STOLARSKY–TOBEY MEAN IN \( n \) VARIABLES

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Abstract. In this paper, an \( n \)-dimensional weighted Stolarsky–Tobey mean is defined via measure. This mean includes as special cases various generalizations of the logarithmic mean. Some elementary properties are listed and various inequalities derived. Attention is given to the case when the mean is specialized to Dirichlet measure. Relations to hypergeometric function are exhibited. An explicit form is given for the mean in the special case when all variables have equal weights.

1. Introduction and notation

Throughout the paper we assume that \( \mathbb{R} \) is a set of real numbers and \( \mathbb{R}_+ \) is a set of strictly positive real numbers. Let \( \mathbf{x} \) denote \( n \)-tuple \( (x_1, x_2 \ldots, x_n) \in \mathbb{R}^n_+ \) and let \( E_{n-1} \subset \mathbb{R}^{n-1} \) represent the simplex

\[
E_{n-1} = \left\{ (u_1, u_2, \ldots, u_{n-1}) : u_i \geq 0 \ (1 \leq i \leq n - 1), \ \sum_{i=1}^{n-1} u_i \leq 1 \right\}
\]

and \( \mathbf{u} = (u_1, u_2, \ldots, u_n) \), where \( u_n = 1 - \sum_{i=1}^{n-1} u_i \).

Let \( du = du_1 \ldots du_{n-1} \) denote the differential of the volume in \( E_{n-1} \), and \( \mu \) be a probability measure on \( E_{n-1} \).

The power mean of order \( r \) of positive numbers \( x_1, x_2, \ldots, x_n \) with weights \( u_1, u_2, \ldots, u_n \) is

\[
M_r(\mathbf{x}; \mathbf{u}) = \begin{cases} 
\left( \sum_{i=1}^{n} u_i x_i^r \right)^{\frac{1}{r}} & \text{if } r \neq 0, \\
\prod_{i=1}^{n} x_i^{u_i} & \text{if } r = 0.
\end{cases}
\]

The logarithmic mean \( L(x, y) \) of positive numbers \( x \) and \( y \) is

\[
L(x, y) = \begin{cases} 
\frac{x - y}{\ln x - \ln y} & \text{if } x \neq y, \\
x & \text{if } x = y.
\end{cases}
\]
Following Carlson’s integral representation [4] for logarithmic mean

\[ L(x, y) = \left[ \int_0^1 \frac{1}{tx + (1-t)y} \, dt \right]^{-1}, \]

Pittenger [11] studied unweighted logarithmic mean in \( n \) variables:

\[ L(x_1, x_2, \ldots, x_n) = \left[ (n-1)! \int_{E_{n-1}} \left( \sum_{i=1}^n u_i x_i \right)^{-1} \, du \right]^{-1}. \quad (3) \]

Further generalization of \( L \) are the weighted logarithmic mean

\[
L_r(x; \mu) = \begin{cases} 
\left[ \int_{E_{n-1}} \left( \sum_{i=1}^n x_i u_i \right)^r d\mu(u) \right]^{\frac{1}{r}} & \text{if } r \neq 0, \\
\exp \left( \int_{E_{n-1}} \ln \left( \sum_{i=1}^n x_i u_i \right) d\mu(u) \right) & \text{if } r = 0,
\end{cases}
\quad (4)
\]

so that, for \( d\mu(u) = (n-1)!du, \quad L_{-1}(x; \mu) = L(x_1, x_2, \ldots, x_n). \)

Neuman [8] gives another integral formula for logarithmic mean

\[ L(x, y) = \int_0^1 x^t y^{1-t} \, dt, \]

and considers the weighted logarithmic mean of several numbers:

\[ \mathcal{L}(x; \mu) = \int_{E_{n-1}} \prod_{i=1}^n x_i^{u_i} \, d\mu(u). \quad (5) \]

Stolarsky mean [12], for distinct positive numbers \( x \) and \( y \) is

\[
E_{r,s}(x, y) = \begin{cases} 
\left[ \frac{r y^s - x^s}{s y^r - x^r} \right]^{\frac{1}{r-s}} & \text{if } r(s-r) \neq 0, \\
\left[ \frac{1}{r \ln y - \ln x} \right]^{\frac{1}{s}} & \text{if } r \neq 0, \quad s = 0, \\
e^{-1/r} \left( \frac{x^r}{y^s} \right)^{s-r} & \text{if } s = r \neq 0, \\
\sqrt{xy} & \text{if } r = s = 0
\end{cases}
\quad (6)
\]

and

\[ E_{r,s}(x, x) = x. \]
An integral representation for Stolarsky mean (6) is given in [10]. Alzer ([1], [2]) considers generalized logarithmic mean \( F_r(x, y) \) as a special case of the Stolarsky mean for two positive numbers \( x \) and \( y \):

\[
F_r(x, y) = \begin{cases} 
\frac{r x^{r+1} - y^{r+1}}{r + 1} & \text{if } r \neq 0, 1, \ x \neq y, \\
\frac{x - y}{\ln x - \ln y} & \text{if } r = 0, \ x \neq y, \\
\frac{\ln x - \ln y}{x - y} & \text{if } r = -1, \ x \neq y, \\
x & \text{if } x = y,
\end{cases}
\]

so that \( L(x, y) = F_0(x, y) \).

Set \( m_{r,x}(u) = M_r(x; u) \). An integral representation for (7) is given in [9]. It can be verified easily that

\[
F_r(x, y) = \int_0^1 m_{r,(x,y)}(t) \ dt. \tag{8}
\]

Also, in [9] a multidimensional weighted generalized logarithmic mean is defined as

\[
F_r(x; \mu) = \int_{E_{n-1}} m_{r,x}(u) \ d\mu(u) = \begin{cases} 
\int_{E_{n-1}} \left( \sum_{i=1}^n u_i x_i^r \right)^{\frac{1}{r}} d\mu(u) & \text{if } r \neq 0, \\
\int_{E_{n-1}} \prod_{i=1}^n x_i^{u_i} d\mu(u) & \text{if } r = 0,
\end{cases}
\]

so that \( F_0(x; \mu) = L(x; \mu) \).

2. The weighted Stolarsky-Tobey mean

In the previous part of the paper we listed some mean values which have been discussed by various authors, mostly since 1975. It should be noted, however, that all the means mentioned are special cases of a two-parameter homogeneous mean value, which has been studied by Tobey ([13], 1967). With some change of notation, his definition is:

**Definition 1.** For \( x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}_+^n \) and \( r, t \in \mathbb{R} \), the two-parameter homogeneous mean value is

\[
L_{r,t}(x; \mu) = \overline{M}_t(m_{r,x}(u); \mu), \tag{10}
\]

where \( \overline{M}_t \) is an integral power mean.

The integral power mean ([6], Chapter 3) of a positive real function \( f \) on \( E_{n-1} \) with probability measure \( \mu \) on \( E_{n-1} \) is

\[
\overline{M}_t(f; \mu) = \begin{cases} 
\left[ \int_{E_{n-1}} (f(u))^t \ d\mu(u) \right]^{\frac{1}{t}} & \text{if } t \neq 0, \\
\exp \left( \int_{E_{n-1}} \ln(f(u)) \ d\mu(u) \right) & \text{if } t = 0,
\end{cases}
\]

(11)
where we suppose that all expressions are well defined.

Now, a weighted Stolarsky-Tobey mean in \( n \) variables will be defined.

**Definition 2.** For \( \mathbf{x} = (x_1, x_2, \ldots, x_n) \in \mathbb{R}_{+}^{n} \) and \( r, s \in \mathbb{R} \), the Stolarsky-Tobey mean is

\[
E_{r,s}(\mathbf{x}; \mu) = L_{r,s-r}(\mathbf{x}, \mu),
\]

that is

\[
E_{r,s}(\mathbf{x}; \mu) = \begin{cases} 
\left[ \int_{E_{n-1}} \left( \sum_{i=1}^{n} u_i x_i^s \right)^{\frac{s-r}{r}} d\mu(u) \right]^{\frac{1}{s-r}}, & \text{if } r(s-r) \neq 0, \\
\exp \left( \int_{E_{n-1}} \ln \left( \sum_{i=1}^{n} u_i x_i^s \right) d\mu(u) \right), & \text{if } s = r \neq 0, \\
\left[ \int_{E_{n-1}} \left( \prod_{i=1}^{n} x_i^{u_i} \right)^s d\mu(u) \right]^{\frac{1}{s}}, & \text{if } r = 0, s \neq 0, \\
\exp \left( \int_{E_{n-1}} \ln(\prod_{i=1}^{n} x_i^{u_i}) d\mu(u) \right), & \text{if } s = r = 0. 
\end{cases}
\]

**Remark 2.1.** The Stolarsky-Tobey mean includes weighted logarithmic means \( L_r(\mathbf{x}; \mu) \) and \( \mathcal{F}_r(\mathbf{x}; \mu) \) defined in (4) and (9), ie.

\[
\begin{align*}
(i) \quad L_r(\mathbf{x}; \mu) &= E_{1,r+1}(\mathbf{x}; \mu) \quad \text{and} \quad E_{1,0}(\mathbf{x}; \mu) = L(\mathbf{x}; \mu); \\
(ii) \quad \mathcal{F}_r(\mathbf{x}; \mu) &= E_{r,r+1}(\mathbf{x}; \mu) \quad \text{and} \quad E_{0,1}(\mathbf{x}; \mu) = \mathcal{L}(\mathbf{x}; \mu).
\end{align*}
\]

Following ( [13], Theorem 1.-3.), we list below some properties which are peculiar for the mean \( E_{r,s}(\mathbf{x}; \mu) \). The next properties for \( E_{r,s}(\mathbf{x}; \mu) \) follow from the definition:

\[
\begin{align*}
(i) \quad \lim_{r \to \infty} E_{r,s}(\mathbf{x}; \mu) &= \lim_{s \to \infty} E_{r,s}(\mathbf{x}; \mu) = x_{\max}, \\
(ii) \quad \lim_{r \to -\infty} E_{r,s}(\mathbf{x}; \mu) &= \lim_{s \to -\infty} E_{r,s}(\mathbf{x}; \mu) = x_{\min}, \\
(iii) \quad \lim_{s \to 0} E_{r,s}(\mathbf{x}; \mu) &= E_{r,r}(\mathbf{x}; \mu); \quad \lim_{r \to 0} E_{r,s}(\mathbf{x}; \mu) = E_{0,s}(\mathbf{x}; \mu); \\
(iv) \quad \lim_{r \to 0} E_{r,r}(\mathbf{x}; \mu) &= E_{0,0}(\mathbf{x}; \mu); \\
(v) \quad E_{t,s}(\mathbf{x}; \mu) &= E_{r,t/r,s/t}(\mathbf{x}'; \mu), \quad t \neq 0,
\end{align*}
\]

where \( \mathbf{x}' = (x'_1, x'_2, \ldots, x'_n) \).

By application of L'Hôpital’s rule, it is not difficult to calculate the limits in (iii), and (iv). Differentiation with respect to \( r \) and \( s \) under the integral sign is allowed. Thus \( E_{r,s}(\mathbf{x}; \mu) \) is continuous in both \( r \) and \( s \).

The next properties for \( E_{r,s}(\mathbf{x}; \mu) \) follow from properties of mean (1) and (11), see [6]:

\[
\begin{align*}
(vi) \quad \min\{x_i : 1 \leq i \leq n\} \leq E_{r,s}(\mathbf{x}; \mu) \leq \max\{x_i : 1 \leq i \leq n\}; \\
(vii) \quad E_{r,s}(\mathbf{x}, \mathbf{x}, \ldots, \mathbf{x}; \mu) = \mathbf{x}, \quad x > 0;
\end{align*}
\]
(viii) \( E_{r,s}(tx; \mu) = tE_{r,s}(x; \mu), \ t > 0, \) where \( tx := (tx_1, \ldots, tx_n); \)

(ix) If \( x_i \leq y_i \) for all \( i \) and \( x_j < y_j \) for some \( j \), then \( E_{r,s}(x; \mu) < E_{r,s}(y; \mu), \) i.e. \( E_{r,s} \) is a strictly increasing function of \( x. \)

(x) If \( x_{\text{min}} < x_{\text{max}}, \) then \( E_{r,s} \) is strictly increasing function of \( s. \)

(xi) If \( x_{\text{min}} < x_{\text{max}} \) and \( s_1 - s_2 < r_1 - r_2 < 0, \) then \( E_{r_1,s_1}(x; \mu) < E_{r_2,s_2}(x; \mu). \)

Denote by

\[
    w_i = \int_{E_{n-1}} u_i d\mu(u) \quad (1 \leq i \leq n)
\]

the \( i \)-th weight associated to probability measure \( \mu \) on \( E_{n-1}. \) Clearly, \( w_i > 0 \) (1 \( \leq i \leq n) \) and \( w_1 + \cdots + w_n = 1. \)

The following theorem shows that many means are special case of the \( E_{r,s}. \)

**Theorem 2.2.** If \( w_i \) denotes weights defined by (14) and \( w = (w_1, \ldots, w_n), \) then

\[
    E_{r,2r}(x; \mu) = M_r(x; w)
\]

is a power mean, and

\[
    E_{1,2}(x; \mu) = A(x; w), \quad E_{0,0}(x; \mu) = G(x; w), \quad E_{-1,-2}(x; \mu) = H(x; w),
\]

are the weighted arithmetic, geometric and harmonic means, respectively.

**Proof.** If \( r \neq 0 \) then

\[
    E_{r,2r}(x; \mu) = \left[ \int_{E_{n-1}} \left( \sum_{i=1}^{n} u_i x_i^r \right) d\mu(u) \right]^{\frac{1}{r}} = \left[ \sum_{i=1}^{n} x_i^r \int_{E_{n-1}} u_i d\mu(u) \right]^{\frac{1}{r}}
\]

\[
    = \left[ \sum_{i=1}^{n} x_i^r w_i \right]^{\frac{1}{r}} = M_r(x; w).
\]

For \( r = 0 \) we have

\[
    E_{0,0}(x; \mu) = \exp \left( \int_{E_{n-1}} \ln \left( \prod_{i=1}^{n} x_i^{w_i} \right) d\mu(u) \right)
\]

\[
    = \exp \left( \int_{E_{n-1}} \sum_{i=1}^{n} u_i \ln(x_i) \ d\mu(u) \right)
\]

\[
    = \exp \left( \sum_{i=1}^{n} \ln(x_i) \int_{E_{n-1}} u_i \ d\mu(u) \right)
\]

\[
    = \exp \left( \sum_{i=1}^{n} w_i \ln(x_i) \right) = \prod_{i=1}^{n} x_i^{w_i}.
\]

\( \square \)
3. Inequalities for Stolarsky-Tobey mean

In the first theorem of this section we will use the following lemma ([7], Theorem 4.2)

**Lemma 3.1.** If $f$ is a convex function on a closed interval $[a, b]$ and $w_i$ are weights defined in (14), then for $z_1, z_2, \ldots, z_n \in [a, b]$,

$$ \int_{E_{n-1}} f \left( \sum_{i=1}^{n} u_i z_i \right) d\mu(u) \leq \sum_{i=1}^{n} w_i f(z_i). \quad (15) $$

The inequalities in (15) are strict provided $f$ is not a polynomial of degree one or less and they are reversed if $f$ is concave on $[a, b]$.

**Theorem 3.2.** (Jensen) Let $E_{r,s}(x; \mu)$ be a Stolarsky-Tobey mean defined in (13) and let $w_i$ be weights defined in (14). Then

$$ M_r(x; w) \leq E_{r,s}(x; \mu) \leq M_s - r(x; w), \quad \text{if} \quad s \geq 2r $$

and reversed inequalities hold if $s \leq 2r$.

**Proof.** For $f(t) = t^{s-r}$, $r(s-r) \neq 0$, applying (15) the result is obtained. Other cases for $r(s-r) = 0$ follow from continuity of $E_{r,s}$ in $r$ and $s$. \hfill \Box

**Corollary 3.3.** For weighted logarithmic means $L_r(x; \mu)$ and $F_r(x; \mu)$ defined in (4) and (9) we have

(i) $M_1(x; w) \leq L_r(x; \mu) \leq M_r(x; w)$ \quad \text{if} \quad r \geq 1,$

and reversed inequalities hold if $r \leq 1$. For $r = -1$ we have

$$ H(x, w) \leq L_{-1}(x; \mu) = L(x; \mu) \leq A(x, w) $$

where $L(x; \mu)$ is the logarithmic mean defined in (3).

(ii) $M_1(x; w) \leq F_r(x; \mu) \leq M_r(x; w)$ \quad \text{if} \quad r \geq 1,$

and reversed inequalities hold if $r \leq 1$. For $r = 0$ we have

$$ G(x, w) \leq F_0(x; \mu) = \mathcal{L}(x; \mu) \leq A(x, w), $$

where $\mathcal{L}(x; \mu)$ is the logarithmic mean defined in (5).

**Theorem 3.4.** Let $\alpha$ and $\beta$ be real positive numbers with $\alpha + \beta = 1$. For $r, s \in \mathbb{R}$ we have

$$ E_{r,s}(x^\alpha y^\beta; \mu) \leq E_{r,s}(\alpha x + \beta y; \mu), $$

where

$$ x^\alpha y^\beta = (x_1^\alpha y_1^\beta, \ldots, x_n^\alpha y_n^\beta) $$

and

$$ \alpha x + \beta y = (\alpha x_1 + \beta y_1, \ldots, \alpha x_n + \beta y_n). $$
Proof. By the arithmetic-geometric inequality
\[ x^\alpha y^\beta \leq \alpha x + \beta y, \quad x > 0, y > 0, \]
and the fact that \( E_{r,s}(x; \mu) \) is increasing in \( x \), the following is obtained
\[ E_{r,s}(x^\alpha y^\beta; \mu) \leq E_{r,s}(\alpha x + \beta y; \mu). \]

\[ \square \]

The following three theorems are the same as in ([13], Theorem 5.-7.). We give them without proofs. We use the notation
\[ x + y = (x_1 + y_1, \ldots, x_n + y_n) \quad \text{and} \quad xy = (x_1y_1, \ldots, x_ny_n). \]

**Theorem 3.5.** (Minkowski) Let \( x = (x_1, x_2, \ldots, x_n) \) and \( y = (y_1, y_2, \ldots, y_n) \) be \( n \)-tuples with \( x_i > 0 \) and \( y_i > 0, 1 \leq i \leq n \). Then, unless \( r = 1, s = 2 \) or \( x_i = ky_i, 1 \leq i \leq n \) we have
\[ E_{r,s}(x + y; \mu) < E_{r,s}(x; \mu) + E_{r,s}(y; \mu) \quad (\text{for} \quad 1 \leq r \leq s - 1), \]
with reversed inequality if \( s - 1 \leq r \leq 1 \). Equality holds in the exceptional cases.

**Corollary 3.6.** For weighted logarithmic means \( L_r(x; \mu) \) and \( F_r(x; \mu) \) defined in (4) and (9) we have
\begin{enumerate}
  \item[(i)] \( L_r(x + y; \mu) < L_r(x; \mu) + L_r(y; \mu) \quad (\text{for} \quad r > 1), \)
  with reversed inequality if \( r < 1; \)
  \item[(ii)] \( F_r(x + y; \mu) < F_r(x; \mu) + F_r(y; \mu) \quad (\text{for} \quad r > 1), \)
  with reversed inequality if \( r < 1. \)
\end{enumerate}
Equalities hold for \( r = 1 \) or \( x_i = ky_i, (1 \leq i \leq n); \)

**Theorem 3.7.** (Hölder) Let \( x = (x_1, \ldots, x_n), \ y = (y_1, \ldots, y_n) \) be \( n \)-tuples with \( x_i > 0 \) and \( y_i > 0, 1 \leq i \leq n \) and let \( \alpha, \beta \) be real positive numbers such that \( \alpha + \beta = 1 \). Then, unless \( s = r = 0 \) or \( x_i^{1/\alpha} = ky_i^{1/\beta}, 1 \leq i \leq n \)
\[ E_{r,s}(xy; \mu) < \left[ E_{r,s}(x^{1/\alpha}; \mu) \right]^\alpha \left[ E_{r,s}(y^{1/\beta}; \mu) \right]^\beta \quad (\text{for} \quad s \geq r \geq 0), \]
with reversed inequality if \( s \leq r \leq 0 \). Equality holds in the exceptional cases.

**Corollary 3.8.** For weighted logarithmic means \( L_r(x; \mu) \) and \( F_r(x; \mu) \) defined in (4) and (9) we have
\begin{enumerate}
  \item[(i)] \( L_r(xy; \mu) < \left[ L_r(x^{1/\alpha}; \mu) \right]^\alpha \left[ L_r(y^{1/\beta}; \mu) \right]^\beta \quad (\text{for} \quad r \geq 0); \)
  \item[(ii)] \( F_r(xy; \mu) < \left[ F_r(x^{1/\alpha}; \mu) \right]^\alpha \left[ F_r(y^{1/\beta}; \mu) \right]^\beta \quad (\text{for} \quad r \geq 0) \).
\end{enumerate}
Equalities hold for \( x_i^{1/\alpha} = ky_i^{1/\beta}. \)
THEOREM 3.9. (Rennie) Let $x = (x_1, x_2, \ldots, x_n)$ and $0 \leq A \leq x_i \leq B$, $1 \leq i \leq n$, then if $s \neq r$

$$E_{rs}^r(x; \mu) + A^{s-r}B^{s-r}E_{rs-2r}^r(x^{-1}; \mu) \leq A^{s-r} + B^{s-r},$$

with equality if and only if $x_i = A$ or $x_i = B$, $1 \leq i \leq n$.

COROLLARY 3.10. For weighted logarithmic means $L_r(x; \mu)$ and $F_r(x; \mu)$ defined in (4) and (9) we have

(i) $L_r(x; \mu) + A^{r}B^{r}L_{-r}(x; \mu) \leq A^{r} + B^{r}$ if $r \neq 0$;

(ii) $F_r(x; \mu) + ABF_{-r}(x^{-1}; \mu) \leq A + B$.

Equalities hold if and only if $x_i = A$ or $x_i = B$, $1 \leq i \leq n$.

THEOREM 3.11. (Kantorovich) Let $x = (x_1, x_2, \ldots, x_n)$ and $0 < A \leq x_i \leq B$, $1 \leq i \leq n$. Then for $s > r$ we have

$$1 \leq E_{rs}(x; \mu)E_{-r,s-2r}(x^{-1}; \mu) \leq \left[ A^{s-r} + B^{s-r} \right]^{1/(s-r)} \left[ A^{-r} + B^{-r} \right]^{1/(s-r)}$$,

with equality on the left if and only if $x_{\text{max}} = x_{\text{min}}$, and equality on the right if and only if $A = B$. If $s < r$ we have reversed inequality.

COROLLARY 3.12. For weighted logarithmic means $L_r(x; \mu)$ and $F_r(x; \mu)$ defined in (4) and (9) we have

(i) $1 \leq L_r(x; \mu)L_{-r}(x; \mu) \leq \left[ A^{r} + B^{r} \right]^{1/r} \left[ A^{-r} + B^{-r} \right]^{1/r}$ if $r > 0$,

and reversed inequality holds if $r < 0$;

(ii) $1 \leq F_r(x; \mu)F_{-r}(x^{-1}; \mu) \leq \frac{(A + B)^2}{4AB},$

The equalities hold on the left if and only if $x_{\text{max}} = x_{\text{min}}$, and equalities on the right if and only if $A = B$.

4. Stolarsky-Tobey mean and Dirichlet measure

An important example of probability measure on simplex $E_{n-1}$ is Dirichlet measure $\mu_b$ ([5], Sec.4.4), because it is connected with hypergeometric functions.

Let $b = (b_1, b_2, \ldots, b_n) \in \mathbb{R}_+^n$, $n \geq 2$. Beta function of $n$ variables is defined by

$$B(b) := B(b_1, b_2, \ldots, b_n) = \frac{\Gamma(b_1) \cdots \Gamma(b_n)}{\Gamma(b_1 + \cdots + b_n)},$$

where $\Gamma$ is gamma function.

An integral representation for $B$ ([5], Sec.4.3) is

$$B(b_1, b_2, \ldots, b_n) = \int_{E_{n-1}} \prod_{i=1}^{n} u_i^{b_i-1} du_1 \cdots du_{n-1},$$
where \( u_n = 1 - \sum_{i=1}^{n-1} u_i \).

The Dirichlet measure \( \mu_b \) is defined by
\[
d\mu_b(u) = \frac{1}{B(b)} \prod_{i=1}^{n} u_i^{b_i-1} du_1 \cdots du_{n-1},
\]
so that
\[
\int_{E_{n-1}} d\mu_b(u) = 1.
\]

A simple calculation gives
\[
w_i = \int_{E_{n-1}} u_i d\mu_b(u) = \frac{b_i}{\sum_{i=1}^{n} b_i}, \quad 1 \leq i \leq n.
\]

For \( z = (z_1, \ldots, z_n) \in \mathbb{R}_+^n \) the hypergeometric \( R \)-function ([5], Sec.5.9) has an integral representation
\[
R_t(z; b) := R(z_1, \ldots, z_n; b_1, \ldots, b_n) = \int_{E_{n-1}} (\sum_{i=1}^{n} u_i z_i)^t d\mu_b(u)
\]
and its derivative with respect to \( t \) is a
\[
D_t(z; b) := D_t(z_1, \ldots, z_n; b_1, \ldots, b_n) = \int_{E_{n-1}} (\sum_{i=1}^{n} u_i z_i)^t \ln(\sum_{i=1}^{n} u_i z_i) d\mu_b(u).
\]

The confluent hypergeometric \( S \)-function ([5], Sec.5.8) has an integral representation
\[
S(z; b) := S(z_1, \ldots, z_n; b_1, \ldots, b_n) = \int_{E_{n-1}} \exp\left(\sum_{i=1}^{n} u_i z_i\right) d\mu_b(u).
\]

We will write \( E_{r,s}(x; b) \) instead
\[
E_{r,s}(x; b) := E_{r,s}(x_1, \ldots, x_n; b_1, \ldots, b_n).
\]

On the basis of the facts mentioned above, Stolarsky-Tobey mean generated by the Dirichlet measure \( \mu_b \) can be expressed as
\[
E_{r,s}(x; b) = \begin{cases}
R_t^{1/(s-r)}(x^r; b) & \text{if } r(s-r) \neq 0, \\
\exp\left[\frac{1}{r} D_0(x^r; b)\right] & \text{if } s = r \neq 0,
\end{cases}
\]
\[
S(\ln x^s; b) & \text{if } r = 0, s \neq 0,
\]
\[
\exp[S(\ln x, b)] & \text{if, } r = s = 0,
\]
where \( x^r = (x_1^r, x_2^r, \ldots, x_n^r) \) and
\[
\ln(x^s) = (\ln(x_1^s), \ln(x_2^s), \ldots, \ln(x_n^s)).
\]
For logarithmic means defined by (4) and (9) we have

$$L_r(x; b) = \begin{cases} R_{1/r}^1(x; b) & \text{if } r \neq 0 \\ \exp[D_0(x; b)] & \text{if } r = 0. \end{cases}$$  \tag{17}

$$F_r(x; b) = \begin{cases} R_{1/r}^1(x^r; b) & \text{if } r \neq 0 \\ S(\ln x^{r+1}; b) & \text{if } r = 0. \end{cases}$$  \tag{18}

The relation between Stolarsky-Tobey mean and hypergeometric functions makes possible that some properties for $E_{r,s}(x; b)$ are derived from the known results for hypergeometric functions. The following theorem is an example of this. From \cite{3}, Theorem 1 and from the fact that $E_{r,s}$ is continuous in both $s$ and $r$ we have

**THEOREM 4.1.** Let $c, w_1, w_2, \ldots, w_n$ be strictly positive real numbers. Let us assume that $\sum_{i=1}^n w_i = 1$, and define $cw = (cw_1, cw_2, \ldots, cw_n)$. Then

$$\lim_{c \to 0} E_{r,s}(x; cw) = M_{s-r}(x; w) \quad \text{and} \quad \lim_{c \to \infty} E_{r,s}(x; cw) = M_r(x; w).$$

**COROLLARY 4.2.** Let $c$, and $cw$ be as in the previous theorem and let $L_r$ and $F_r$ be means defined in (4) and (9), respectively. Then, we have

$$\lim_{c \to 0} F_r(x; cw) = M_1(x; w) = A(x; w), \quad \text{and} \quad \lim_{c \to \infty} F_r(x; cw) = M_r(x; w);$$

$$\lim_{c \to 0} L_r(x; cw) = M_r(x; w), \quad \text{and} \quad \lim_{c \to \infty} L_r(x; cw) = M_1(x; w) = A(x; w).$$

The next theorem shows that the multiple integral in the definition of Stolarsky-Tobey mean $(13)$ generated by $\mu_b$ can be reduced to a single integral.

**THEOREM 4.3.** Let $a = (b_1, \ldots, b_{n-1}) \in \mathbb{R}^{n-1}$, $\alpha = b_1 + \cdots + b_{n-1}, \beta = b_n$ and let $\mu_{(\alpha, \beta)}$ be Dirichlet measure on $E_1$, that is

$$d\mu_{(\alpha, \beta)}(v) = \frac{v^{\alpha-1}(1-v)^{\beta-1}}{B(\alpha, \beta)} \, dv$$

and for $i = 1, 2, \ldots, n-1$ we define

$$y_i = \begin{cases} [x_i^r + (1-v)x_n^r]^{1/r} & \text{if } r \neq 0 \\ x_i^r x_n^{1-v} & \text{if } r = 0. \end{cases}$$

Then

$$E_{r,s}(x; b) = \begin{cases} \left( \int_0^1 E_{r,s}^{1/r}(y; a) \mu_{(\alpha, \beta)}(v) \, dv \right)^{1/s} & \text{if } r(s - r) \neq 0, \\ \exp \left( \int_0^1 \ln E_{r,s}(y; a) \mu_{(\alpha, \beta)}(v) \, dv \right) & \text{if } s = r \neq 0, \\ \left( \int_0^1 E_{0,s}(y; a) \mu_{(\alpha, \beta)}(v) \, dv \right)^{1/s} & \text{if } r = 0, \, s \neq 0, \\ \exp \left( \int_0^1 \ln E_{0,0}(y; a) \mu_{(\alpha, \beta)}(v) \, dv \right) & \text{if } s = r = 0. \end{cases}$$
Proof. Let \( v = \sum_{i=1}^{n-1} u_i \) and \( p_i = \frac{u_i}{v}, \ i = 1, \ldots, n-1 \). Then \( u_n = 1 - v \) and \( \sum_{i=1}^{n-1} p_i = 1 \). For this substitution Jacobian is \( v^{n-2} \). We will use the notation \( dp = dp_1 dp_2 \cdots dp_{n-2} \) for the differential of the volume in \( E_{n-2} \).

A simple computation gives

\[
B(b) = B(a)B(\alpha, \beta).
\]

If \( r(s - r) \neq 0 \) then

\[
E^{s-r}_{r,s}(x; b) = \int_{E_{n-1}} \left( \sum_{i=1}^{n} u_i x_i \right)^{\frac{s-r}{r}} \ d\mu_b(u)
\]

\[
= \frac{1}{\int_{E_{n-2}} \left( \sum_{i=1}^{n-1} v p_i x_i^r + (1 - v) x_n^r \right)^{\frac{s-r}{r}} \prod_{i=1}^{n-1} (vp_i)^{b_i-1} (1 - v)^{b_n-1}} \ B(b) v^{n-2} \ dv \ dp
\]

\[
= \int_{E_{n-2}} \left[ \int_{E_{n-2}} \left( \sum_{i=1}^{n-1} p_i y_i^r \right)^{\frac{s-r}{r}} \ d\mu_a(p) \right] \mu(\alpha, \beta)(v) \ dv
\]

\[
= \int_0^1 E^{s-r}_{r,s}(y; a) \mu(\alpha, \beta)(v) \ dv.
\]

For \( s = r \neq 0 \) we have

\[
\ln \left[ E_{r,r}(x; b) \right] = \int_{E_{n-1}} \ln \left( \sum_{i=1}^{n} u_i x_i \right)^{1/r} \ d\mu_b(u)
\]

\[
= \frac{1}{\int_{E_{n-2}} \ln \left( \sum_{i=1}^{n-1} (vp_i x_i^r + (1 - v) x_n^r) \right)^{1/r} \prod_{i=1}^{n-1} (vp_i)^{b_i-1} (1 - v)^{b_n-1}} \ B(b) v^{n-2} \ dv \ dp
\]

\[
= \int_{E_{n-2}} \left[ \int_{E_{n-2}} \ln \left( \sum_{i=1}^{n-1} p_i y_i^r \right)^{1/r} \ d\mu_a(p) \right] \mu(\alpha, \beta)(v) \ dv
\]

\[
= \int_0^1 \ln [E_{r,r}(y; a)] \mu(\alpha, \beta)(v) \ dv.
\]
For \( r = 0 \) and \( s \neq 0 \) we have

\[
[E_{0,s}(x; b)]^s = \int_{E_{n-1}} \left( \prod_{i=1}^{n} x_i^{p_i} \right)^s d\mu_b(u) = \\
= \frac{1}{n!} \int_{E_{n-2}} \left( \prod_{i=1}^{n} x_i^{p_i} \right)^s \left( \prod_{i=1}^{n-1} (y_i^{p_i})^{b_i-1} \right) \left( \prod_{i=1}^{n-1} \frac{1}{B(b)} \right) v^{n-2} dp \\
= \int_{E_{n-2}} \left( \prod_{i=1}^{n} y_i^{p_i} \right)^s \mu_a(p) \mu(\alpha, \beta)(v) dv = \\
= \int E_{0,s}(y; a) \mu(\alpha, \beta)(v) dv.
\]

Finally, we have

\[
\ln[E_{0,0}(x; b)] = \int_{E_{n-1}} \ln \left( \prod_{i=1}^{n} x_i^{p_i} \right) d\mu_b(u) = \\
= \frac{1}{n!} \int_{E_{n-2}} \ln \left( \prod_{i=1}^{n} x_i^{p_i} \right) \left( \prod_{i=1}^{n-1} (y_i^{p_i})^{b_i-1} \right) \left( \prod_{i=1}^{n-1} \frac{1}{B(b)} \right) v^{n-2} dp \\
= \int_{E_{n-2}} \ln \left( \prod_{i=1}^{n} y_i^{p_i} \right) \mu_a(p) \mu(\alpha, \beta)(v) dv = \\
= \int \ln[E_{0,0}(y; a)] \mu(\alpha, \beta)(v) dv.
\]

In the rest of this section, several relations which are peculiar to the important case \( n = 2 \) will be established. In this case, many of the most important special functions can be represented as R-function or S-function ([5], Sec.5), in some cases as Stolarsky-Tobey means. Here are some examples. We use Appell’s symbol for shifted factorial:

\[(a, 0) = 1, (a, n) = a(a + 1)(a + 2) \cdots (a + n - 1), n \geq 1.\]

Let us recall that ([5], Sec.2) \(_2F_1\) is the Gauss’s hypergeometric function represented by hypergeometric series as

\[_2F_1(a, b; c; z) = \sum_{m=0}^{\infty} \frac{(a, m)(b, m)}{(c, m)} \frac{z^m}{m!}, \quad |z| < 1, \quad c \neq 0, -1, -2, \ldots \]
and $1_F_1$ is the Kummer’s confluent hypergeometric function represented by hypergeometric series as

$$1_F_1(a, b, z) = \sum_{m=0}^{\infty} \frac{(a, m) z^m}{(b, m) m!}, \quad b \neq 0, -1, -2, \ldots$$

Applying results from ([5], Sec.5) on (16) we obtain the following theorem.

**THEOREM 4.4.** Let $x = (x, y) \in \mathbb{R}_+^2$, $b = (\alpha, \beta) \in \mathbb{R}_+^2$ and $x < y$

(i) For $s \neq r(n+1), n \in \mathbb{N}$ we have

$$E_{r,s}(x, y; \alpha, \beta) = \begin{cases} y^{s-r} 2_F_1 \left( -\frac{s-r}{r}, \alpha; \alpha + \beta; 1 - \frac{y}{r} \right) & \text{if } r > 0 \\ x^{s-r} 2_F_1 \left( -\frac{s-r}{r}, \alpha; \alpha + \beta; 1 - \frac{x}{r} \right) & \text{if } r < 0 \\ y^s 1_F_1 \left( \alpha; \alpha + \beta; s \ln \frac{y}{x} \right) & \text{if } r = 0; \end{cases}$$

(ii) For $s = r(n+1), n \in \mathbb{N}$ and $r \neq 0$ we have

$$E_{r,r(n+1)}(x, y; \alpha, \beta) = y^{rn} \sum_{m=0}^{n} \binom{n}{m} \frac{(\alpha, n-m)(\beta, m)}{\alpha + \beta, n} \left( \frac{x}{y} \right)^{rn};$$

(iii)

$$E_{0,0}(x, y; \alpha, \beta) = x^\alpha y^\beta.$$

**5. Explicit form for unweighted Stolarsky-Tobey mean**

When a measure $\mu_b$ reduces to Lebesgue measure, then

$$d\mu_b(u) = (n-1)! du_1 \cdots du_{n-1} = (n-1)! du.$$

A sample calculation gives

$$\int_{E_{n-1}} du_1 \cdots du_{n-1} = \frac{1}{(n-1)!} \quad \text{and} \quad w_i = \int_{E_{n-1}} u_i d\mu(u) = \frac{1}{n}.\$$

In this case, we will write $E_{r,s}(x)$ for $E_{r,s}(x; \mu).$ In the proof of the next theorem a well-known relations is used:

$$[t_1, t_2, \ldots, t_n] f = \sum_{i=1}^{n} \frac{f(t_i)}{\prod_{j=1}^{i-1}(t_i - t_j)} = \int_{E_{n-1}} f^{(n-1)} \left( \sum_{i=1}^{n} u_it_i \right) du, \quad (19)$$

where $[t_1, \ldots, t_n] f$ stands for the divided difference of order $n-1$ of $f$ with knots at $t_1, \ldots, t_n$, and $f \in C^{n-1}(a, b), a = \min(t_i), b = \max(t_i), 1 \leq i \leq n$. Also, the following lemma [11] will be used.
LEMA 5.1. Let \( a_1, \ldots, a_n \) be distinct, nonzero real numbers. For \( 0 \leq k \leq n - 2 \),
\[
\sum_{i=1}^{n} \frac{a_i^k}{\prod_{j=1}^{n} (a_i - a_j)} = 0.
\] (20)

THEOREM 5.2. For \( r, s \in \mathbb{R} \), \( \mathbf{x} = (x_1, x_2, \ldots, x_n) \in \mathbb{R}_+^n \) and \( x_i \neq x_j \) (for \( i \neq j \)) we have:

(i) \( E_{r,s}(\mathbf{x}) = \left[ \frac{(n-1)! r^{n-1}}{s(s+r) \cdots (s+(n-2)r)} \sum_{i=1}^{n} \frac{x_i^{s+(n-2)r}}{\prod_{j=1}^{n} (x_i - x_j)} \right]^{\frac{1}{s-r}} \), \( r \neq 0, s \neq -kr, 0 \leq k \leq n - 2 \);

(ii) \( E_{-kr, r}(\mathbf{x}) = \left[ (-1)^k (k+1) \binom{n-1}{k+1} \sum_{i=1}^{n} \frac{x_i^{(n-k-2)r} \ln(x_i)}{\prod_{j=1}^{n} (x_i - x_j)} \right]^{\frac{1}{(k+1)r}} \), \( r \neq 0, 0 \leq k \leq n - 2 \);

(iii) \( E_{0,s}(\mathbf{x}) = \left[ \frac{(n-1)!}{s^{n-1}} \sum_{i=1}^{n} \frac{x_i^s}{\prod_{j=1}^{n} \ln(x_i/x_j)} \right]^{\frac{1}{s}} \), \( s \neq 0 \);

(iv) \( E_{r,r}(\mathbf{x}) = \exp \left( \frac{1}{r} \sum_{i=1}^{n} \frac{x_i^{r(n-1)} \left( \ln x_i - \sum_{k=1}^{n-1} \frac{1}{k} \right)}{\prod_{j=1}^{n} (x_i - x_j)} \right), r \neq 0 \);

(v) \( E_{0,0}(\mathbf{x}) = \left( \prod_{i=1}^{n} x_i \right)^{\frac{1}{n}} \).

Proof. (i) For \( f(t) = t^{q+n-2} \), \( q = \frac{r}{s}, n \geq 2 \), we have
\[
f^{(n-1)}(t) = q(q+1) \cdots (q+n-2)t^{q+1}
\]
and
\[
E_{r,s}(\mathbf{x}) = \left[ \frac{(n-1)!}{q(q+1) \cdots (q+n-2)} \int_{E_{n-1}} f^{(n-1)} \left( \sum_{i=1}^{n} u_i x_i' \right) du \right]^{\frac{1}{s-r}} .
\]
Using (19) for \( t_i = x_i^r \) \((1 \leq i \leq n)\) we obtain \((i)\).

\((ii)\) From the explicit form \((i)\) for \( E_{r,s}(x) \) and using (20), we obtain a formula for \( E_{r,-kr} \) \((0 \leq k \leq n-2)\) as a limit:

\[
E_{r,-kr}(x) = \lim_{s \to -kr} E_{r,s}(x)
\]

\[
= \left[ \frac{(n-1)!}{q(q+1) \cdots (q+k) \cdots (q+n-2)} \sum_{i=1}^{n} \frac{x_i^{(q+n-2)r - x_i^{(q-k)r}}}{\prod_{j=1 \atop j \neq i}^{n} (x_i^r - x_j^r)} \right] \frac{1}{(k+1)r}.
\]

This completes the proof of \((ii)\). In particular, for \( k = 0 \) we obtain

\[
E_{r,0} = \left[ (n-1) \sum_{i=1}^{n} \frac{x_i^{(n-2)r} \ln(x_i)}{\prod_{j=1 \atop j \neq i}^{n} (x_i^r - x_j^r)} \right]^{-\frac{1}{s}} = L_{\frac{1}{s}}(x_1^r, x_2^r, \ldots, x_n^r). \quad (21)
\]

\((iii)\) Explicit form for \( E_{0,s}(x) \) we obtain as a limit

\[
E_{0,s}(x) = \lim_{r \to 0} E_{r,s}(x) = \left[ \frac{(n-1)!}{s^{n-1}} \sum_{i=1}^{n} \frac{x_i^{(n-2)r}}{\prod_{j=1 \atop j \neq i}^{n} (x_i^r - x_j^r)} \right]^{1/s}.
\]
(iv) 
\[ E_{r,r}(x) = \exp\left( (n-1)! \int_{E_{n-1}} \ln \left( \sum_{i=1}^{n} x_i^r u_i \right) du \right), \]
that is
\[ \ln(E_{r,r}(x)) = \frac{(n-1)!}{r} \int_{E_{n-1}} \ln \left( \sum_{i=1}^{n} x_i^r u_i \right) du. \]

For
\[ f(t) = \frac{t^{n-1}}{(n-1)!} \left( \ln(t) - \sum_{k=1}^{n-1} \frac{1}{k} \right), \quad n \geq 2, \]
we have \( f^{(n-1)}(t) = \ln t \) and from (19) we obtain (iv).

(v) 
\[ E_{0,0}(x) = \exp\left( (n-1)! \int_{E_{n-1}} \sum_{i=1}^{n} u_i \ln(x_i) du \right) = \exp\left( \sum_{i=1}^{n} w_i \ln(x_i) \right) = \exp\left( \sum_{i=1}^{n} \frac{\ln(x_i)}{n} \right) = \left( \prod_{i=1}^{n} x_i \right)^{1/n}. \]

\[\square\]

**Remark 5.3.** For \( r = 1 \) from (21) we have a explicit form for logarithmic mean in \( n \) variables given in [11].

**Remark 5.4.** For \( s = 1 \) from (iii) we have the result given in [8]:
\[ E_{0,1}(x) = \mathcal{L}(x) = (n-1)! \sum_{i=1}^{n} \frac{x_i}{\prod_{j=1}^{n} \ln(x_i/x_j)}. \]

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