_2 = \{a^{\frac{t}{2}}(a^{\frac{-t}{2}}b^p a^{\frac{-t}{2}})^s a^{\frac{t}{2}}\}^{\frac{q}{(p-t)s+t}} > 0$ in (3.1). Then applying (i) of Theorem B for $a_2 \geqslant b_2 > 0$ for $1 \geqslant q \geqslant t \geqslant 0$, $p \geqslant q$ and *any* $s \geqslant 1$ by (3.1), so we have

$$a_2^{1+r_2} \geqslant (a_2^{\frac{r_2}{2}} b_2^{p_2} a_2^{\frac{r_2}{2}})^{\frac{1+r_2}{p_2+r_2}} \quad \text{holds for } p_2 \geqslant 1 \text{ and } r_2 \geqslant 0. \tag{3.5}$$

We have only to put $r_2 = \frac{r-t}{q} \geqslant 0$ and $p_2 = \frac{(p-t)s+t}{q} \geqslant 1$ in (3.5) to obtain the desired inequality (i) in Theorem 1. (ii) in Theorem 1 is obtained by the similar method. Whence the proof of Theorem 1 is complete.

Proof of Corollary 2.

(i) We have only to put $t = 0$, $q = 1$ and $s = 1$ in (i) of Theorem 1.

(ii) $a \geqslant b > 0$ implies $b^{-1} \geqslant a^{-1} > 0$ by Lemma D, so by (i)

$$(b^{\frac{-r}{2}} b^{-p} b^{\frac{-r}{2}})^{\frac{1+r}{p+r}} \geqslant (b^{\frac{-r}{2}} a^{-p} b^{\frac{-r}{2}})^{\frac{1+r}{p+r}} \quad \text{if } a \geqslant b > 0$$

taking inverses of both sides

$$(b^{\frac{r}{2}} a^p b^{\frac{r}{2}})^{\frac{1+r}{p+r}} \geqslant (b^{\frac{r}{2}} b^p b^{\frac{r}{2}})^{\frac{1+r}{p+r}} \quad \text{if } a \geqslant b > 0,$$

holds by Lemma D, so that we obtain (ii).

(iii) We have only to put $t = 0$, $q = 1$ and $s = 1$ in (ii) of Theorem 1.

(iv) (iv) follows from (iii) by the similar way as the proof of $(i) \Longrightarrow (ii)$.

REMARK 2. In Corollary 2, by scrutinizing $(i) \Longrightarrow (ii)$ and the reverse implication $(ii) \Longrightarrow (i)$ holds, we remark that $(i) \Longleftrightarrow (ii)$ and also $(iii) \Longleftrightarrow (iv)$ holds by the similar method.

REMARK 3. (ii) of Corollary 2 is essentially equivalent to (i) of Theorem B since Theorem B for $1 \geqslant p \geqslant 0$ is obvious by Theorem A and Lemma E. Also (iv) of Corollary 2 is essentially equivalent to (ii) of Theorem B by the same reason. Consequently Corollary 2 is essentially equivalent to Theorem B together with Remark 2.

Proof of Theorem 3. Put $q = t$ in Theorem 1. Then if $a \geqslant b > 0$, then for each $t \in [0,1]$ and $p \geqslant t$

$$a^r \geqslant \{a^{\frac{r}{2}}(a^{\frac{-t}{2}} b^p a^{\frac{-t}{2}})^s a^{\frac{r}{2}}\}^{\frac{r}{(p-t)s+r}} > 0 \quad \text{for } s \geqslant 1 \text{ and } r \geqslant t. \tag{3.6}$$

(a) *Decreasing of* $G_{p,q,t}(a,b,r,s)$ *for* s. Put $d = a^{\frac{-t}{2}} b^p a^{\frac{-t}{2}}$ in (3.6). We remark that $d = a^{\frac{-t}{2}} b^p a^{\frac{-t}{2}} > 0$ by (3.0). Applying Lemma C to (3.6), and then by using Lemma D, Lemma E and Theorem A, we obtain for each $t \in [0,1]$, $p \geqslant t$ and $r \geqslant t$

$$(d^{\frac{s}{2}} a^r d^{\frac{s}{2}})^{\frac{(p-t)w}{(p-t)s+r}} \geqslant d^w > 0 \quad \text{for } s \geqslant w > 0. \tag{3.7}$$

Then we have

$$
\begin{aligned}
f(s) &= \{a^{\frac{r}{2}}(a^{\frac{-t}{2}}b^p a^{\frac{-t}{2}})^s a^{\frac{r}{2}}\}^{\frac{q-t+r}{(p-t)s+r}} \\
&= (a^{\frac{r}{2}}d^s a^{\frac{r}{2}})^{\frac{q-t+r}{(p-t)s+r}} \\
&= \{(a^{\frac{r}{2}}d^s a^{\frac{r}{2}})^{\frac{(p-t)(s+w)+r}{(p-t)s+r}}\}^{\frac{q-t+r}{(p-t)(s+w)+r}} \\
&= \{a^{\frac{r}{2}}d^{\frac{s}{2}}(d^{\frac{s}{2}}a^r d^{\frac{s}{2}})^{\frac{(p-t)w}{(p-t)s+r}}d^{\frac{s}{2}}a^{\frac{r}{2}}\}^{\frac{q-t+r}{(p-t)(s+w)+r}} \quad \text{by Lemma C} \\
&\geqslant (a^{\frac{r}{2}}d^{s+w}a^{\frac{r}{2}})^{\frac{q-t+r}{(p-t)(s+w)+r}} \\
&= f(s+w)
\end{aligned}
$$

and the last inequality holds by (3.7), Lemma E and Theorem A since $\frac{q-t+r}{(p-t)(s+w)+r} \in [0,1]$ holds, so the proof of (a) is complete by Lemma E since $G_{p,q,t}(a,b,r,s) = a^{\frac{-r}{2}}f(s)a^{\frac{-r}{2}}$.

(b) *Decreasing of* $G_{p,q,t}(a,b,r,s)$ *for* r. Applying Theorem A to (3.6), if $a \geqslant b > 0$, then for each $t \in [0,1]$, $p \geqslant t$ and $s \geqslant 1$

$$
a^u \geqslant \{a^{\frac{r}{2}}d^s a^{\frac{r}{2}}\}^{\frac{u}{(p-t)s+r}} > 0 \quad \text{for } r \geqslant u > 0. \tag{3.8}
$$

Then we have

$$
\begin{aligned}
G_{p,q,t}(a,b,r,s) &= a^{\frac{-r}{2}}\{a^{\frac{r}{2}}(a^{\frac{-t}{2}}b^p a^{\frac{-t}{2}})^s a^{\frac{r}{2}}\}^{\frac{q-t+r}{(p-t)s+r}}a^{\frac{-r}{2}} \\
&= d^{\frac{s}{2}}(d^{\frac{s}{2}}a^r d^{\frac{s}{2}})^{\frac{q-t-(p-t)s}{(p-t)s+r}}d^{\frac{s}{2}} \quad \text{by Lemma C} \\
&= d^{\frac{s}{2}}\{(d^{\frac{s}{2}}a^r d^{\frac{s}{2}})^{\frac{(p-t)s+r+u}{(p-t)s+r}}\}^{\frac{q-t-(p-t)s}{(p-t)s+r+u}}d^{\frac{s}{2}} \\
&= d^{\frac{s}{2}}\{d^{\frac{s}{2}}a^{\frac{r}{2}}(a^{\frac{r}{2}}d^s a^{\frac{r}{2}})^{\frac{u}{(p-t)s+r}}a^{\frac{r}{2}}d^{\frac{s}{2}}\}^{\frac{q-t-(p-t)s}{(p-t)s+r+u}}d^{\frac{s}{2}} \quad \text{by Lemma C} \\
&\geqslant d^{\frac{s}{2}}(d^{\frac{s}{2}}a^{r+u}d^{\frac{s}{2}})^{\frac{q-t-(p-t)s}{(p-t)s+r+u}}d^{\frac{s}{2}} \\
&= G_{p,q,t}(a,b,r+u,s)
\end{aligned}
$$

and the last inequality holds by (3.8), Lemma E , Theorem A and Lemma D since $\frac{q-t-(p-t)s}{(p-t)s+r+u} \in [-1,0]$. Consequently $G_{p,q,t}(a,b,r,s)$ is decreasing for $r \geqslant t$. Whence the proof of Theorem 2 is complete by (a) and (b).

Proof of Corollary 4. We have only to put $q = 1$ in Theorem 1 and Theorem 3 respectively in order to obtain the first half and the latter half of Corollary 4.

REMARK 4. In Theorem 3 and Corollary 4, when the hypothesis $a \geqslant b > 0$ is replaced by $a > b > 0$, it turns out by scrutinizing of the proof of Theorem 3 that $G_{p,q,t}(a,b,r,s)$ is strictly decreasing of r and s.

REMARK 5. Take p,q,r and t in Theorem 3 as follows; $1 \geqslant q \geqslant t \geqslant 0$, $p \geqslant q$, and $r \geqslant t$. Then we have

$$
a^{q-t} \geqslant a^{\frac{-t}{2}}b^q a^{\frac{-t}{2}} = G_{p,q,t}(a,b,t,1) \geqslant G_{p,q,t}(a,b,r,s)
$$

and the second inequality follows by the monotonicity of $G_{p,q,t}(a,b,r,s)$ and the first inequality follows by Theorem A and Lemma E, so that we have (i) of Theorem 1. (ii) of Theorem 1 is also obtained by the similar method together with Remark 4. Whence Theorem 3 can be considered as an extension of Theorem 1.

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