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STOLARSKY–TOBEY MEAN IN n VARIABLES

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(*communicated by B. Carlson*)

Abstract. In this paper, an n–dimensional weighted Stolarsky–Tobey mean is defined via measure. This mean includes as special cases various generalizations of the logarithmic mean. Some elementary properties are listed and various inequalities derived. Attention is given to the case when the mean is specialized to Dirichlet measure. Relations to hypergeometric function are exhibited. An explicit form is given for the mean in the special case when all variables have equal weights.

1. Introduction and notation

Throughout the paper we assume that $\mathbf{R}$ is a set of real numbers and $\mathbf{R}_+$ is a set of strictly positive real numbers. Let $\mathbf{x}$ denote n-tuple $(x_1, x_2 \dots, x_n) \in \mathbf{R}_+^n$ and let $E_{n-1} \subset \mathbf{R}^{n-1}$ represent the simplex

$$E_{n-1} = \{(u_1, u_2, \dots, u_{n-1}) : \ u_i \geqslant 0 \ \ (1 \leqslant i \leqslant n-1), \ \sum_{i=1}^{n-1} u_i \leqslant 1\}$$

and $\mathbf{u} = (u_1, u_2, \dots, u_n)$, where $u_n = 1 - \sum_{i=1}^{n-1} u_i$.

Let $du = du_1 \dots du_{n-1}$ denote the differential of the volume in E_{n-1}, and μ be a probability measure on E_{n-1}.

The power mean of order r of positive numbers $x_1, x_2, \dots, x_n$ with weights $u_1, u_2, \dots, u_n$ is

$$M_r(\mathbf{x};\mathbf{u}) = \begin{cases} \left(\sum_{i=1}^n u_i x_i^r\right)^{\frac{1}{r}} & \text{if} \quad r \neq 0, \\ \prod_{i=1}^n x_i^{u_i} & \text{if} \quad r = 0. \end{cases} \tag{1}$$

The logarithmic mean $L(x, y)$ of positive numbers x and y is

$$L(x,y) = \begin{cases} \dfrac{x-y}{\ln x - \ln y} & \text{if} \quad x \neq y\,, \\ x & \text{if} \quad x = y. \end{cases} \tag{2}$$

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Following Carlson's integral representation [4] for logarithmic mean

$$L(x,y)=\left[\int_0^1 \frac{1}{tx+(1-t)y}\,dt\right]^{-1},$$

Pittenger [11] studied unweighted logarithmic mean in n variables:

$$L(x_1,x_2,\dots,x_n)=\left[(n-1)!\int_{E_{n-1}}\left(\sum_{i=1}^n u_ix_i\right)^{-1}du\right]^{-1}. \tag{3}$$

Further generalization of L are the weighted logarithmic mean

$$L_r(\mathbf{x};\mu)=\begin{cases}\left[\int_{E_{n-1}}\left(\sum_{i=1}^n x_iu_i\right)^r d\mu(u)\right]^{\frac{1}{r}} & \text{if } r\neq 0\,,\\ \exp\left(\int_{E_{n-1}}\ln\left(\sum_{i=1}^n x_iu_i\right)d\mu(u)\right) & \text{if } r=0,\end{cases} \tag{4}$$

so that, for $d\mu(u)=(n-1)!du,\quad L_{-1}(\mathbf{x};\mu)=L(x_1,x_2,\dots,x_n)$.

Neuman [8] gives another integral formula for logarithmic mean

$$L(x,y)=\int_0^1 x^ty^{1-t}\,dt,$$

and considers the weighted logarithmic mean of several numbers :

$$\mathcal{L}(\mathbf{x};\mu)=\int_{E_{n-1}}\prod_{i=1}^n x_i^{u_i}\,d\mu(u). \tag{5}$$

Stolarsky mean [12], for distinct positive numbers x and y is

$$E_{r,s}(x,y)=\begin{cases}\left[\frac{r}{s}\frac{y^s-x^s}{y^r-x^r}\right]^{\frac{1}{s-r}} & \text{if}\quad r(s-r)\neq 0,\\ \left[\frac{1}{r}\frac{y^r-x^r}{\ln y-\ln x}\right]^{\frac{1}{r}} & \text{if}\quad r\neq 0,\quad s=0,\\ e^{-1/r}\left(\frac{x^{x^r}}{y^{y^r}}\right)^{\frac{1}{x^r-y^r}} & \text{if}\quad s=r\neq 0,\\ \sqrt{xy} & \text{if}\quad r=s=0\end{cases} \tag{6}$$

and

$$E_{r,s}(x,x)=x.$$

An integral representation for Stolarsky mean (6) is given in [10]. Alzer ([1], [2]) considers generalized logarithmic mean $F_r(x,y)$ as a special case of the Stolarsky mean for two positive numbers x and y :

$$F_r(x,y)=\begin{cases}\dfrac{r}{r+1}\dfrac{x^{r+1}-y^{r+1}}{x^r-y^r} & \text{if} \quad r\neq 0,1, \quad x\neq y,\\ \dfrac{x-y}{\ln x-\ln y} & \text{if} \quad r=0, \quad x\neq y,\\ xy\dfrac{\ln x-\ln y}{x-y} & \text{if} \quad r=-1, \quad x\neq y,\\ x & \text{if} \quad x=y,\end{cases} \tag{7}$$

so that $L(x,y)=F_0(x,y)$.

Set $m_{r;\mathbf{x}}(\mathbf{u})=M_r(\mathbf{x};\mathbf{u})$. An integral representation for (7) is given in [9]. It can be verified easily that

$$F_r(x,y)=\int_0^1 m_{r,(x,y)}(t)\,dt. \tag{8}$$

Also, in [9] a multidimensional weighted generalized logarithmic mean is defined as

$$\mathcal{F}_r(\mathbf{x};\mu)=\int_{E_{n-1}} m_{r;\mathbf{x}}(\mathbf{u})\,d\mu(u)=\begin{cases}\displaystyle\int_{E_{n-1}}\left(\sum_{i=1}^n u_i x_i^r\right)^{\frac{1}{r}} d\mu(u) & \text{if} \quad r\neq 0,\\ \displaystyle\int_{E_{n-1}}\prod_{i=1}^n x_i^{u_i}\,d\mu(u) & \text{if} \quad r=0,\end{cases} \tag{9}$$

so that $\mathcal{F}_0(\mathbf{x};\mu)=\mathcal{L}(\mathbf{x};\mu)$.

2. The weighted Stolarsky-Tobey mean

In the previous part of the paper we listed some mean values which have been discussed by various authors, mostly since 1975. It should be noted, however, that all the means mentioned are special cases of a two-parameter homogeneous mean value, which has been studied by Tobey ([13], 1967). With some change of notation, his definition is:

Definition 1. For $\mathbf{x}=(x_1,x_2,\ldots,x_n)\in\mathbf{R}_+^n$ *and* $r,t\in\mathbf{R}$, the two-parameter homogeneous mean value is

$$\mathbf{L}_{r,t}(\mathbf{x};\mu)=\overline{M}_t(m_{r,\mathbf{x}}(\mathbf{u});\mu), \tag{10}$$

where $\overline{M}_t$ is a integral power mean.

The integral power mean ([6], Chapter 3) of a positive real function f on E_{n-1} with probability measure μ on E_{n-1} is

$$\overline{M}_t(f;\mu)=\begin{cases}\left[\displaystyle\int_{E_{n-1}}(f(u))^t\,d\mu(u)\right]^{\frac{1}{t}}, & \text{if } t\neq 0,\\ \exp\left(\displaystyle\int_{E_{n-1}}\ln(f(u))\,d\mu(u)\right), & \text{if } t=0,\end{cases} \tag{11}$$

where we suppose that all expressions are well defined.

Now, a weighted Stolarsky-Tobey mean in n variables will be defined.

Definition 2. For $\mathbf{x} = (x_1, x_2, \dots, x_n) \in \mathbf{R}^n_+$ *and* $r, s \in \mathbf{R}$, the Stolarsky-Tobey mean is

$$E_{r,s}(\mathbf{x};\mu) = \mathbf{L}_{r,s-r}(\mathbf{x},\mu), \tag{12}$$

that is

$$E_{r,s}(\mathbf{x};\mu) = \begin{cases} \left[\int_{E_{n-1}} \left(\sum_{i=1}^n u_i x_i^r\right)^{\frac{s-r}{r}} d\mu(u)\right]^{\frac{1}{s-r}}, & \text{if} \quad r(s-r) \neq 0, \\ \exp\left(\int_{E_{n-1}} \ln \left(\sum_{i=1}^n u_i x_i^r\right)^{\frac{1}{r}} d\mu(u)\right), & \text{if} \quad s = r \neq 0, \\ \left[\int_{E_{n-1}} \left(\prod_{i=1}^n x_i^{u_i}\right)^s d\mu(u)\right]^{\frac{1}{s}}, & \text{if} \quad r = 0, s \neq 0, \\ \exp\left(\int_{E_{n-1}} \ln(\prod_{i=1}^n x_i^{u_i})\, d\mu(u)\right), & \text{if} \quad s = r = 0. \end{cases} \tag{13}$$

Remark 2.1. The Stolarsky-Tobey mean includes weighted logarithmic means $L_r(\mathbf{x};\mu)$ and $\mathcal{F}_r(\mathbf{x};\mu)$ defined in (4) and (9), ie.

$$(i) \qquad L_r(\mathbf{x};\mu) = E_{1,r+1}(\mathbf{x};\mu) \quad \text{and} \quad E_{1,0}(\mathbf{x};\mu) = L(\mathbf{x};\mu);$$
$$(ii) \qquad \mathcal{F}_r(\mathbf{x};\mu) = E_{r,r+1}(\mathbf{x};\mu) \quad \text{and} \quad E_{0,1}(\mathbf{x};\mu) = \mathcal{L}(\mathbf{x};\mu) \quad .$$

Following ([13], Theorem 1.-3.), we list below some properties which are peculiar for the mean $E_{r,s}(\mathbf{x};\mu)$. The next properties for $E_{r,s}(\mathbf{x};\mu)$ follow from the definition:

$$(i) \qquad \lim_{r\to\infty} E_{r,s}(\mathbf{x};\mu) = \lim_{s\to\infty} E_{r,s}(\mathbf{x};\mu) = x_{max};$$
$$(ii) \qquad \lim_{r\to-\infty} E_{r,s}(\mathbf{x};\mu) = \lim_{s\to-\infty} E_{r,s}(\mathbf{x};\mu) = x_{min};$$
$$(iii) \qquad \lim_{s\to r} E_{r,s}(\mathbf{x};\mu) = E_{r,r}(\mathbf{x};\mu); \lim_{r\to 0} E_{r,s}(\mathbf{x};\mu) = E_{o,s}(\mathbf{x};\mu);$$
$$(iv) \qquad \lim_{r\to 0} E_{r,r}(\mathbf{x};\mu) = E_{0,0}(\mathbf{x};\mu);$$
$$(v) \qquad E^t_{r,s}(\mathbf{x};\mu) = E_{r/t,s/t}(\mathbf{x}^t;\mu), \quad t \neq 0,$$

where $\mathbf{x}^t = (x_1^t, x_2^t, \dots, x_n^t)$.

By application of L'Hôspital's rule, it is not difficult to calculate the limits in (iii), and (iv). Differentiation with respect to r and s under the integral sign is allowed. Thus $E_{r,s}(\mathbf{x};\mu)$ is continuous in both r and s.

The next properties for $E_{r,s}(\mathbf{x};\mu)$ follow from properties of mean (1) and (11), see [6]:

$$(vi) \qquad \min\{x_i : 1 \leqslant i \leqslant n\} \leqslant E_{r,s}(\mathbf{x};\mu) \leqslant \max\{x_i : 1 \leqslant i \leqslant n\};$$
$$(vii) \qquad E_{r,s}(x, x, \dots, x;\mu) = x,\ x > 0;$$

$(viii)$ $E_{r,s}(t\mathbf{x};\mu) = tE_{r,s}(\mathbf{x};\mu)$, $t>0$, where $t\mathbf{x} := (tx_1,\dots,tx_n)$;

(ix) If $x_i \leqslant y_i$ for all i and $x_j < y_j$ for some j, then $E_{r,s}(x;\mu) < E_{r,s}(y;\mu)$, i.e. $E_{r,s}$ is a strictly increasing function of x.

(x) If $x_{\min} < x_{\max}$, then $E_{r,s}$ is strictly increasing function of s.

(xi) If $x_{\min} < x_{\max}$ and $s_1 - s_2 < r_1 - r_2 < 0$, then

$$E_{r_1,s_1}(\mathbf{x};\mu) < E_{r_2,s_2}(\mathbf{x};\mu).$$

Denote by

$$w_i = \int_{E_{n-1}} u_i d\mu(u) \qquad (1 \leqslant i \leqslant n) \tag{14}$$

the i-th weight associated to probability measure μ on E_{n-1}. Clearly, $w_i > 0$ $(1 \leqslant i \leqslant n)$ and $w_1 + \cdots + w_n = 1$.

The following theorem shows that many means are special case of the $E_{r,s}$.

THEOREM 2.2. *If w_i denotes weights defined by* (14) *and* $\mathbf{w} = (w_1,\dots,w_n)$, *then*

$$E_{r,2r}(\mathbf{x};\mu) = M_r(\mathbf{x};\mathbf{w})$$

is a power mean, and

$$E_{1,2}(\mathbf{x};\mu) = A(\mathbf{x};\mathbf{w}), \quad E_{0,0}(\mathbf{x};\mu) = G(\mathbf{x};\mathbf{w}), \quad E_{-1,-2}(\mathbf{x};\mu) = H(\mathbf{x};\mathbf{w}),$$

are the weighted arithmetic, geometric and harmonic means, respectively.

Proof. If $r \neq 0$ then

$$\begin{aligned} E_{r,2r}(\mathbf{x};\mu) &= \left[\int_{E_{n-1}} \Big(\sum_{i=1}^n u_i x_i^r\Big)\, d\mu(u)\right]^{\frac{1}{r}} = \left[\sum_{i=1}^n x_i^r \int_{E_{n-1}} u_i\, d\mu(u)\right]^{\frac{1}{r}} \\ &= \left[\sum_{i=1}^n x_i^r w_i\right]^{\frac{1}{r}} = M_r(\mathbf{x};\mathbf{w}). \end{aligned}$$

For $r=0$ we have

$$\begin{aligned} E_{0,0}(\mathbf{x};\mu) &= \exp\left(\int_{E_{n-1}} \ln\Big(\prod_{i=1}^n x_i^{u_i}\Big)\, d\mu(u)\right) \\ &= \exp\left(\int_{E_{n-1}} \sum_{i=1}^n u_i \ln(x_i)\, d\mu(u)\right) \\ &= \exp\left(\sum_{i=1}^n \ln(x_i) \int_{E_{n-1}} u_i\, d\mu(u)\right) \\ &= \exp\left(\sum_{i=1}^n w_i \ln(x_i)\right) = \prod_{i=1}^n {x_i}^{w_i}. \end{aligned}$$

□

3. Inequalities for Stolarsky-Tobey mean

In the first theorem of this section we will use the the following lemma ([7], Theorem 4.2)

LEMMA 3.1. *If f is a convex function on a closed interval $[a,b]$ and w_i are weights defined in (14), then for $z_1, z_2, \dots, z_n \in [a,b]$*

$$f\left(\sum_{i=1}^{n} w_i z_i\right) \leqslant \int_{E_{n-1}} f\left(\sum_{i=1}^{n} u_i z_i\right) d\mu(u) \leqslant \sum_{i=1}^{n} w_i f(z_i). \tag{15}$$

The inequalities in(15) are strict provided f is not a polynomial of degree one or less and they are reversed if f is concave on $[a,b]$.

THEOREM 3.2. (Jensen) *Let $E_{r,s}(\mathbf{x};\mu)$ be a Stolarsky-Tobey mean defined in (13) and let w_i be weights defined in (14). Then*

$$M_r(\mathbf{x};\mathbf{w}) \leqslant E_{r,s}(\mathbf{x};\mu) \leqslant M_{s-r}(\mathbf{x};\mathbf{w}), \quad \text{if} \quad s \geqslant 2r$$

and reversed inequalities hold if $s \leqslant 2r$.

Proof. For $f(t) = t^{\frac{s-r}{r}}, \quad r(s-r) \neq 0,$ applying (15) the result is obtained. Other cases for $r(s-r) = 0$ follow from continuity of $E_{r,s}$ in r and s.

□

COROLLARY 3.3. *For weighted logarithmic means $L_r(\mathbf{x};\mu)$ and $\mathcal{F}_r(\mathbf{x};\mu)$ defined in (4) and (9) we have*

(i)

$$M_1(\mathbf{x};\mathbf{w}) \leqslant L_r(\mathbf{x};\mu) \leqslant M_r(\mathbf{x};\mathbf{w}) \qquad \text{if} \quad r \geqslant 1,$$

and reversed inequalities hold if $r \leqslant 1$. For $r = -1$ we have

$$H(\mathbf{x},\mathbf{w}) \leqslant L_{-1}(\mathbf{x};\mu) = L(\mathbf{x};\mu) \leqslant A(\mathbf{x},\mathbf{w})$$

where $L(\mathbf{x};\mu)$ is the logarithmic mean defined in (3).

(ii)

$$M_1(\mathbf{x};\mathbf{w}) \leqslant \mathcal{F}_r(\mathbf{x};\mu) \leqslant M_r(\mathbf{x};\mathbf{w}) \qquad \text{if} \quad r \geqslant 1,$$

and reversed inequalities hold if $r \leqslant 1$. For $r = 0$ we have

$$G(\mathbf{x},\mathbf{w}) \leqslant \mathcal{F}_0(\mathbf{x};\mu) = \mathcal{L}(\mathbf{x};\mu) \leqslant A(\mathbf{x},\mathbf{w}),$$

where $\mathcal{L}(\mathbf{x};\mu)$ is the logarithmic mean defined in (5).

THEOREM 3.4. *Let α and β be real positive numbers with $\alpha+\beta = 1$. For $r,s \in$ $\mathbf{R}$ we have*

$$E_{r,s}(\mathbf{x}^{\alpha}\mathbf{y}^{\beta};\mu) \leqslant E_{r,s}(\alpha\mathbf{x}+\beta\mathbf{y};\mu),$$

where

$$\mathbf{x}^{\alpha}\mathbf{y}^{\beta} = (x_1^{\alpha}y_1^{\beta}, \dots, x_n^{\alpha}y_n^{\beta})$$

and

$$\alpha\mathbf{x}+\beta\mathbf{y} = (\alpha x_1+\beta y_1, \dots, \alpha x_n+\beta y_n).$$

Proof. By the arithmetic-geometric inequality

$$x^{\alpha}y^{\beta} \leqslant \alpha x + \beta y, \quad x > 0, y > 0,$$

and the fact that $E_{r,s}(\mathbf{x};\mu)$ is increasing in $\mathbf{x}$, the following is obtained

$$E_{r,s}(\mathbf{x}^{\alpha}\mathbf{y}^{\beta};\mu) \leqslant E_{r,s}(\alpha\mathbf{x} + \beta\mathbf{y};\mu).$$

□

The following three theorems are the same as in ([13], Theorem 5.-7.). We give them without proofs. We use the notation

$$\mathbf{x} + \mathbf{y} = (x_1 + y_1, \ldots, x_n + y_n) \quad \text{and} \quad \mathbf{xy} = (x_1y_1, \ldots, x_ny_n).$$

THEOREM 3.5. (Minkowski) *Let* $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ *and* $\mathbf{y} = (y_1, y_2, \ldots, y_n)$ *be* n*-tuples with* $x_i > 0$ *and* $y_i > 0$, $1 \leqslant i \leqslant n$. *Then, unless* $r = 1, s = 2$ *or* $x_i = ky_i$, $1 \leqslant i \leqslant n$ *we have*

$$E_{r,s}(\mathbf{x} + \mathbf{y};\mu) < E_{r,s}(\mathbf{x};\mu) + E_{r,s}(\mathbf{y};\mu) \qquad (for \quad 1 \leqslant r \leqslant s-1),$$

with reversed inequality if $s - 1 \leqslant r \leqslant 1$. *Equality holds in the exceptional cases.*

COROLLARY 3.6. *For weighted logarithmic means* $L_r(\mathbf{x};\mu)$ *and* $\mathcal{F}_r(\mathbf{x};\mu)$ *defined in* (4) *and* (9) *we have*

(*i*) $\quad L_r(\mathbf{x} + \mathbf{y};\mu) < L_r(\mathbf{x};\mu) + L_r(\mathbf{y};\mu) \qquad (for \quad r > 1),$

with reversed inequality if $r < 1$;

(*ii*) $\quad \mathcal{F}_r(\mathbf{x} + \mathbf{y};\mu) < \mathcal{F}_r(\mathbf{x};\mu) + \mathcal{F}_r(\mathbf{y};\mu) \qquad (for \quad r > 1),$

with reversed inequality if $r < 1$.

Equalities hold for $r = 1$ *or* $x_i = ky_i, (1 \leqslant i \leqslant n)$;

THEOREM 3.7. (Hölder) *Let* $\mathbf{x} = (x_1, \ldots, x_n)$, $\mathbf{y} = (y_1, \ldots, y_n)$ *be* n*-tuples with* $x_i > 0$ *and* $y_i > 0$, $1 \leqslant i \leqslant n$ *and let* α, β *be real positive numbers such that* $\alpha + \beta = 1$. *Then, unless* $s = r = 0$ *or* $x_i^{1/\alpha} = ky_i^{1/\beta}$, $1 \leqslant i \leqslant n$

$$E_{r,s}(\mathbf{xy};\mu) < \Big[E_{r,s}(\mathbf{x}^{1/\alpha};\mu)\Big]^{\alpha} \ \Big[E_{r,s}(\mathbf{y}^{1/\beta};\mu)\Big]^{\beta} \quad (for \quad s \geqslant r \geqslant 0),$$

with reversed inequality if $s \leqslant r \leqslant 0$. *Equality holds in the exceptional cases.*

COROLLARY 3.8. *For weighted logarithmic means* $L_r(\mathbf{x};\mu)$ *and* $\mathcal{F}_r(\mathbf{x};\mu)$ *defined in* (4) *and* (9) *we have*

(*i*) $\quad L_r(\mathbf{xy};\mu) < \left[L_r(\mathbf{x}^{1/\alpha};\mu)\right]^{\alpha} \ \left[L_r(\mathbf{y}^{1/\beta};\mu)\right]^{\beta} \quad (for \quad r \geqslant 0);$

(*ii*) $\quad \mathcal{F}_r(\mathbf{xy};\mu) < \left[\mathcal{F}_r(\mathbf{x}^{1/\alpha};\mu)\right]^{\alpha} \ \left[\mathcal{F}_r(\mathbf{y}^{1/\beta};\mu)\right]^{\beta} \quad (for \quad r \geqslant 0) \quad .$

Equalities hold for $x_i^{1/\alpha} = ky_i^{1/\beta}$.

THEOREM 3.9. (Rennie) *Let* $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ *and* $0 \leqslant A \leqslant x_i \leqslant B, 1 \leqslant i \leqslant n$, *then if* $s \neq r$

$$E_{r,s}^{s-r}(\mathbf{x};\mu) + A^{s-r}B^{s-r}E_{-r,s-2r}^{s-r}(\mathbf{x}^{-1};\mu) \leqslant A^{s-r} + B^{s-r},$$

with equality if and only if $x_i = A$ *or* $x_i = B, 1 \leqslant i \leqslant n$.

COROLLARY 3.10. *For weighted logarithmic means* $L_r(\mathbf{x};\mu)$ *and* $\mathcal{F}_r(\mathbf{x};\mu)$ *defined in* (4) *and* (9) *we have*

(*i*) $L_r^r(\mathbf{x};\mu) + A^rB^rL_{-r}^{-r}(\mathbf{x};\mu) \leqslant A^r + B^r$ *if* $r \neq 0$;
(*ii*) $\mathcal{F}_r(\mathbf{x};\mu) + AB\mathcal{F}_{-r}(\mathbf{x}^{-1};\mu) \leqslant A + B$.

Equalities hold if and only if $x_i = A$ *or* $x_i = B, 1 \leqslant i \leqslant n$.

THEOREM 3.11. (Kantorovich) *Let* $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ *and* $0 < A \leqslant x_i \leqslant B$, $1 \leqslant i \leqslant n$. *Then for* $s > r$ *we have*

$$1 \leqslant E_{r,s}(\mathbf{x};\mu)E_{-r,s-2r}(\mathbf{x}^{-1};\mu) \leqslant \left[\frac{A^{s-r}+B^{s-r}}{2}\right]^{1/(s-r)} \left[\frac{A^{r-s}+B^{r-s}}{2}\right]^{1/(s-r)},$$

with equality on the left if and only if $x_{\max} = x_{\min}$, *and equality on the right if and only if* $A = B$. *If* $s < r$ *we have reversed inequality.*

COROLLARY 3.12. *For weighted logarithmic means* $L_r(\mathbf{x};\mu)$ *and* $\mathcal{F}_r(\mathbf{x};\mu)$ *defined in* (4) *and* (9) *we have*

$$(i) \quad 1 \leqslant L_r(\mathbf{x};\mu)L_{-r}(\mathbf{x};\mu) \leqslant \left[\frac{A^r+B^r}{2}\right]^{1/r} \left[\frac{A^{-r}+B^{-r}}{2}\right]^{1/r} \quad \text{if } r > 0,$$

and reversed inequality holds if $r < 0$;

$$(ii) \quad 1 \leqslant \mathcal{F}_r(\mathbf{x};\mu)\mathcal{F}_{-r}^{-1}(\mathbf{x}^{-1};\mu) \leqslant \frac{(A+B)^2}{4AB},$$

The equalities hold on the left if and only if $x_{\max} = x_{\min}$, *and equalities on the right if and only if* $A = B$.

4. Stolarsky-Tobey mean and Dirichlet measure

An important example of probability measure on simplex E_{n-1} is Dirichlet measure μ_b ([5], Sec.4.4), because it is connected with hypergeometric functions.

Let $\mathbf{b} = (b_1, b_2, \ldots, b_n) \in \mathbf{R}_+^n$, $n \geqslant 2$. Beta function of n variables is defined by

$$B(\mathbf{b}) := B(b_1, b_2, \ldots, b_n) = \frac{\Gamma(b_1)\cdots\Gamma(b_n)}{\Gamma(b_1+\cdots+b_n)},$$

where Γ is gamma function.

An integral representation for B ([5], Sec.4.3) is

$$B(b_1, b_2, \ldots, b_n) = \int_{E_{n-1}} \prod_{i=1}^{n} u_i^{b_i-1} du_1 \cdots du_{n-1},$$

where $u_n = 1 - \sum_{i=1}^{n-1} u_i$.

The Dirichlet measure μ_b is defined by

$$d\mu_b(u) = \frac{1}{B(\mathbf{b})} \prod_{i=1}^{n} u_i^{b_i-1} du_1 \cdots du_{n-1} ,$$

so that

$$\int_{E_{n-1}} d\mu_b(u) = 1.$$

A simple calculation gives

$$w_i = \int_{E_{n-1}} u_i d\mu_b(u) = \frac{b_i}{\sum_{i=1}^{n} b_i}, \quad 1 \leqslant i \leqslant n.$$

For $\mathbf{z} = (z_1, \ldots, z_n) \in \mathbf{R}_+^n$ the hypergeometric $R-$ function ([5],Sec.5.9) has an integral representation

$$R_t(\mathbf{z};\mathbf{b}) := R_t(z_1, \ldots, z_n; b_1, \ldots, b_n) = \int_{E_{n-1}} (\sum_{i=1}^{n} u_i z_i)^t d\mu_b(u)$$

and its derivative with respect to t is a

$$D_t(\mathbf{z};\mathbf{b}) := D_t(z_1, \ldots, z_n; b_1, \ldots, b_n) = \int_{E_{n-1}} (\sum_{i=1}^{n} u_i z_i)^t \ln(\sum_{i=1}^{n} u_i z_i) d\mu_b(u).$$

The confluent hypergeometric S-function ([5], Sec.5.8) has an integral representation

$$S(\mathbf{z};\mathbf{b}) := S(z_1, \ldots, z_n; b_1, \ldots, b_n) = \int_{E_{n-1}} \exp\left(\sum_{i=1}^{n} u_i z_i\right) d\mu_b(u).$$

We will write $E_{r,s}(\mathbf{x};\mathbf{b})$ instead

$$E_{r,s}(\mathbf{x};\mu_b) := E_{r,s}(x_1, \ldots, x_n; b_1, \ldots, b_n).$$

On the basis of the facts mentioned above, Stolarsky-Tobey mean generated by the Dirichlet measure μ_b can be expressed as

$$E_{r,s}(\mathbf{x};\mathbf{b}) = \begin{cases} R^{1/(s-r)}_{\frac{s-r}{r}}(\mathbf{x}^r;\mathbf{b}) & \text{if} \quad r(s-r) \neq 0 \\ \exp\left[\frac{1}{r} D_0(\mathbf{x}^r;\mathbf{b}\right] & \text{if} \quad s = r \neq 0, \\ S(\ln \mathbf{x}^s;\mathbf{b}) & \text{if} \quad r = 0, s \neq 0 \\ \exp\left[S(\ln \mathbf{x},\mathbf{b})\right] & \text{if}, r = s = 0, \end{cases} \tag{16}$$

where $\mathbf{x}^r = (x_1^r, x_2^r, \ldots, x_n^r)$ and

$$\ln(\mathbf{x}^s) = (\ln(x_1^s), \ln(x_2^s), \ldots, \ln(x_n^s)).$$

For logarithmic means defined by (4) and (9) we have

$$L_r(\mathbf{x};\mathbf{b}) = \begin{cases} R_r^{1/r}(\mathbf{x};\mathbf{b}) & \text{if} \quad r \neq 0 \\ \exp\left[D_0(\mathbf{x};\mathbf{b})\right] & \text{if} \quad r = 0. \end{cases} \tag{17}$$

$$\mathcal{F}_r(\mathbf{x};\mathbf{b}) = \begin{cases} R_{1/r}(\mathbf{x}^r;\mathbf{b}) & \text{if} \quad r \neq 0 \\ S(\ln \mathbf{x}^{r+1};\mathbf{b}) & \text{if} \quad r = 0. \end{cases} \tag{18}$$

The relation between Stolarsky-Tobey mean and hypergeometric functions makes possible that some properties for $E_{r,s}(\mathbf{x};\mathbf{b})$ are derived from the known results for hypergeometric functions. The following theorem is an example of this. From ([3], Theorem 1) and from the fact that $E_{r,s}$ is continuous in both s and r we have

THEOREM 4.1. *Let $c, w_1, w_2, \ldots, w_n$ be strictly positive real numbers. Let us assume that $\sum_{i=1}^{n} w_i = 1$, and define $c\mathbf{w} = (cw_1, cw_2, \ldots, cw_n)$. Then*

$$\lim_{c\to 0} E_{r,s}(\mathbf{x};c\mathbf{w}) = M_{s-r}(\mathbf{x};\mathbf{w}) \quad \text{and} \quad \lim_{c\to\infty} E_{r,s}(\mathbf{x};c\mathbf{w}) = M_r(\mathbf{x};\mathbf{w}) \quad .$$

COROLLARY 4.2. *Let c, and $c\mathbf{w}$ be as in the previous theorem and let L_r and $\mathcal{F}_r$ be means defined in (4) and (9), respectively. Then, we have*

$$\lim_{c\to 0} \mathcal{F}_r(\mathbf{x};c\mathbf{w}) = M_1(\mathbf{x};\mathbf{w}) = A(\mathbf{x};\mathbf{w}), \quad \text{and} \quad \lim_{c\to\infty} \mathcal{F}_r(\mathbf{x};c\mathbf{w}) = M_r(\mathbf{x};\mathbf{w});$$

$$\lim_{c\to 0} L_r(\mathbf{x};c\mathbf{w}) = M_r(\mathbf{x};\mathbf{w}), \quad \text{and} \quad \lim_{c\to\infty} L_r(\mathbf{x};c\mathbf{w}) = M_1(\mathbf{x};\mathbf{w}) = A(\mathbf{x};\mathbf{w}) \quad .$$

The next theorem shows that the multiple integral in the definition of Stolarsky-Tobey mean (13) generated by μ_b can be reduced to a single integral.

THEOREM 4.3. *Let $\mathbf{a} = (b_1, \ldots, b_{n-1}) \in \mathbf{R}_+^{n-1}, \alpha = b_1 + \cdots + b_{n-1}, \beta = b_n$ and let $\mu_{(\alpha,\beta)}$ be Dirichlet measure on E_1, that is*

$$d\mu_{(\alpha,\beta)}(v) = \frac{v^{\alpha-1}(1-v)^{\beta-1}}{B(\alpha,\beta)}\, dv$$

and for $i = 1, 2, \ldots, n-1$ we define

$$y_i = \begin{cases} [vx_i^r + (1-v)x_n^r]^{1/r} & \text{if} \quad r \neq 0 \\ x_i^v x_n^{1-v} & \text{if} \quad r = 0. \end{cases}$$

Then

$$E_{r,s}(\mathbf{x};\mathbf{b}) = \begin{cases} \left(\int_0^1 E_{r,s}^{s-r}(\mathbf{y};\mathbf{a})\mu_{(\alpha,\beta)}(v)\, dv\right)^{\frac{1}{s-r}} & \text{if } r(s-r) \neq 0, \\ \exp\left(\int_0^1 \ln E_{r,r}(\mathbf{y};\mathbf{a})\mu_{(\alpha,\beta)}(v)\, dv\right) & \text{if } s = r \neq 0, \\ \left(\int_0^1 E_{0,s}^{s}(\mathbf{y};\mathbf{a})\mu_{(\alpha,\beta)}(v)\, dv\right)^{1/s} & \text{if } r = 0,\ s \neq 0, \\ \exp\left(\int_0^1 \ln E_{0,0}(\mathbf{y};\mathbf{a})\mu_{(\alpha,\beta)}(v)\, dv\right) & \text{if } s = r = 0. \end{cases}$$

Proof. Let $v=\sum_{i=1}^{n-1}u_i$ and $p_i=\frac{u_i}{v}$, $i=1,\dots,n-1$. Then $u_n=1-v$ and $\sum_{i=1}^{n-1}p_i=1$. For this substitution Jacobian is v^{n-2}. We will use the notation $dp=dp_1dp_2\cdots dp_{n-2}$ for the differential of the volume in E_{n-2}.

A simple computation gives

$$B(\mathbf{b})=B(\mathbf{a})B(\alpha,\beta).$$

If $r(s-r)\neq 0$ then

$$\begin{aligned}
E_{r,s}^{s-r}(\mathbf{x};\mathbf{b}) &= \int_{E_{n-1}}\left(\sum_{i=1}^{n}u_ix_i^r\right)^{\frac{s-r}{r}}d\mu_b(u)\\
&= \int_0^1\int_{E_{n-2}}\left(\sum_{i=1}^{n-1}vp_ix_i^r+(1-v)x_n^r\right)^{\frac{s-r}{r}}\frac{\prod_{i=1}^{n-1}(vp_i)^{b_i-1}(1-v)^{b_n-1}}{B(\mathbf{b})}v^{n-2}\,dvdp\\
&= \int_0^1\left[\int_{E_{n-2}}\left(\sum_{i=1}^{n-1}p_iy_i^r\right)^{\frac{s-r}{r}}d\mu_a(p)\right]\mu_{(\alpha,\beta)}(v)\,dv\\
&= \int_0^1 E_{r,s}^{s-r}(\mathbf{y};\mathbf{a})\mu_{(\alpha,\beta)}(v)\,dv.
\end{aligned}$$

For $s=r\neq 0$ we have

$$\begin{aligned}
\ln\left[E_{r,r}(\mathbf{x};\mathbf{b})\right] &= \int_{E_{n-1}}\ln\left(\sum_{i=1}^{n}u_ix_i^r\right)^{1/r}d\mu_b(u)\\
&= \int_0^1\int_{E_{n-2}}\ln\left(\sum_{i=1}^{n-1}(vp_ix_i^r+(1-v)x_n^r)\right)^{1/r}\frac{\prod_{i=1}^{n-1}(vp_i)^{b_i-1}(1-v)^{b_n-1}}{B(\mathbf{b})}v^{n-2}\,dvdp\\
&= \int_0^1\left[\int_{E_{n-2}}\ln\left(\sum_{i=1}^{n-1}p_iy_i^r\right)^{1/r}d\mu_a(p)\right]\mu_{(\alpha,\beta)}(v)\,dv\\
&= \int_0^1\ln\left[E_{r,r}(\mathbf{y};\mathbf{a})\right]\mu_{(\alpha,\beta)}(v)\,dv.
\end{aligned}$$

For $r=0$ and $s\neq 0$ we have

$$\begin{aligned}
[E_{0,s}(\mathbf{x};\mathbf{b})]^s &= \int_{E_{n-1}} \left(\prod_{i=1}^{n} x_i^{u_i}\right)^s d\mu_b(u) \\
&= \int_0^1 \int_{E_{n-2}} \left(\prod_{i=1}^{n-1} x_i^{vp_i}\right)^s (x_n^{1-v})^s \frac{\prod_{i=1}^{n-1} (vp_i)^{b_i-1}(1-v)^{b_n-1}}{B(\mathbf{b})} v^{n-2}\, dvdp \\
&= \int_0^1 \left[\int_{E_{n-2}} \left(\prod_{i=1}^{n-1} y_i^{p_i}\right)^s d\mu_a(p)\right] \mu_{(\alpha,\beta)}(v)\, dv \\
&= \int_0^1 E_{0,s}^s(\mathbf{y};\mathbf{a})\mu_{(\alpha,\beta)}(v)\, dv.
\end{aligned}$$

Finally, we have

$$\begin{aligned}
\ln [E_{0,0}(\mathbf{x};\mathbf{b})] &= \int_{E_{n-1}} \ln \left(\prod_{i=1}^{n} x_i^{u_i}\right) d\mu_b(u) \\
&= \int_0^1 \left[\int_{E_{n-2}} \ln \left(\prod_{i=1}^{n-1} x_i^{vp_i} x_n^{1-v}\right)\right] \frac{\prod_{i=1}^{n-1} (vp_i)^{b_i-1}(1-v)^{b_n-1}}{B(\mathbf{b})} v^{n-2}\, dvdp \\
&= \int_0^1 \left[\int_{E_{n-2}} \ln \left(\prod_{i=1}^{n-1} y_i^{p_i}\right) d\mu_a(p)\right] \mu_{(\alpha,\beta)}(v)\, dv \\
&= \int_0^1 \ln [E_{0,0}(\mathbf{y};\mathbf{a})]\, \mu_{(\alpha,\beta)}(v)\, dv.
\end{aligned}$$

□

In the rest of this section, several relations which are peculiar to the important case $n=2$ will be established. In this case, many of the most important special functions can be represented as R-function or S-function ([5], Sec.5), in some cases as Stolarsky-Tobey means. Here are some examples. We use Appell's symbol for shifted factorial:

$$(a,0)=1, (a,n)=a(a+1)(a+2)\cdots(a+n-1), n\geqslant 1.$$

Let us recall that ([5], Sec.2) ${}_2F_1$ is the Gauss's hypergeometric function represented by hypergeometric series as

$${}_2F_1(a,b;c;z) = \sum_{m=0}^{\infty} \frac{(a,m)(b,m)}{(c,m)} \frac{z^m}{m!}, \qquad |z|<1, \quad c\neq 0,-1,-2,\ldots$$

and ${}_1F_1$ is the Kummer's confluent hypergeometric function represented by hypergeometric series as

$$ {}_1F_1(a,b,z) = \sum_{m=0}^{\infty} \frac{(a,m)}{(b,m)} \frac{z^m}{m!}, \qquad b \neq 0,-1,-2,\dots $$

Applying results from ([5], Sec.5) on (16) we obtain the following theorem.

THEOREM 4.4. *Let* $\mathbf{x} = (x,y) \in \mathbf{R}_+^2$, $\mathbf{b} = (\alpha,\beta) \in \mathbf{R}_+^2$ *and* $x < y$
(i) *For* $s \neq r(n+1), n \in \mathbf{N}$ *we have*

$$ E_{r,s}^{s-r}(x,y;\alpha,\beta) = \begin{cases} y^{s-r}\, {}_2F_1\left(-\frac{s-r}{r}, \alpha; \alpha+\beta; 1-\frac{x^r}{y^r}\right) & \text{if } r>0 \\ x^{s-r}\, {}_2F_1\left(-\frac{s-r}{r}, \alpha; \alpha+\beta; 1-\frac{y^r}{x^r}\right) & \text{if } r<0 \\ y^{s}\, {}_1F_1\left(\alpha; \alpha+\beta; s\ln\frac{x}{y}\right) & \text{if } r=0; \end{cases} $$

(ii) *For* $s = r(n+1), n \in \mathbf{N}$ *and* $r \neq 0$ *we have*

$$ E_{r,r(n+1)}^{rn}(x,y;\alpha,\beta) = y^{nr} \sum_{m=0}^{n} \binom{n}{m} \frac{(\alpha,n-m)(\beta,m)}{(\alpha+\beta,n)} \left(\frac{x}{y}\right)^{rm}; $$

(iii)

$$ E_{0,0}(x,y;\alpha,\beta) = x^{\frac{\alpha}{\alpha+\beta}} y^{\frac{\beta}{\alpha+\beta}}; $$

5. Explicit form for unweighted Stolarsky-Tobey mean

When a measure μ_b reduces to Lebesgue measure, then

$$ d\mu_b(u) = (n-1)!du_1 \cdots du_{n-1} = (n-1)!du . $$

A sample calculation gives

$$ \int_{E_{n-1}} du_1 \cdots du_{n-1} = \frac{1}{(n-1)!} \text{ and } w_i = \int_{E_{n-1}} u_i d\mu(u) = \frac{1}{n} . $$

In this case, we will write $E_{r,s}(\mathbf{x})$ for $E_{r,s}(\mathbf{x};\mu)$. In the proof of the next theorem a well-known relations is used:

$$ [t_1,t_2,\dots,t_n]f = \sum_{i=1}^{n} \frac{f(t_i)}{\prod\limits_{\substack{j=1\\ j\neq i}}^{n}(t_i-t_j)} = \int_{E_{n-1}} f^{(n-1)}\left(\sum_{i=1}^{n} u_i t_i\right) du, \qquad (19) $$

where $[t_1,\dots,t_n]f$ stands for the divided difference of order $n-1$ of f with knots at $t_1,\dots,t_n$, and $f \in C^{(n-1)}(a,b)$, $a = \min(t_i)$, $b = \max(t_i)$, $1 \leqslant i \leqslant n$. Also, the following lemma [11] will be used.

LEMMA 5.1. *Let* $a_1,\dots,a_n$ *be distinct, nonzero real numbers. For* $0 \leqslant k \leqslant n-2$,

$$\sum_{i=1}^{n} \frac{a_i^k}{\prod\limits_{\substack{j=1\\ j\neq i}}^{n}(a_i^r - a_j^r)} = 0 \ . \tag{20}$$

THEOREM 5.2. *For* $r,s \in \mathbf{R}$, $\mathbf{x} = (x_1, x_2 \dots, x_n) \in \mathbf{R}_+^n$ *and* $x_i \neq x_j$ (*for* $i \neq j$) *we have:*

$$(i) \quad E_{r,s}(\mathbf{x}) = \left[\frac{(n-1)!r^{n-1}}{s(s+r)\dots(s+(n-2)r)} \sum_{i=1}^{n} \frac{x_i^{s+(n-2)r}}{\prod\limits_{\substack{j=1\\ j\neq i}}^{n}(x_i^r - x_j^r)} \right]^{\frac{1}{s-r}},$$

$r \neq 0,\ s \neq -kr,\ 0 \leqslant k \leqslant n-2;$

$$(ii) \quad E_{r,-kr}(\mathbf{x}) = \left[(-1)^k(k+1)\binom{n-1}{k+1} \sum_{i=1}^{n} \frac{x_i^{(n-k-2)r}\ln(x_i)}{\prod\limits_{\substack{j=1\\ j\neq i}}^{n}(x_i^r - x_j^r)} \right]^{-\frac{1}{(k+1)r}},$$

$r \neq 0\ ,0 \leqslant k \leqslant n-2;$

$$(iii) \quad E_{0,s}(\mathbf{x}) = \left[\frac{(n-1)!}{s^{n-1}} \sum_{i=1}^{n} \frac{x_i^s}{\prod\limits_{\substack{j=1\\ j\neq i}}^{n}\ln(\frac{x_i}{x_j})} \right]^{\frac{1}{s}}, \qquad s \neq 0;$$

$$(iv) \quad E_{r,r}(\mathbf{x}) = \exp\left(\frac{1}{r} \sum_{i=1}^{n} \frac{x_i^{r(n-1)}\left(\ln x_i - \sum\limits_{k=1}^{n-1}\frac{1}{k}\right)}{\prod\limits_{\substack{j=1\\ j\neq i}}^{n}(x_i^r - x_j^r)} \right), r \neq 0;$$

$$(v) \quad E_{0,0}(\mathbf{x}) = \left(\prod_{i=1}^{n} x_i \right)^{\frac{1}{n}} .$$

Proof. (i) For $f(t) = t^{q+n-2}$, $q = \frac{s}{r}, n \geqslant 2,$ we have

$$f^{(n-1)}(t) = q(q+1)\cdots(q+n-2)t^{q+1}$$

and

$$E_{r,s}(\mathbf{x}) = \left[\frac{(n-1)!}{q(q+1)\cdots(q+n-2)} \int_{E_{n-1}} f^{(n-1)}\left(\sum_{i=1}^{n} u_i x_i^r \right) du \right]^{\frac{1}{s-r}} .$$

Using (19) for $t_i = x_i^r \ (1 \leqslant i \leqslant n)$ we obtain (i).

(ii) From the explicit form (i) for $E_{r,s}(\mathbf{x})$ and using (20), we obtain a formula for $E_{r,-kr}$ $(0 \leqslant k \leqslant n-2)$ as a limit:

$$\begin{aligned}
E_{r,-kr}(\mathbf{x}) &= \lim_{s \to -kr} E_{r,s}(\mathbf{x}) \\
&= \left[\lim_{q \to -k} \frac{(n-1)!}{q(q+1)\cdots(q+k)\cdots(q+n-2)} \sum_{i=1}^{n} \frac{x_i^{(q+n-2)r}}{\prod\limits_{\substack{j=1 \\ j \neq i}}^{n} (x_i^r - x_j^r)} \right]^{-\frac{1}{(k+1)r}} \\
&= \left[\frac{(n-1)!}{(-1)^k k!(n-k-2)!} \sum_{i=1}^{n} \lim_{q \to -k} \frac{x_i^{(q+n-2)r} - x_i^{(n-k-2)r}}{(q+k) \prod\limits_{\substack{j=1 \\ j \neq i}}^{n} (x_i^r - x_j^r)} \right]^{-\frac{1}{(k+1)r}} \\
&= \left[(-1)^k (k+1) \binom{n-1}{k+1} \sum_{i=1}^{n} \frac{x_i^{(n-k-2)r} \ln(x_i)}{\prod\limits_{\substack{j=1 \\ j \neq i}}^{n} (x_i^r - x_j^r)} \right]^{-\frac{1}{(k+1)r}} .
\end{aligned}$$

This completes the proof of (ii). In particular, for $k = 0$ we obtain

$$E_{r,0} = \left[(n-1) \sum_{i=1}^{n} \frac{x_i^{(n-2)r} \ln(x_i)}{\prod\limits_{\substack{j=1 \\ j \neq i}}^{n} (x_i^r - x_j^r)} \right]^{-\frac{1}{r}} = L^{\frac{1}{r}} \left(x_1^r, x_2^r, \ldots x_n^r \right) . \tag{21}$$

(iii) Explicit form for $E_{o,s}(\mathbf{x})$ we obtain as a limit

$$\begin{aligned}
E_{o,s}(\mathbf{x}) = \lim_{r \to 0} E_{r,s}(\mathbf{x}) &= \left[\frac{(n-1)!}{s^{n-1}} \sum_{i=1}^{n} \frac{x_i^s}{\prod\limits_{\substack{j=1 \\ j \neq i}}^{n} \lim\limits_{r \to 0} \frac{x_i^r - x_j^r}{r}} \right]^{1/s} \\
&= \left[\frac{(n-1)!}{s^{n-1}} \sum_{i=1}^{n} \frac{x_i^s}{\prod\limits_{\substack{j=1 \\ j \neq i}}^{n} \ln(x_i / x_j)} \right]^{1/s}
\end{aligned}$$

(iv)

$$E_{r,r}(\mathbf{x}) = \exp\left((n-1)!\int_{E_{n-1}} \ln\left(\sum_{i=1}^{n} x_i^r u_i\right)^{1/r} du\right),$$

that is

$$\ln(E_{r,r}(\mathbf{x})) = \frac{(n-1)!}{r}\int_{E_{n-1}} \ln\left(\sum_{i=1}^{n} x_i^r u_i\right) du.$$

For

$$f(t) = \frac{t^{n-1}}{(n-1)!}\left(\ln(t) - \sum_{k=1}^{n-1}\frac{1}{k}\right),\ n \geqslant 2,$$

we have $f^{(n-1)}(t) = \ln t$ and from (19) we obtain (iv).

(v)

$$\begin{aligned} E_{0,0}(\mathbf{x}) &= \exp\left((n-1)!\int_{E_{n-1}} \sum_{i=1}^{n} u_i \ln(x_i) du\right) \\ &= \exp\left(\sum_{i=1}^{n} w_i \ln(x_i)\right) = exp\left(\sum_{i=1}^{n}\frac{\ln(x_i)}{n}\right) \\ &= \left(\prod_{i=1}^{n} x_i\right)^{1/n}. \end{aligned}$$

□

Remark 5.3. For $r = 1$ from (21) we have a explicit form for logarithmic mean in n variables given in [11].

Remark 5.4. For $s = 1$ from (iii) we have the result given in [8]:

$$E_{0,1}(\mathbf{x}) = \mathcal{L}(\mathbf{x}) = (n-1)!\sum_{i=1}^{n}\frac{x_i}{\prod_{\substack{j=1\\ j\neq i}}^{n}\ln(x_i/x_j)}.$$

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