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SEVERAL APPROXIMATIONS OF $\pi(x)$

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Abstract. In this paper several new inequalities on the function $\pi(x)$ (numbers of primes not exceeding x) are presented. In the proofs, essentially the well-known results of Rosser and Schoenfeld are used.

Legendre conjectured that $x/(\log x - A)$ (with $A = 1.08366...$) is a good approximation for $\pi(x)$. We prove that, for $x > 10^6$, the function considered by Legendre is actually an upper bound.

L. Locker-Ernst affirms that $\frac{n}{h(n)}$, with $h(n) = \frac{1}{3} + \frac{1}{4} + ... + \frac{1}{n}$ is very close to $\pi(n)$. We precize the above statement by proving that, for $n \geqslant 1429$, $\frac{n}{h(n)}$ is actually a lower bound for $\pi(n)$.

1. Introduction

Studying the number $\pi(x)$ of primes not greater than x, Legendre conjectured that there exist some constants A and B such that the aproximation

$$\pi(x) = \frac{x}{A\log x + B} \tag{1}$$

is valid when x is great enough. In 1808 Legendre also conjectured that

$$\pi(x) = \frac{x}{\log x - A(x)} \tag{2}$$

with $\lim_{x\to\infty} A(x) = 1,08366\ldots$ (see [5]).

Since then it has been proved that $\lim_{x\to\infty} A(x) = 1$ (see [7]), but for $x < 10^6$, $A(x)$ also takes values around $1.08366\ldots$ In 1959, L. Locker-Ernst [2] observed that a good approximation to $\pi(n)$ is given by $\dfrac{n}{h(n)}$ when $n > 50$, with $h(n) = \dfrac{1}{3} + \dfrac{1}{4} + \cdots + \dfrac{1}{n}$.

In order to make clear these statements it is necessary to use inequalities of the form

$$\frac{x}{\log x - m} < \pi(x) < \frac{x}{\log x - M}$$

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Several such results are known. Most of them have been obtained by a very delicate analysis by Rosser and Scoenfeld, and were published in a series of papers. In 1941, they showed in [6] that for $x \geqslant 55$ we can take $m = -2$ and $M = 4$. In 1962, they proved in [7] that for $x \geqslant 67$ one can take $m = \frac{1}{2}$ and for $x > e^{1.5}$, the inequality holds for $M = \frac{3}{2}$. Related to these, they improved the results of [6] and [7] in the papers [8] and [9], published in 1975 and 1976. The improvements are concerned to estimates for the Chebyshev functions $\psi(x)$ and $\theta(x)$. In the present paper we will use this kind of result. In 1989, Costa Pereira used in [4] elementary methods for improving previous estimates for $\psi(x)$. Besides the mentioned results, many other ones are announced in [3]. However, no one of these is strong enough for giving a precise answer to the problems posed by Legendre and L. Locker-Ernst. We will essentially use the papers of Rosser and Schoenfeld to improve their results concerned to the distance from 1 to m and M.

2. The inequality between $\pi(n)$ and $\dfrac{n}{h(n)}$

A table from [1] suggests that in general the inequality $\pi(n) > \dfrac{n}{h(n)}$ holds. We shall prove this inequality and we shall also determine the lowest value of n for which it is valid.

We need the following classical results obtained by Rosser and Schoenfeld [7]:

$$\theta(x) < x \quad \text{for } 0 < x < 10^8, \text{ where } \theta(x) = \sum_{p \leqslant x} \log p \tag{3}$$

$$\pi(x) > \frac{x}{\log x}\left(1 + \frac{1}{2\log x}\right) \quad \text{for } x \geqslant 59 \tag{4}$$

$$\pi(x) > \int_{\sqrt{x}}^{x} \frac{dt}{\log t} \quad \text{for } 11 \leqslant x \leqslant 10^8 \tag{5}$$

Also, the table from [9], p. 359 shows that

$$\theta(x) > x\left(1 - \frac{1}{29\log x}\right) \qquad \text{for } x > 315,437 \tag{6}$$

Using these results we can now prove

LEMMA. *The inequality*

$$\pi(x) > \frac{x}{\log x - \frac{28}{29}}$$

holds for $x \geqslant 3299$.

Proof. The following well-known relation holds

$$\pi(x) = \frac{\theta(x)}{\log x} + \int_2^x \frac{\theta(t)}{t\log^2 t}dt$$

For $x > e^{12.7} > 315.437$ we get

$$\pi(x) = \pi(e^{12.7}) - \frac{\theta(e^{12.7})}{12.7} + \frac{\theta(x)}{\log x} + \int_{e^{12.7}}^x \frac{\theta(t)}{t\log^2 t}dt$$

Since $e^{12.7} > 59$ and $11 < e^{12.7} < 10^8$, by applying (4) and (5) we obtain that

$$\pi(e^{12.7}) - \frac{\theta(e^{12.7})}{12.7} > \frac{e^{12.7}}{2\cdot(12.7)^2}$$

Then using (6), we see that

$$\pi(x) > \frac{x}{\log x} - \frac{x}{29\log^2 x} + \frac{e^{12.7}}{2\cdot(12.7)^2} + \int_{e^{12.7}}^x \frac{dt}{\log^2 t} - $$
$$- \frac{1}{29}\int_{e^{12.7}}^x \frac{dt}{\log^3 t}, \quad \text{when } x \geqslant e^{12.7}$$

Integrating by parts the previous relations we get

$$\pi(x) > \frac{x}{\log x}\left(1 + \frac{28}{29\log x}\right) + \frac{57}{29}\int_{e^{12.7}}^x \frac{dt}{\log^3 t} - \frac{e^{12.7}}{2\cdot(12.7)^2} \tag{7}$$

Let us denote

$$f(x) = \frac{57}{29}\int_{e^{12.7}}^x \frac{dt}{\log^3 t} - \frac{x}{\log^3 x} - \frac{e^{12.7}}{2\cdot(12.7)^2} \quad \text{for } x > e^{16}$$

It follows that $f'(x) = \frac{28\log x + 87}{29\log^4 x} > 0$. In order to prove that

$$f(x) > 0 \tag{8}$$

we still need to show that $f(e^{16}) > 0$.

We have

$$f(e^{16}) > \frac{57}{59}\cdot\frac{e^{16} - e^{12.7}}{16^3} - \frac{e^{16}}{16^3} - \frac{e^{12.7}}{2(12.7)^2} >$$
$$> e^{12.7}(0.01253 - 0.00662 - 0.0032) > 0.$$

From (7) and (8) it follows that

$$\pi(x) > \frac{x}{\log x}\left(1 + \frac{28}{29\log x} + \frac{1}{\log^2 x}\right) \text{ when } x \geqslant e^{16}$$

hence we get that

$$\pi(x) > \frac{x}{\log x - \frac{28}{29}}, \text{ when } x \geqslant e^{16} \tag{9}$$

Let us consider now $x < e^{16}$. Since $e^{16} < 10^8$ we make use of (5). We have $\pi(x) > \int\limits_{\sqrt{x}}^{x} \frac{dt}{\log t}$ when $11 \leqslant x \leqslant e^{16}$ and by integration it follows that

$$\pi(x) > \frac{x}{\log x} + \frac{x}{\log^2 x} - \frac{2\sqrt{x}}{\log x}\left(1 + \frac{2}{\log x}\right) + 2\int\limits_{\sqrt{x}}^{x} \frac{dt}{\log^3 t},$$

when $11 \leqslant x \leqslant e^{16}$.

Since $\int\limits_{\sqrt{x}}^{x} \frac{dt}{\log^3 t} > \frac{x - \sqrt{x}}{\log^3 x}$, we have

$$\begin{aligned}\pi(x) &> \frac{x}{\log x}\left(1 + \frac{1}{\log x}\right) - \frac{2\sqrt{x}}{\log x}\left(1 + \frac{2}{\log x}\right) + 2\frac{x - \sqrt{x}}{\log^3 x} = \\ &= \frac{x}{\log x}\left(1 + \frac{1}{\log x} + \frac{1}{\log^2 x}\right) + \frac{\sqrt{x}}{\log^3 x}(\sqrt{x} - 2\log^2 x - 4\log x - 2),\end{aligned}$$

for $11 \leqslant x \leqslant e^{16}$.

Let us denote $f(x) = \sqrt{x} - 2\log^2 x - 4\log x - 2$ for $11 \leqslant x \leqslant e^{16}$. It may be easily shown that $f'(x) > 0$ for $x \geqslant 6100$ and $f(10^5) > 0$, hence $f(x) > 0$ for $10^5 \leqslant x \leqslant e^{16}$. Therefore

$$\pi(x) > \frac{x}{\log x}\left(1 + \frac{1}{\log x} + \frac{1}{\log^2 x}\right) > \frac{x}{\log x - \frac{28}{29}}$$

The relationship has been tested with a computer for $x < 10^5$.

□

A comparison between this result and the inequality (4) shows that the relationship that we proved is important by itself, apart from its application to other problems.

We may now state

THEOREM 1. *We have* $\pi(n) > \dfrac{n}{h(n)}$ *for* $n \geqslant 1429$.

Proof. The following well-known inequality

$$c+\frac{1}{2n}>1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}-\log n>c \tag{10}$$

holds, where $c=0.57721566\ldots$ is Euler's constant.

Let us denote $x_n=\log n-\dfrac{n}{\pi(n)}$. From the Lemma it follows that

$$x_n>\frac{28}{29}=0.9655\cdots>0.9228>\frac{3}{2}-c,\ \text{for } n\geqslant 3299$$

The inequality $x_n>\frac{3}{2}-c$ means that $\pi(n)>\frac{n}{\log n-\frac{3}{2}+c}$. Since the sequence $y_n=\frac{n}{\log n-\frac{3}{2}+c}$ is increasing for $n\geqslant 8$, it is enough to check the inequality for $n=p-1$, where p is a prime number. Using a simple computing program we see that the inequality $\pi(n)>y_n$ holds for $1429\leqslant n\leqslant 3298$. Since (10) implies that $\dfrac{3}{2}-c>\log n-h_n$, we get that $x_n>\log n-h_n$ for $n\geqslant 1429$, hence $h_n>\dfrac{n}{\pi(n)}$.

For $n=1428$, the inequality is not valid because

$$h_{1428}<c-1.5+\log 1428+\frac{1}{2\cdot 1428}<6.3415959$$

and so $\dfrac{1428}{h_{1428}}>225.17>225=\pi(1428)$.

□

3. The inequality between $\pi(x)$ and $\dfrac{x}{\log x\ -a}$

As regards the conjecture made by Legendre, we shall prove

THEOREM 2. *We have* $\pi(x)<\dfrac{x}{\log x-a}$ *for* $x>10^6$, *where a=1.08366.*

Proof. We shall need upper bounds for the two integrals of convex functions (see [1]). We remind that for $\alpha<\beta$ and f a continuous convex function, we have that

$$\int_{\alpha}^{\beta} f(t)dt\leqslant(\beta-\alpha)\frac{f(\alpha)+f(\beta)}{2}$$

so then

$$\int_{c}^{d} f(t)dt\leqslant h\left(\frac{f(c)+f(d)}{2}+\sum_{i=1}^{n-1} f(c+ih)\right),\ \text{where } h=\frac{d-c}{n}$$

Take $f(t) = \dfrac{1}{\log^3 t}$, $c = 2$, $d = e$ and $n = 10^4$, we obtain

$$\int_2^e \frac{1}{\log^3 t} dt < 1.230759$$

then similarly

$$\int_e^{e^2} \frac{1}{\log^3 t} dt < 1.476944$$

$$\int_{e^2}^{e^3} \frac{1}{\log^3 t} dt < 0.797243$$

We continue and adding the obtained equations we find

$$\int_2^{e^{17}} \frac{dt}{\log^3 t} < 6091.58594 \tag{11}$$

In a similar way we have that

$$\int_2^{e^{15.5}} \frac{dt}{\log^2 t} < 26119.61112 \tag{12}$$

We remind that $\theta(x) < x$ for $1 < x < 10^8$ (3), so that

$$\pi(x) = \frac{\theta(x)}{\log x} + \int_2^x \frac{\theta(t)dt}{t\log^2 t} < \frac{x}{\log x} + \int_2^x \frac{dt}{\log^2 t}$$

Put

$$g(x) = \frac{x}{\log x - a} - \frac{x}{\log x} - \int_2^x \frac{dt}{\log^2 t}$$

and it follows that

$$g'(x) = \frac{(a-1)\log x - a^2}{\log x(\log x - a)^2} > 0 \quad \text{for } x \geqslant 15,$$

hence g is a monotonically increasing function.

Since

$$g(e^{15.5}) = \frac{ae^{15.5}}{(15.5-a)\cdot 15.5} - \int_2^{e^{15.5}} \frac{dt}{\log^2 t} > 26137.9 - 26119.9 > 0$$

we have $g(x) > 0$ for $e^{15.5} \leqslant x \leqslant 10^8$, hence $\pi(x) < \dfrac{x}{\log x - a}$.

Checking all the primes p, $10^6 \leqslant p \leqslant e^{15.5} = 5389698.476\ldots$, we get that the inequality $\pi(x) < \dfrac{x}{\log x - a}$ holds for $10^6 \leqslant x \leqslant 10^8$. Since $e^{15.5} < 2^{32}$, there are no problems in writing the computation program we used to check the inequality.

It remains to prove this inequality for $x \geqslant 10^8$.

In [9], p 360 it is shown that $|\theta(x) - x| < \dfrac{bx}{\log x}$ for $x \geqslant 1.04 \cdot 10^7$, where $b = 0.0077629$. From (3) it follows that

$$\theta(x) < x + \frac{bx}{\log x} \quad \text{for } x > 0, \tag{13}$$

hence we get

$$\begin{aligned}
\pi(x) &< \frac{x}{\log x} + \frac{bx}{\log^2 x} + \int_2^x \left(\frac{1}{\log^2 t} + \frac{b}{\log^3 t} \right) dt \\
&= \frac{x}{\log x} + \frac{(b+1)x}{\log^2 x} - \frac{2}{\log^2 2} + (b+2) \int_2^x \frac{dt}{\log^3 t}.
\end{aligned}$$

For $x \geqslant e^{17}$ let us consider

$$h(x) = \frac{x}{\log x - a} - \frac{x}{\log x} - \frac{(b+1)x}{\log^2 x} + \frac{2}{\log^2 2} - (b+2) \int_2^x \frac{dt}{\log^3 t}$$

Then

$$\begin{aligned}
h'(x) &= \frac{(a-b-1)\log^3 x - (a^2 - 2ab - b)\log^2 x - ab(a+b)\log x + a^2 b}{\log^3 x (\log x - a)^2} > \\
&> \frac{0.075 \log^2 x - 1.15 \log x - 0.03}{\log^2 x (\log x - a)^2} > 0
\end{aligned}$$

because $\log x \geqslant 17$.

Therefore h is an increasing function and using (11) we find

$$\begin{aligned}
h(e^{17}) &= \frac{ae^{17}}{17(17-a)} - \frac{(b+1)e^{17}}{17^2} + \frac{2}{\log^2 2} - (b+2) \cdot 6091.59 > \\
&> 96.740 - 84.230 + 4 - 12.231 > 0
\end{aligned}$$

We get $\pi(x) < \dfrac{x}{\log x - a}$ also for $x \geqslant e^{17}$.

Since $e^{17} < 10^8$, we have completed the proof. □

A straightforward check shows that

COROLLARY. $\pi(x) < \frac{x}{\log x - 1.11}$ *for any* $x \geqslant 4$.

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