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SOME APPLICATIONS OF TANAHASHI'S RESULT ON THE BEST POSSIBILITY OF FURUTA INEQUALITY

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Abstract. We shall give some applications of Tanahashi's result which states the best possibility of Furuta inequality. Firstly, we shall discuss the best possibility of a well-known characterization of chaotic order: $\log A \geqslant \log B$ *if and only if* $A^r \geqslant (A^{\frac{r}{2}}B^pA^{\frac{r}{2}})^{\frac{r}{p+r}}$ *holds for all* $p \geqslant 0$ *and* $r \geqslant 0$. Secondly, we shall discuss the best possibility of p-hyponormality of generalized Aluthge transformation $\tilde{T}_{s,t} = |T|^sU|T|^t$ for p-hyponormal or log-hyponormal operator T whose polar decomposition is $T = U|T|$.

1. Introduction

A capital letter means a bounded linear operator on a complex Hilbert space H. An operator T is said to be positive (denoted by $T \geqslant 0$) if $(Tx, x) \geqslant 0$ for all $x \in H$ and also an operator T is said to be strictly positive (denoted by $T > 0$) if T is positive and invertible. The following Theorem A is an extension of the celebrated Löwner-Heinz theorem: $A \geqslant B \geqslant 0$ *ensures* $A^\alpha \geqslant B^\alpha$ *for any* $\alpha \in [0, 1]$.

THEOREM A. (Furuta inequality [8])

If $A \geqslant B \geqslant 0$, *then for each* $r \geqslant 0$,

(*i*) $$(B^{\frac{r}{2}}A^pB^{\frac{r}{2}})^{\frac{1}{q}} \geqslant (B^{\frac{r}{2}}B^pB^{\frac{r}{2}})^{\frac{1}{q}}$$

and

(*ii*) $$(A^{\frac{r}{2}}A^pA^{\frac{r}{2}})^{\frac{1}{q}} \geqslant (A^{\frac{r}{2}}B^pA^{\frac{r}{2}})^{\frac{1}{q}}$$

hold for $p \geqslant 0$ *and* $q \geqslant 1$ *with*

$$(1 + r)q \geqslant p + q.$$

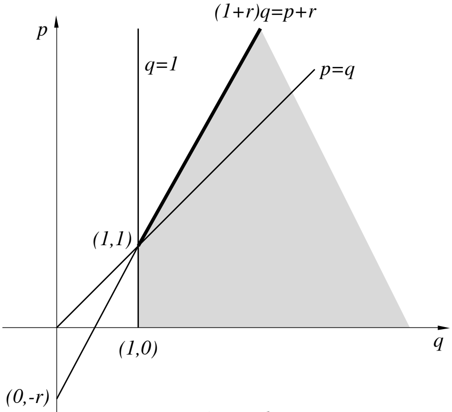

Figure 1

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We remark that Theorem A yields Löwner-Heinz theorem when we put $r = 0$ in (i) or (ii) stated above. Alternative proofs of Theorem A are given in [3] and [13], and also an elementary one-page proof in [9]. Associated with Theorem A, Tanahashi [14] shows the following result.

THEOREM B. ([14]) *Let $p > 0$, $q > 0$ and $r > 0$. If $0 < q < 1$ or $(1+r)q < p+r$, there exist positive and invertible operators A and B on $\mathbb{R}^2$ such that $A \geqslant B > 0$ and*

$$A^{\frac{p+r}{q}} \ngeqslant (A^{\frac{r}{2}} B^p A^{\frac{r}{2}})^{\frac{1}{q}}. \tag{1.1}$$

Theorem B states that the domain of the parameters p, q and r in Theorem A is the best possible for the inequalities (i) and (ii) under the assumption $A \geqslant B \geqslant 0$. We remark that invertibility of A and B are not mentioned in [14] but it is obvious by scrutinizing the proof of Theorem B in [14].

For positive and invertible operators A and B, chaotic order is defined by $\log A \geqslant \log B$. Chaotic order is weaker than usual order $A \geqslant B$ since $\log t$ is an operator monotone function. Ando [2] shows that $\log A \geqslant \log B$ if and only if $A^p \geqslant (A^{\frac{p}{2}} B^p A^{\frac{p}{2}})^{\frac{1}{2}}$ for all $p \geqslant 0$. As a generalization of this result, the following characterization of chaotic order is given by using Theorem A.

THEOREM C'. ([4],[5],[10]) *For positive and invertible operators A and B, $\log A \geqslant \log B$ if and only if*

$$A^r \geqslant (A^{\frac{r}{2}} B^p A^{\frac{r}{2}})^{\frac{r}{p+r}} \tag{1.2}$$

holds for all $p \geqslant 0$ and $r \geqslant 0$.

For positive operators A and B, we consider an order $A^\delta \geqslant B^\delta$ for $\delta > 0$. By Löwner-Heinz theorem, $A^{\delta_1} \geqslant B^{\delta_1}$ implies $A^{\delta_2} \geqslant B^{\delta_2}$ for $\delta_1 \geqslant \delta_2 > 0$. And we remark that the order $A^\delta \geqslant B^\delta$ coincides with usual order $A \geqslant B$ in case $\delta = 1$, and approaches chaotic order $\log A \geqslant \log B$ as letting $\delta \to +0$. The following result is obtained by applying Theorem A to positive operators A and B which satisfy $A^\delta \geqslant B^\delta$ for $\delta > 0$.

THEOREM D. ([6],[7]) *For positive operators A and B, $A^\delta \geqslant B^\delta$ for a fixed $\delta > 0$ if and only if*

$$A^{\frac{p+r}{q}} \geqslant (A^{\frac{r}{2}} B^p A^{\frac{r}{2}})^{\frac{1}{q}}$$

holds for all $p \geqslant 0$, $r \geqslant 0$ and $q \geqslant 1$ with $(\delta + r)q \geqslant p + r$.

Theorem D can be considered as a connection with Theorem A and Theorem C' via the order $A^\delta \geqslant B^\delta$ since Theorem C' can be rewritten as follows.

THEOREM C. *For positive and invertible operators A and B, $\log A \geqslant \log B$ if and only if*

$$A^{\frac{p+r}{q}} \geqslant (A^{\frac{r}{2}} B^p A^{\frac{r}{2}})^{\frac{1}{q}}$$

holds for all $p \geqslant 0$ and $r \geqslant 0$ with $rq \geqslant p + r$.

Figure 2 [7] shows the domains of the parameters p, q and r on which the inequality $A^{\frac{p+r}{q}} \geqslant (A^{\frac{r}{2}}B^pA^{\frac{r}{2}})^{\frac{1}{q}}$ holds under the assumptions $A \geqslant B$, $A^\delta \geqslant B^\delta$ for $\delta \in (0,1)$ and $\log A \geqslant \log B$, respectively. We remark that the domain drawn for p, q and r in Figure 2 gets smaller as the order gets weaker.

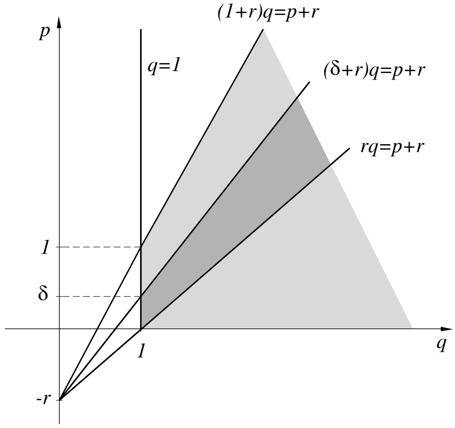

Figure 2

On the other hand, an operator T is said to be p-hyponormal for $p > 0$ if $(T^*T)^p \geqslant (TT^*)^p$ and an operator T is said to be log-hyponormal if T is invertible and $\log T^*T \geqslant \log TT^*$. p-hyponormal and log-hyponormal operators are defined as extensions of hyponormal one, i.e., $T^*T \geqslant TT^*$. It is easily obtained that every p-hyponormal operator is q-hyponormal for $p > q > 0$ by Löwner-Heinz theorem, and every p-hyponormal operator is log-hyponormal since $\log t$ is an operator monotone function.

Let T be a p-hyponormal operator whose polar decomposition is $T = U|T|$. Aluthge [1] introduced the operator $\tilde{T} = |T|^{\frac{1}{2}}U|T|^{\frac{1}{2}}$, which is called Aluthge transformation, and also showed the following result by applying Theorem A.

THEOREM E. ([1]) *Let $T = U|T|$ be the polar decomposition of a p-hyponormal operator for $0 < p < 1$ and U be unitary. Then the following assertions hold:*
(*i*) $\tilde{T} = |T|^{\frac{1}{2}}U|T|^{\frac{1}{2}}$ *is $(p+\frac{1}{2})$-hyponormal if* $0 < p < \frac{1}{2}$.
(*ii*) $\tilde{T} = |T|^{\frac{1}{2}}U|T|^{\frac{1}{2}}$ *is hyponormal if* $\frac{1}{2} \leqslant p < 1$.

As a natural generalization of Aluthge transformation, the operator $\tilde{T}_{s,t} = |T|^sU|T|^t$ for $s > 0$ and $t > 0$ can be considered. The following Theorem F on $\tilde{T}_{s,t}$ is a generalization of Theorem E on $\tilde{T}$.

THEOREM F. ([11][12][17]) *Let $T = U|T|$ be the polar decomposition of a p-hyponormal operator for $p > 0$. Then the following assertions hold:*
(*i*) $\tilde{T}_{s,t} = |T|^sU|T|^t$ *is* $\frac{p+\min\{s,t\}}{s+t}$*-hyponormal for $s > 0$ and $t > 0$ such that* $\max\{s,t\} \geqslant p$.
(*ii*) $\tilde{T}_{s,t} = |T|^sU|T|^t$ *is hyponormal for $s > 0$ and $t > 0$ such that* $p \geqslant \max\{s,t\}$.

We remark that Theorem F yields Theorem E when putting $s = t = \frac{1}{2}$ and the proof of [11] is cited under the condition $N(T) = N(T^*)$. As a parallel result to Theorem F for log-hyponormal operators, the following Theorem G is given in [16].

THEOREM G. ([16]) *Let $T = U|T|$ be the polar decomposition of a log-hyponormal operator. Then $\tilde{T}_{s,t} = |T|^sU|T|^t$ is* $\frac{\min\{s,t\}}{s+t}$*-hyponormal for $s > 0$ and $t > 0$.*

We remark that Theorem G is a parallel result to Theorem F. In fact, Theorem G corresponds to the case $p \to +0$ of Theorem F since p-hyponormality of T (i.e., $(T^*T)^p \geqslant (TT^*)^p$) approaches log-hyponormality of T (i.e., $\log T^*T \geqslant \log TT^*$) as $p \to +0$.

In this paper, we shall show some applications of Theorem B which states the best possibility of Theorem A. In fact, we shall discuss the best possibilities of some applications of Theorem A.

Firstly, we shall discuss a characterization of chaotic order. In fact, we shall prove the best possibilities of Theorem D and Theorem C, which are parallel results to Theorem B.

Secondly, we shall discuss generalized Aluthge transformation for p-hyponormal operators and log-hyponormal operators. In fact, we shall prove the best possibilities of Theorem F and Theorem G by using the best possibilities of Theorem D and Theorem C, respectively.

2. On a characterization of chaotic order

We show the following Theorem 1 and Theorem 2, which state the best possibilities of Theorem D and Theorem C, respectively. We remark that the fact of Theorem 1 was pointed out in [15].

THEOREM 1. *Let $p > 0$, $q > 0$, $r > 0$ and $\delta > 0$. If $0 < q < 1$ or $(\delta + r)q < p + r$, there exist positive and invertible operators A and B on $\mathbb{R}^2$ such that $A^\delta \geqslant B^\delta$ and*

$$A^{\frac{p+r}{q}} \ngeqslant (A^{\frac{r}{2}} B^p A^{\frac{r}{2}})^{\frac{1}{q}}. \tag{1.1}$$

THEOREM 2. *Let $p > 0$, $q > 0$ and $r > 0$. If $rq < p + r$, there exist positive and invertible operators A and B on $\mathbb{R}^2$ such that $\log A \geqslant \log B$ and*

$$A^{\frac{p+r}{q}} \ngeqslant (A^{\frac{r}{2}} B^p A^{\frac{r}{2}})^{\frac{1}{q}}. \tag{1.1}$$

We remark that Theorem 1 and Theorem 2 are parallel results to Theorem B. We also remark that Theorem 2 can be rewritten in the following form.

THEOREM 2'. *Let $p > 0$ and $r > 0$. If $\alpha > 1$, there exist positive and invertible operators A and B on $\mathbb{R}^2$ such that $\log A \geqslant \log B$ and*

$$A^{r\alpha} \ngeqslant (A^{\frac{r}{2}} B^p A^{\frac{r}{2}})^{\frac{r\alpha}{p+r}}. \tag{2.1}$$

Theorem 2' states that the outside exponents on both sides of (1.2) in Theorem C' are the best possible.

Proof of Theorem 1. Assume $0 < q < 1$ or $(\delta + r)q < p + r$. Put $p_1 = \frac{p}{\delta} > 0$ and $r_1 = \frac{r}{\delta} > 0$, then $(\delta + r)q < p + r$ is equivalent to $(1 + r_1)q < p_1 + r_1$. By Theorem B, there exist positive and invertible operators A_1 and B_1 such that $A_1 \geqslant B_1 > 0$ and

$$A_1^{\frac{p_1+r_1}{q}} \ngeqslant (A_1^{\frac{r_1}{2}} B_1^{p_1} A_1^{\frac{r_1}{2}})^{\frac{1}{q}}. \tag{2.2}$$

Here we put $A = A_1^{\frac{1}{\delta}} > 0$ and $B = B_1^{\frac{1}{\delta}} > 0$, then $A_1 = A^\delta$ and $B_1 = B^\delta$, so that $A_1 \geqslant B_1$ is equivalent to $A^\delta \geqslant B^\delta$ and (2.2) is equivalent to the following (1.1):

$$A^{\frac{p+r}{q}} \not\geqslant (A^{\frac{r}{2}} B^p A^{\frac{r}{2}})^{\frac{1}{q}}, \tag{1.1}$$

therefore A and B satisfy both $A^\delta \geqslant B^\delta$ and (1.1). Hence the proof of Theorem 1 is complete. □

Proof of Theorem 2. Assume $rq < p+r$. Since $0 < \frac{p+r}{q} - r$, there exists a $\delta > 0$ such that $0 < \delta < \frac{p+r}{q} - r$, that is, $(\delta + r)q < p+r$. By Theorem 1, there exist positive and invertible operators A and B such that $A^\delta \geqslant B^\delta$ and

$$A^{\frac{p+r}{q}} \not\geqslant (A^{\frac{r}{2}} B^p A^{\frac{r}{2}})^{\frac{1}{q}}. \tag{1.1}$$

$A^\delta \geqslant B^\delta$ ensures $\log A \geqslant \log B$ since $\log t$ is operator monotone, so that A and B satisfy both $\log A \geqslant \log B$ and (1.1). Hence the proof of Theorem 2 is complete. □

3. On generalized Aluthge transformation

By using Theorem 1 and Theorem 2 in the previous section, we show the best possibility of Theorem F and Theorem G, respectively.

THEOREM 3. *Let $p > 0$, $s > 0$ and $t > 0$. And let $T = U|T|$ be the polar decomposition of T. Then the following assertions hold:*

(i) *In case $\max\{s,t\} \geqslant p$, if $\alpha > \frac{p+\min\{s,t\}}{s+t}$, there exists a p-hyponormal operator T such that $\tilde{T}_{s,t} = |T|^s U |T|^t$ is not α-hyponormal.*

(ii) *In case $p \geqslant \max\{s,t\}$, if $\alpha > 1$, there exists a p-hyponormal operator T such that $\tilde{T}_{s,t} = |T|^s U |T|^t$ is not α-hyponormal.*

THEOREM 4. *Let $s > 0$ and $t > 0$. If $\alpha > \frac{\min\{s,t\}}{s+t}$, there exists a log-hyponormal operator T such that $\tilde{T}_{s,t} = |T|^s U |T|^t$ is not α-hyponormal, where $T = U|T|$ is the polar decomposition of T.*

In order to prove Theorem 3 and Theorem 4, we prepare the following lemma.

LEMMA 1. *For positive operators A and B on a Hilbert space H, define the operator T on $\bigoplus_{k=-\infty}^{\infty} H$ as follows:*

$$T = \begin{pmatrix} \ddots & & & & & & \\ \ddots & 0 & & & & & \\ & B^{\frac{1}{2}} & 0 & & & & \\ & & B^{\frac{1}{2}} & \boxed{0} & & & \\ & & & A^{\frac{1}{2}} & 0 & & \\ & & & & A^{\frac{1}{2}} & 0 & \\ & & & & & \ddots & \ddots \end{pmatrix}, \tag{3.1}$$

where $\boxed{}$ *shows the place of the* $(0,0)$ *matrix element. Then the following assertions hold:*

(i) T *is p-hyponormal for* $p>0$ *if and only if* $A^p \geqslant B^p$.

(ii) T *is log-hyponormal if and only if* A *and* B *are invertible and* $\log A \geqslant \log B$.

Furthermore, define $\tilde{T}_{s,t} = |T|^s U|T|^t$ *for* $s>0$ *and* $t>0$ *where* $T=U|T|$ *is the polar decomposition of* T. *Then the following assertion holds:*

(iii) $\tilde{T}_{s,t} = |T|^s U|T|^t$ *is* α*-hyponormal for* $\alpha>0$ *if and only if both*

$$(B^{\frac{t}{2}} A^s B^{\frac{t}{2}})^{\alpha} \geqslant B^{(s+t)\alpha}$$

and

$$A^{(s+t)\alpha} \geqslant (A^{\frac{s}{2}} B^t A^{\frac{s}{2}})^{\alpha}$$

hold.

Proof. By easy calculation, we have

$$T^*T = \begin{pmatrix} \ddots & & & & & \\ & B & & & & \\ & & B & & & \\ & & & \boxed{A} & & & \\ & & & & A & & \\ & & & & & A & \\ & & & & & & \ddots \end{pmatrix} \quad \text{and} \quad TT^* = \begin{pmatrix} \ddots & & & & & \\ & B & & & & \\ & & B & & & \\ & & & \boxed{B} & & & \\ & & & & A & & \\ & & & & & A & \\ & & & & & & \ddots \end{pmatrix},$$

so that (i) and (ii) are obvious. Let $T=U|T|$ be the polar decomposition of T, then it turns out that the partial isometry operator U can be written as following:

$$U = \begin{pmatrix} \ddots & & & & & & \\ \ddots & 0 & & & & & \\ & W_2 & 0 & & & & \\ & & W_2 & \boxed{0} & & & \\ & & & W_1 & 0 & & \\ & & & & W_1 & 0 & \\ & & & & & \ddots & \ddots \end{pmatrix},$$

where the polar decompositions of $A^{\frac{1}{2}}$ and $B^{\frac{1}{2}}$ are $A^{\frac{1}{2}} = W_1 A^{\frac{1}{2}}$ and $B^{\frac{1}{2}} = W_2 B^{\frac{1}{2}}$, respectively. Therefore we have

$$\tilde{T}_{s,t}^* \tilde{T}_{s,t} = |T|^t U^* |T|^{2s} U |T|^t = \begin{pmatrix} \ddots & & & & & & \\ & B^{s+t} & & & & & \\ & & B^{\frac{t}{2}} A^s B^{\frac{t}{2}} & & & & \\ & & & \boxed{A^{s+t}} & & & \\ & & & & A^{s+t} & & \\ & & & & & A^{s+t} & \\ & & & & & & \ddots \end{pmatrix}$$

and

$$\tilde{T}_{s,t}\tilde{T}_{s,t}^* = |T|^s U|T|^{2t}U^*|T|^s = \begin{pmatrix} \ddots & & & & & & \\ & B^{s+t} & & & & & \\ & & B^{s+t} & & & & \\ & & & \boxed{A^{\frac{s}{2}}B^tA^{\frac{s}{2}}} & & & \\ & & & & A^{s+t} & & \\ & & & & & A^{s+t} & \\ & & & & & & \ddots \end{pmatrix}.$$

By comparing both the (-1,-1) and (0,0) matrix elements of $(\tilde{T}_{s,t}^*\tilde{T}_{s,t})^\alpha$ and $(\tilde{T}_{s,t}\tilde{T}_{s,t}^*)^\alpha$, we have (iii). Hence the proof of Lemma 1 is complete. $\square$

Proof of Theorem 3. In the process of the proof of (i), we divide the case (i) into $(i\text{-}a)$ and $(i\text{-}b)$ as follows.

$(i\text{-}a)$ Case $t \geqslant s > 0$ and $t = \max\{s,t\} \geqslant p > 0$. Assume $\alpha > \frac{p+\min\{s,t\}}{s+t} = \frac{p+s}{s+t}$. Put $q = \frac{1}{\alpha} > 0$ and $\delta = p > 0$, then $(\delta + s)q < t + s$. By Theorem 1, there exist positive operators A and B on a Hilbert space H such that $A^\delta \geqslant B^\delta$ and

$$A^{\frac{t+s}{q}} \ngeqslant (A^{\frac{s}{2}}B^tA^{\frac{s}{2}})^{\frac{1}{q}}. \tag{3.2}$$

Since $q = \frac{1}{\alpha}$ and $\delta = p$, $A^\delta \geqslant B^\delta$ is equivalent to $A^p \geqslant B^p$ and (3.2) is equivalent to

$$A^{(s+t)\alpha} \ngeqslant (A^{\frac{s}{2}}B^tA^{\frac{s}{2}})^\alpha.$$

Here we define the operator T on $\bigoplus_{k=-\infty}^{\infty} H$ as (3.1) in Lemma 1. Then T is p-hyponormal and $\tilde{T}_{s,t}$ is not α-hyponormal by (i) and (iii) of Lemma 1.

$(i\text{-}b)$ Case $s \geqslant t > 0$ and $s = \max\{s,t\} \geqslant p > 0$. Assume $\alpha > \frac{p+\min\{s,t\}}{s+t} = \frac{p+t}{s+t}$. Put $q = \frac{1}{\alpha} > 0$ and $\delta = p > 0$, then $(\delta + t)q < s + t$. By Theorem 1, there exist positive and invertible operators A_1 and B_1 on a Hilbert space H such that $A_1^\delta \geqslant B_1^\delta$ and

$$A_1^{\frac{s+t}{q}} \ngeqslant (A_1^{\frac{t}{2}}B_1^sA_1^{\frac{t}{2}})^{\frac{1}{q}}. \tag{3.3}$$

Put $A = B_1^{-1}$ and $B = A_1^{-1}$, then $A_1^\delta \geqslant B_1^\delta$ is equivalent to $A^\delta \geqslant B^\delta$ and (3.3) is equivalent to the following (3.4):

$$(B^{\frac{t}{2}}A^sB^{\frac{t}{2}})^{\frac{1}{q}} \ngeqslant B^{\frac{s+t}{q}}. \tag{3.4}$$

Since $q = \frac{1}{\alpha}$ and $\delta = p$, $A^\delta \geqslant B^\delta$ is equivalent to $A^p \geqslant B^p$ and (3.4) is equivalent to

$$(B^{\frac{t}{2}}A^sB^{\frac{t}{2}})^\alpha \ngeqslant B^{(s+t)\alpha}.$$

Here we define the operator T on $\bigoplus_{k=-\infty}^{\infty} H$ as (3.1) in Lemma 1. Then T is p-hyponormal and $\tilde{T}_{s,t}$ is not α-hyponormal by (i) and (iii) of Lemma 1.

(ii) Case $p \geqslant \max\{s,t\}$, i.e., $p \geqslant s > 0$ and $p \geqslant t > 0$. Assume $\alpha > 1$. Put $q = \frac{1}{\alpha} > 0$ and $\delta = p > 0$, then $0 < q < 1$. By Theorem 1, there exist positive operators A and B on a Hilbert space H such that $A^\delta \geqslant B^\delta$ and

$$A^{\frac{t+s}{q}} \ngeqslant (A^{\frac{s}{2}}B^tA^{\frac{s}{2}})^{\frac{1}{q}}. \tag{3.5}$$

Since $q=\frac{1}{\alpha}$ and $\delta=p$, $A^\delta \geqslant B^\delta$ is equivalent to $A^p \geqslant B^p$ and (3.5) is equivalent to

$$A^{(s+t)\alpha} \not\geqslant (A^{\frac{s}{2}}B^tA^{\frac{s}{2}})^\alpha.$$

Here we define the operator T on $\bigoplus_{k=-\infty}^{\infty} H$ as (3.1) in Lemma 1. Then T is p-hyponormal and $\tilde{T}_{s,t}$ is not α-hyponormal by (i) and (iii) of Lemma 1.

Consequently the proof of Theorem 3 is complete. □

Proof of Theorem 4.

(a) Case $t \geqslant s > 0$. Assume $\alpha > \frac{\min\{s,t\}}{s+t} = \frac{s}{s+t}$. Put $q=\frac{1}{\alpha}>0$, then $sq < t+s$. By Theorem 2, there exist positive and invertible operators A and B on a Hilbert space H such that $\log A \geqslant \log B$ and

$$A^{\frac{t+s}{q}} \not\geqslant (A^{\frac{s}{2}}B^tA^{\frac{s}{2}})^{\frac{1}{q}}. \tag{3.6}$$

Since $q=\frac{1}{\alpha}$, (3.6) is equivalent to

$$A^{(s+t)\alpha} \not\geqslant (A^{\frac{s}{2}}B^tA^{\frac{s}{2}})^\alpha.$$

Here we define the operator T on $\bigoplus_{k=-\infty}^{\infty} H$ as (3.1) in Lemma 1, then T is log-hyponormal and $\tilde{T}_{s,t}$ is not α-hyponormal by (ii) and (iii) of Lemma 1.

(b) Case $s \geqslant t > 0$. Assume $\alpha > \frac{\min\{s,t\}}{s+t} = \frac{t}{s+t}$. Put $q=\frac{1}{\alpha}>0$, then $tq < s+t$. By Theorem 2, there exist positive and invertible operators A_1 and B_1 on a Hilbert space H such that $\log A_1 \geqslant \log B_1$ and

$$A_1^{\frac{s+t}{q}} \not\geqslant (A_1^{\frac{t}{2}}B_1^sA_1^{\frac{t}{2}})^{\frac{1}{q}}. \tag{3.7}$$

Put $A=B_1^{-1}$ and $B=A_1^{-1}$, then $\log A_1 \geqslant \log B_1$ is equivalent to $\log A \geqslant \log B$ and (3.7) is equivalent to the following (3.8):

$$(B^{\frac{t}{2}}A^sB^{\frac{t}{2}})^{\frac{1}{q}} \not\geqslant B^{\frac{s+t}{q}}. \tag{3.8}$$

Since $q=\frac{1}{\alpha}$, (3.8) is equivalent to

$$(B^{\frac{t}{2}}A^sB^{\frac{t}{2}})^\alpha \not\geqslant B^{(s+t)\alpha}.$$

Here we define the operator T on $\bigoplus_{k=-\infty}^{\infty} H$ as (3.1) in Lemma 1, then T is log-hyponormal and $\tilde{T}_{s,t}$ is not α-hyponormal by (ii) and (iii) of Lemma 1.

Consequently the proof of Theorem 4 is complete. □

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