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ON NEW ESTIMATION OF THE REMAINDER IN GENERALIZED TAYLOR'S FORMULA

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(*communicated by T. Rassias*)

Abstract. We derive some estimations of the remainder in perturbed generalized Taylor's formula and apply them to calculations of the logarithmic function.

1. Introduction

Recently, S. S. Dragomir in [2] has obtained the following result:

THEOREM 1. *Let $f : I \to R$ ($I \subset R$ is a closed interval, $a \in I$) be such that $f^{(n)}$ is absolutely continuous. Then we have the Taylor's perturbed formula:*

$$f(x) = T_n(f;a,x) + \frac{(x-a)^{n+1}}{(n+1)!}\left[f^{(n)};a,x\right] + G_n(f;a,x), \tag{1.1}$$

where

$$T_n(f;a,x) := \sum_{k=0}^{n} \frac{(x-a)^k}{k!} f^{(k)}(a) \tag{1.2}$$

and

$$\left[f^{(n)};a,x\right] := \frac{f^{(n)}(x) - f^{(n)}(a)}{x-a}. \tag{1.3}$$

The remainder $G_n(f;a,x)$ satisfies the estimation:

$$|G_n(f;a,x)| \leqslant \frac{(x-a)^{n+1}}{4(n!)}[\Gamma(x) - \gamma(x)], \tag{1.4}$$

where

$$\Gamma(x) := \sup_{t\in[a,x]} f^{(n+1)}(t), \quad \gamma(x) := \inf_{t\in[a,x]} f^{(n+1)}(t) \tag{1.5}$$

for all $x \geqslant a$, $x \in I$.

In this paper we shall give improvement and generalization of this, as well as of some other results from [2].

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2. Generalized Taylor's formula

In this section we consider a formula which can be regarded as generalized Taylor's formula.

THEOREM 2. *Let* $\{P_n(x)\}$ *be a harmonic sequence of polynomials, that is*

$$P_n'(x) = P_{n-1}(x), \text{ for } n \in \mathbf{N};\ P_0(x) = 1.$$

Further, let $I \subset \mathbf{R}$ *be a closed interval and* $a \in I$. *If* $f : I \to R$ *is any function such that, for some* $n \in \mathbf{N}$, $f^{(n)}$ *is absolutely continuous, then for any* $x \in I$

$$f(x) = f(a) + \sum_{k=1}^{n} (-1)^{k+1} \left[P_k(x) f^{(k)}(x) - P_k(a) f^{(k)}(a) \right] + R_n(f; a, x), \tag{2.1}$$

where

$$R_n(f; a, x) = (-1)^n \int_a^x P_n(t) f^{(n+1)}(t) dt. \tag{2.2}$$

Proof. By integration by parts we have:

$$\begin{aligned} & (-1)^n \int_a^x P_n(t) f^{(n+1)}(t) dt \\ = \ & (-1)^n \, P_n(t) f^{(n)}(t) \Big|_a^x + (-1)^{n-1} \int_a^x P_{n-1}(t) f^{(n)}(t) dt \\ = \ & (-1)^n \left[P_n(x) f^{(n)}(x) - P_n(a) f^{(n)}(a) \right] + (-1)^{n-1} \int_a^x P_{n-1}(t) f^{(n)}(t) dt. \end{aligned}$$

Clearly, we can apply the same procedure to the term $(-1)^{n-1} \int_a^x P_{n-1}(t) f^{(n)}(t) dt$. So, by successive integration by parts we obtain

$$(-1)^n \int_a^x P_n(t) f^{(n+1)}(t) dt = \sum_{k=1}^{n} (-1)^k \left[P_k(x) f^{(k)}(x) - P_k(a) f^{(k)}(a) \right] + f(x) - f(a)$$

and this is equivalent to (2.1). □

We can call (2.1) the generalized Taylor's formula. Namely, if we set in (2.1)

$$P_n(t) = \frac{(t-x)^n}{n!},$$

then we get the classical Taylor's formula:

$$f(x) = f(a) + \sum_{k=1}^{n} \frac{(x-a)^k}{k!} f^{(k)}(a) + R_n^T(f; a, x), \tag{2.3}$$

where

$$R_n^T(f; a, x) := \frac{1}{n!} \int_a^x (x-t)^n f^{(n+1)}(t) dt. \tag{2.4}$$

For

$$P_n(t) = \frac{1}{n!}\left(t - \frac{a+x}{2}\right)^n$$

we have

$$f(x) = T_n^M(f;a,x) + R_n^M(f;a,x),$$

where

$$T_n^M(f;a,x) := f(a) + \sum_{k=1}^{n} \frac{(x-a)^k}{2^k k!}\left[f^{(k)}(a) - (-1)^k f^{(k)}(x)\right] \tag{2.5}$$

and

$$R_n^M(f;a,x) := \frac{(-1)^n}{n!}\int_a^x \left(t - \frac{a+x}{2}\right)^n f^{(n+1)}(t)dt. \tag{2.6}$$

Here we give another special case of (2.1) by using the well known Bernoulli polynomials $B_n(t)$. These polynomials can be defined by the expansion

$$\frac{xe^{tx}}{e^x - 1} = \sum_{n=0}^{\infty} \frac{B_n(t)}{n!}x^n, \quad |x| < 2\pi, \quad t \in \mathbf{R}.$$

We have

$$B_0(t) = 1,\ B_1(t) = t - \frac{1}{2},\ B_2(t) = t^2 - t + \frac{1}{6},\ B_3(t) = t^3 - \frac{3}{2}t^2 + \frac{1}{2}t,\ \cdots.$$

The numbers $B_n := B_n(0)$ are called Bernoulli numbers. The polynomials $B_n(t)$ and the numbers B_n have many interesting properties. It can be shown that the polynomials $B_n(t)$ are uniquely determined by the following two properties ([1, 23.1.5 and 23.1.6]):

$$B_n'(t) = nB_{n-1}(t),\ n \in \mathbf{N};\ B_0(t) = 1 \tag{2.7}$$

and

$$B_n(t+1) - B_n(t) = nt^{n-1}, \quad n \in \mathbf{N}. \tag{2.8}$$

If we set

$$P_n(t) = \frac{(x-a)^n}{n!}B_n\left(\frac{t-a}{x-a}\right),\ n \in \mathbf{N},\ P_0(t) = 1,$$

then it is easy to see, using (2.7), that $\{P_n(t)\}$ is a harmonic sequence of polynomials. So, we can apply (2.1) to obtain

$$f(x) = f(a) + \sum_{k=1}^{n}(-1)^{k+1}\frac{(x-a)^k}{k!}\left[B_k(1)f^{(k)}(x) - B_k(0)f^{(k)}(a)\right] + R_n^B(f;a,x),$$

where

$$R_n^B(f;a,x) := (-1)^n\frac{(x-a)^n}{n!}\int_a^x B_n\left(\frac{t-a}{x-a}\right)f^{(n+1)}(t)dt. \tag{2.9}$$

Setting $t = 0$ in (2.8) we get $B_n(1) - B_n(0) = 0$, for $n \neq 1$, that is $B_n(1) = B_n(0) = B_n$, for $n \neq 1$. Also, $B_1(1) = -B_1(0) = 1/2$ so that we have

$$\begin{aligned} f(x) &= f(a) + \frac{x-a}{2}[f'(x) + f'(a)] \\ &\quad + \sum_{k=2}^{n} (-1)^{k+1} \frac{(x-a)^k}{k!} B_k \left[f^{(k)}(x) - f^{(k)}(a) \right] + R_n^B(f; a, x). \end{aligned}$$

Finally, we can use the fact that $B_{2k+1} = 0$ for $k = 1, 2, \cdots$, ([1, 23.1.19]), so that

$$f(x) = T_n^B(f; a, x) + R_n^B(f; a, x),$$

where

$$\begin{aligned} & T_n^B(f; a, x) \qquad (2.10) \\ := \; & f(a) + \frac{x-a}{2}[f'(x) + f'(a)] - \sum_{k=1}^{[\frac{n}{2}]} \frac{(x-a)^{2k}}{(2k)!} B_{2k} \left[f^{(2k)}(x) - f^{(2k)}(a) \right] \end{aligned}$$

and $R_n^B(f; a, x)$ is given by (2.9). (Here, as well as in the rest of paper, $[z]$ denotes the greatest integer less than or equal to z.)

Instead of Bernoulli polynomials $B_n(t)$ we can use Euler polynomials $E_n(t)$ which have the properties similar to those of Bernoulli polynomials. Euler polynomials can be defined by the expansion

$$\frac{2e^{tx}}{e^x + 1} = \sum_{n=0}^{\infty} \frac{E_n(t)}{n!} x^n, \quad |x| < \pi, \quad t \in \mathbf{R}.$$

We have

$$E_0(t) = 1, \; E_1(t) = t - \frac{1}{2}, \; E_2(t) = t^2 - t, \; E_3(t) = t^3 - \frac{3}{2}t^2 + \frac{1}{4}, \; \cdots .$$

It can be shown that the polynomials $E_n(t)$ are uniquely determined by the following two properties ([1, 23.1.5 and 23.1.6]):

$$E_n'(t) = nE_{n-1}(t), \; n \in \mathbf{N}; \; E_0(t) = 1 \qquad (2.11)$$

and

$$E_n(t+1) + E_n(t) = 2t^n, \;\; n \in \mathbf{N}. \qquad (2.12)$$

Using (2.11) we see that

$$P_n(t) = \frac{(x-a)^n}{n!} E_n\left(\frac{t-a}{x-a}\right), \; n \in \mathbf{N}, \; P_0(t) = 1$$

form a harmonic sequence of polynomials so that (2.1) yields

$$f(x) = f(a) + \sum_{k=1}^{n} (-1)^{k+1} \frac{(x-a)^k}{k!} \left[E_k(1) f^{(k)}(x) - E_k(0) f^{(k)}(a) \right] + R_n^E(f; a, x),$$

where

$$R_n^E(f;a,x) := (-1)^n \frac{(x-a)^n}{n!} \int_a^x E_n\left(\frac{t-a}{x-a}\right) f^{(n+1)}(t)dt. \tag{2.13}$$

Further, since ([1, 23.1.20])

$$E_n(0) = -E_n(1) = -\frac{2}{n+1}(2^{n+1}-1)B_{n+1}, \text{ for } n \in \mathbf{N},$$

we get

$$\begin{aligned} f(x) &= f(a) + 2\sum_{k=1}^{n}(-1)^{k+1}\frac{(x-a)^k(2^{k+1}-1)}{(k+1)!}B_{k+1}\left[f^{(k)}(x)+f^{(k)}(a)\right] \\ &\quad + R_n^E(f;a,x). \end{aligned}$$

Finally, since $B_{2k+1} = 0$ for $k = 1, 2, \cdots$, we get

$$f(x) = T_n^E(f;a,x) + R_n^E(f;a,x),$$

where

$$\begin{aligned} &T_n^E(f;a,x) \\ &:= f(a) + 2\sum_{k=1}^{\left[\frac{n+1}{2}\right]}\frac{(x-a)^{2k-1}(4^k-1)}{(2k)!}B_{2k}\left[f^{(2k-1)}(x)+f^{(2k-1)}(a)\right] \end{aligned} \tag{2.14}$$

and $R_n^E(f;a,x)$ is given by (2.13).

The following result is a generalization of Corollary 1.2 from [2].

COROLLARY 1. *Under the assumptions and with the notation of Theorem 2, we have the estimations:*

$$|R_n(f;a,x)| \leqslant \max_{t\in[a,x]} |P_n(t)| \int_a^x \left|f^{(n+1)}(s)\right| ds \tag{2.15}$$

and

$$|R_n(f;a,x)| \leqslant \max_{t\in[a,x]} \left|f^{(n+1)}(t)\right| \int_a^x |P_n(s)|\, ds. \tag{2.16}$$

Also,

$$|R_n(f;a,x)| \leqslant \left(\int_a^x P_n^q(t)dt\right)^{\frac{1}{q}} \left(\int_a^x \left|f^{(n+1)}(t)\right|^p dt\right)^{\frac{1}{p}}, \tag{2.17}$$

where $p > 1,\ \frac{1}{p}+\frac{1}{q} = 1.$

Proof. The estimations (2.15) and (2.16) are obvious, while the estimation (2.17) is a simple consequence of Hölder inequality. □

3. Main results

Let $a \leqslant x$ and let $g, h : [a,x] \to \mathbf{R}$ be two integrable functions. Define

$$T(g,h) := \frac{1}{x-a}\int_a^x g(t)h(t)dt - \frac{1}{(x-a)^2}\int_a^x g(t)dt \int_a^x h(t)dt$$

Then $T(g,g) \geqslant 0, T(h,h) \geqslant 0$ and the following inequality is valid: (see [3, p. 209])

$$T^2(g,h) \leqslant T(g,g)T(h,h). \tag{3.1}$$

On the other hand, if

$$\alpha \leqslant g(t) \leqslant A, \ \ \beta \leqslant h(t) \leqslant B, \ \ \forall t \in [a,x],$$

for some constants α, A, β and $B,$ then the well known Grüss' inequality

$$|T(g,h)| \leqslant \frac{1}{4}(A-\alpha)(B-\beta) \tag{3.2}$$

holds (see [3, p. 206]). We can combine the inequalities (3.1) and (3.2) to obtain the following result.

LEMMA 1. *Let $a \leqslant x$ and let $g, h : [a,x] \to \mathbf{R}$ be two integrable functions. If*

$$\alpha \leqslant g(t) \leqslant A, \ \ \forall t \in [a,x],$$

for some constants α and A, then

$$|T(g,h)| \leqslant \frac{1}{2}(A-\alpha)\sqrt{T(h,h)}. \tag{3.3}$$

Proof. Setting $h = g$ in (3.2) we get

$$T(g,g) = |T(g,g)| \leqslant \frac{1}{4}(A-\alpha)^2.$$

Combining this with (3.1) we get

$$T^2(g,h) \leqslant \frac{1}{4}(A-\alpha)^2 T(h,h),$$

which is equivalent to (3.3) $\square$

Now we give the generalization of the result stated in Theorem 1. As we shall see, our result also improves the estimation (1.4).

THEOREM 3. *Let $\{P_n(x)\}$ be a harmonic sequence of polynomials. Let $I \subset \mathbf{R}$ be a closed interval and $a \in I$. Suppose $f : I \to R$, is such that $f^{(n)}$ is absolutely continuous. Then for any $x \in I$ we have the generalized Taylor's perturbed formula:*

$$f(x) = \tilde{T}_n(f;a,x) + (-1)^n\left[P_{n+1}(x) - P_{n+1}(a)\right]\left[f^{(n)};a,x\right] + \tilde{G}_n(f;a,x), \tag{3.4}$$

where

$$\tilde{T}_n(f;a,x) = f(a) + \sum_{k=1}^{n}(-1)^{k+1}\left[P_k(x)f^{(k)}(x) - P_k(a)f^{(k)}(a)\right] \tag{3.5}$$

and $\left[f^{(n)};a,x\right]$ *is defined by (1.3). For* $x \geqslant a$ *the remainder* $\tilde{G}(f;a,x)$ *satisfies the estimation*

$$\left|\tilde{G}(f;a,x)\right| \leqslant \frac{x-a}{2}\sqrt{T(P_n,P_n)}[\Gamma(x)-\gamma(x)], \tag{3.6}$$

where $\Gamma(x)$ *and* $\gamma(x)$ *are defined by (1.5)*

Proof. Taylor's generalized formula (2.1) can be rewritten as

$$f(x) = \tilde{T}_n(f;a,x) + (-1)^n\left[P_{n+1}(x) - P_{n+1}(a)\right]\left[f^{(n)};a,x\right] + \tilde{G}_n(f;a,x),$$

where

$$\tilde{G}_n(f;a,x) := R_n(f;a,x) - (-1)^n\left[P_{n+1}(x) - P_{n+1}(a)\right]\left[f^{(n)};a,x\right],$$

and this is just the representation (3.4). By (2.2) we have

$$\begin{aligned}&\tilde{G}_n(f;a,x) \\ =\ &(-1)^n\left\{\int_a^x P_n(t)f^{(n+1)}(t)dt - \left[P_{n+1}(x) - P_{n+1}(a)\right]\left[f^{(n)};a,x\right]\right\}\end{aligned} \tag{3.7}$$

On the other hand, setting $g = f^{(n+1)}$ and $h = P_n$ in Lemma 1 we get

$$\begin{aligned}&\left|\frac{1}{x-a}\int_a^x P_n(t)f^{(n+1)}(t)dt - \frac{1}{(x-a)^2}\int_a^x P_n(t)dt\int_a^x f^{(n+1)}(t)dt\right| \\ \leqslant\ &\frac{1}{2}[\Gamma(x)-\gamma(x)]\sqrt{T(P_n,P_n)}\end{aligned} \tag{3.8}$$

Note that

$$\int_a^x P_n(t)dt = \int_a^x P'_{n+1}(t)dt = P_{n+1}(x) - P_{n+1}(a)$$

and

$$\int_a^x f^{(n+1)}(t)dt = f^{(n)}(x) - f^{(n)}(a) = (x-a)\left[f^{(n)};a,x\right]$$

so that, after multiplying (3.8) by $x-a$, we have

$$\begin{aligned}&\left|\int_a^x P_n(t)f^{(n+1)}(t)dt - \left[P_{n+1}(x) - P_{n+1}(a)\right]\left[f^{(n)};a,x\right]\right| \\ &\qquad \leqslant \frac{x-a}{2}[\Gamma(x)-\gamma(x)]\sqrt{T(P_n,P_n)}.\end{aligned}$$

Combining this with (3.7) we get the estimation (3.6). □

The above result gives the following improvement of the estimation (1.4).

COROLLARY 2. *Let the assumptions of Theorem 1 be satisfied. Then the remainder* $G_n(f;a,x)$ *defined by (1.1) satisfies the estimation*

$$|G_n(f;a,x)| \leqslant \frac{n(x-a)^{n+1}}{2[(n+1)!]\sqrt{2n+1}}[\Gamma(x)-\gamma(x)]. \tag{3.9}$$

Proof. If $P_n(t) = \frac{(t-x)^n}{n!}$, then it is easy to see that

$$\tilde{T}_n(f\,;a,x) = T_n(f\,;a,x) \ \text{ and } \ (-1)^n\left[P_{n+1}(x) - P_{n+1}(a)\right] = \frac{(x-a)^{n+1}}{(n+1)!},$$

so that (3.4) becomes (1.1), that is $\tilde{G}_n(f\,;a,x) = G_n(f\,;a,x)$. Also we have

$$\begin{aligned} T(P_n, P_n) &= \frac{1}{x-a}\int_a^x \frac{(t-x)^{2n}}{(n!)^2}dt - \frac{1}{(x-a)^2}\left(\int_a^x \frac{(t-x)^n}{n!}dt\right)^2 \\ &= \frac{1}{(n!)^2}\left[\frac{1}{x-a}\,\frac{(t-x)^{2n+1}}{2n+1}\Big|_a^x - \frac{1}{(x-a)^2}\left(\frac{(t-x)^{n+1}}{n+1}\Big|_a^x\right)^2\right] \\ &= \frac{n^2(x-a)^{2n}}{[(n+1)!]^2(2n+1)} \end{aligned}$$

or

$$\sqrt{T(P_n, P_n)} = \frac{n(x-a)^n}{(n+1)!\sqrt{2n+1}}. \tag{3.10}$$

Now we apply the inequality (3.6) to obtain the desired result. □

REMARK 1. Denote by Δ_n and $\tilde{\Delta}_n$ the right hand sides of (1.4) and (3.9). respectively. Then we have

$$\tilde{\Delta}_n = \frac{2n}{(n+1)\sqrt{2n+1}}\Delta_n < \Delta_n,$$

since obviously $\frac{2n}{(n+1)\sqrt{2n+1}} < 1$ for all $n \in \mathbf{N}$. Moreover, $\frac{2n}{(n+1)\sqrt{2n+1}}$ tends to zero when n tends to ∞. So the estimation (3.9) is much better than the estimation (1.4).

Using the generalized Taylor's perturbed formula (3.4) we can obtain some other estimations which depend on the choice of the polynomials $P_n(t)$. We first prove the following technical lemma:.

LEMMA 2. *(i) If* $P_n(t) = \frac{1}{n!}\left(t - \frac{a+x}{2}\right)^n$, *then*

$$\sqrt{T(P_n, P_n)} = \frac{(x-a)^n}{n!2^n\sqrt{2n+1}}\left[1 - \frac{1+(-1)^n}{2(n+1)}\right].$$

(ii) Let $P_n(t) = \frac{(x-a)^n}{n!}B_n(\frac{t-a}{x-a})$ *where* $B_n(t)$ *are Bernoulli polynomials. Then*

$$\sqrt{T(P_n, P_n)} = (x-a)^n\sqrt{\frac{|B_{2n}|}{(2n)!}}.$$

(iii) Let $P_n(t) = \frac{(x-a)^n}{n!}E_n(\frac{t-a}{x-a})$ *where* $E_n(t)$ *are Euler polynomials. Then*

$$\sqrt{T(P_n, P_n)} = 2(x-a)^n\sqrt{\frac{(4^{n+1}-1)\,|B_{2n+2}|}{(2n+2)!} - \left[\frac{2(2^{n+2}-1)B_{n+2}}{(n+2)!}\right]^2}$$

Proof. (i) We have

$$\int_a^x P_n^2(t)dt = \frac{1}{(n!)^2}\int_a^x \left(t-\frac{a+x}{2}\right)^{2n} dt$$
$$= \frac{1}{(n!)^2(2n+1)}\left(t-\frac{a+x}{2}\right)^{2n+1}\Bigg|_a^x = \frac{(x-a)^{2n+1}}{(n!)^2 2^{2n}(2n+1)}$$

and

$$\int_a^x P_n(t)dt = \frac{1}{n!}\int_a^x \left(t-\frac{a+x}{2}\right)^{n} dt$$
$$= \frac{1}{(n+1)!}\left(t-\frac{a+x}{2}\right)^{n+1}\Bigg|_a^x = \frac{(x-a)^{n+1}}{(n+1)!2^{n+1}}[1+(-1)^n].$$

So

$$\begin{aligned} T(P_n,P_n) &= \frac{1}{x-a}\int_a^x P_n^2(t)dt - \frac{1}{(x-a)^2}\left(\int_a^x P_n(t)dt\right)^2 \\ &= \frac{(x-a)^{2n}}{(n!)^2 2^{2n}(2n+1)}\left[1-\frac{2n+1}{(n+1)^2}\left(\frac{1+(-1)^n}{2}\right)^2\right] \\ &= \frac{(x-a)^{2n}}{(n!)^2 2^{2n}(2n+1)}\left[1-\frac{1+(-1)^n}{n+1}+\left(\frac{1+(-1)^n}{2(n+1)}\right)^2\right] \\ &= \frac{(x-a)^{2n}}{(n!)^2 2^{2n}(2n+1)}\left[1-\frac{1+(-1)^n}{2(n+1)}\right]^2 \end{aligned}$$

In calculation we used the fact that $[(1+(-1)^n)/2]^2 = (1+(-1)^n)/2$. The desired result follows.

(ii) By the substitution $s=\frac{t-a}{x-a}$ we get

$$\int_a^x P_n^2(t)dt = \frac{(x-a)^{2n}}{(n!)^2}\int_a^x B_n^2\left(\frac{t-a}{x-a}\right)dt = \frac{(x-a)^{2n+1}}{(n!)^2}\int_0^1 B_n^2(s)ds.$$

Bernoulli polynomials have the following property (see [1, 23.1.12])

$$\int_0^1 B_n(s)B_m(s)ds = (-1)^{n-1}\frac{n!m!}{(n+m)!}B_{n+m}, \text{ for } n,m=1,2,\cdots,$$

which for $m=n$ gives

$$\int_0^1 B_n^2(s)ds = (-1)^{n-1}\frac{(n!)^2}{(2n)!}B_{2n} = \frac{(n!)^2}{(2n)!}\left|B_{2n}\right|.$$

So

$$\int_a^x P_n^2(t)dt = \frac{(x-a)^{2n+1}}{(2n)!}\left|B_{2n}\right|.$$

Further

$$\int_a^x P_n(t)dt = \frac{(x-a)^n}{n!}\int_a^x B_n\left(\frac{t-a}{x-a}\right)dt = \frac{(x-a)^{n+1}}{n!}\int_0^1 B_n(s)ds.$$

Using (2.7) we get

$$\int_0^1 B_n(s)ds = \frac{1}{n+1}\int_0^1 B'_{n+1}(s)ds = \frac{B_{n+1}(1)-B_{n+1}(0)}{n+1} = 0,$$

since for $t=0$ (2.8) gives

$$B_{n+1}(1) - B_{n+1}(0) = 0. \tag{3.11}$$

. This implies $\int_a^x P_n(t)dt = 0$ and

$$T(P_n,P_n) = \frac{1}{x-a}\int_a^x P_n^2(t)dt - \frac{1}{(x-a)^2}\left(\int_a^x P_n(t)dt\right)^2 = (x-a)^{2n}\frac{|B_{2n}|}{(2n)!}.$$

The desired result follows.

(iii) The substitution $s = \frac{t-a}{x-a}$ gives

$$\int_a^x P_n^2(t)dt = \frac{(x-a)^{2n}}{(n!)^2}\int_a^x E_n^2\left(\frac{t-a}{x-a}\right)dt = \frac{(x-a)^{2n+1}}{(n!)^2}\int_0^1 E_n^2(s)ds.$$

Euler polynomials have the following property (see [1, 23.1.12])

$$\int_0^1 E_n(s)E_m(s)ds = 4(-1)^n(2^{n+m+2}-1)\frac{n!m!}{(n+m+2)!}B_{n+m+2}, \text{ for } n,m = 0,1,\cdots,$$

which for $m=n$ gives

$$\int_0^1 E_n^2(s)ds = \frac{4(-1)^n(4^{n+1}-1)(n!)^2}{(2n+2)!}B_{2n+2} = \frac{4(4^{n+1}-1)(n!)^2}{(2n+2)!}\left|B_{2n+2}\right|$$

and

$$\int_a^x P_n^2(t)dt = 4(x-a)^{2n+1}\frac{(4^{n+1}-1)\left|B_{2n+2}\right|}{(2n+2)!}.$$

Further

$$\int_a^x P_n(t)dt = \frac{(x-a)^n}{n!}\int_a^x E_n\left(\frac{t-a}{x-a}\right)dt = \frac{(x-a)^{n+1}}{n!}\int_0^1 E_n(s)ds.$$

Using (2.11) we get

$$\int_0^1 E_n(s)ds = \frac{1}{n+1}\int_0^1 E'_{n+1}(s)ds = \frac{E_{n+1}(1)-E_{n+1}(0)}{n+1}.$$

Here we use the following property of Euler polynomials (see [1, 23.1,20])

$$E_n(0) = -E_n(1) = -\frac{2}{n+1}(2^{n+1}-1)B_{n+1}, \text{ for } n = 1,2,\cdots, \tag{3.12}$$

to get

$$\int_0^1 E_n(s)ds = \frac{2E_{n+1}(1)}{n+1} = \frac{4(2^{n+2}-1)B_{n+2}}{(n+1)(n+2)}.$$

This implies

$$\int_a^x P_n(t)dt = \frac{4(x-a)^{n+1}(2^{n+2}-1)B_{n+2}}{(n+2)!}$$

and

$$\begin{aligned} T(P_n,P_n) &= \frac{1}{x-a}\int_a^x P_n^2(t)dt - \frac{1}{(x-a)^2}\left(\int_a^x P_n(t)dt\right)^2 \\ &= 4(x-a)^{2n}\left\{\frac{(4^{n+1}-1)\left|B_{2n+2}\right|}{(2n+2)!} - \left[\frac{2(2^{n+2}-1)B_{n+2}}{(n+2)!}\right]^2\right\}. \end{aligned}$$

The desired result follows. □

COROLLARY 3. *Let $I \subset \mathbf{R}$ be a closed interval and $a,x \in I$, $a \leqslant x$.. Suppose $f : I \to R$, is such that $f^{(n)}$ is absolutely continuous. Let $\Gamma(x)$ and $\gamma(x)$ be defined by (1.5).*

(i) If $T_n^M(f;a,x)$ is defined by (2.5), then

$$f(x) = T_n^M(f;a,x) + \frac{(x-a)^{n+1}[1+(-1)^n]}{(n+1)!2^{n+1}}\left[f^{(n)};a,x\right] + G_n^M(f;a,x)$$

and

$$\left|G_n^M(f;a,x)\right| \leqslant \frac{(x-a)^{n+1}}{n!2^{n+1}\sqrt{2n+1}}\left[1 - \frac{1+(-1)^n}{2(n+1)}\right][\Gamma(x)-\gamma(x)].$$

(ii) If $T_n^B(f;a,x)$ is defined by (2.10), then

$$f(x) = T_n^B(f;a,x) + G_n^B(f;a,x)$$

and

$$\left|G_n^B(f;a,x)\right| \leqslant \frac{(x-a)^{n+1}}{2}\sqrt{\frac{|B_{2n}|}{(2n)!}}[\Gamma(x)-\gamma(x)].$$

(iii) If $T_n^E(f;a,x)$ is defined by (2.14), then

$$f(x) = T_n^E(f;a,x) + \frac{4(-1)^n(x-a)^{n+1}(2^{n+2}-1)B_{n+2}}{(n+2)!}\left[f^{(n)};a,x\right] + G_n^E(f;a,x)$$

and

$$\begin{aligned}&\left|G_n^E(f;a,x)\right|\\ &\leqslant (x-a)^{n+1}\sqrt{\frac{(4^{n+1}-1)\left|B_{2n+2}\right|}{(2n+2)!}-\left[\frac{2(2^{n+2}-1)B_{n+2}}{(n+2)!}\right]^2}[\Gamma(x)-\gamma(x)].\end{aligned}$$

Proof. (i) Set $P_n(t)=\frac{1}{n!}\left(t-\frac{a+x}{2}\right)^n$. We have $\tilde{T}_n(f;a,x)=T_n^M(f;a,x)$ and $\tilde{G}_n(f;a,x)=G_n^M(f;a,x)$. Also

$$(-1)^n\left[P_{n+1}(x)-P_{n+1}(a)\right]=\frac{(x-a)^{n+1}[1+(-1)^n]}{(n+1)!2^{n+1}}.$$

Now apply Theorem 3 and Lemma 2(i).

(ii) Set $P_n(t)=\frac{(x-a)^n}{n!}B_n\left(\frac{t-a}{x-a}\right)$. We have $\tilde{T}_n(f;a,x)=T_n^B(f;a,x)$ and $\tilde{G}_n(f;a,x)=G_n^B(f;a,x)$. Also, by (3.11)

$$(-1)^n\left[P_{n+1}(x)-P_{n+1}(a)\right]=(-1)^n\frac{(x-a)^{n+1}}{(n+1)!}[B_{n+1}(1)-B_{n+1}(0)]=0.$$

Now apply Theorem 3 and Lemma 2(ii).

(iii) Set $P_n(t)=\frac{(x-a)^n}{n!}E_n\left(\frac{t-a}{x-a}\right)$. We have $\tilde{T}_n(f;a,x)=T_n^E(f;a,x)$ and $\tilde{G}_n(f;a,x)=G_n^E(f;a,x)$. Also, using (3.12) we get

$$\begin{aligned}(-1)^n\left[P_{n+1}(x)-P_{n+1}(a)\right] &= (-1)^n\frac{(x-a)^{n+1}}{(n+1)!}[E_{n+1}(1)-E_{n+1}(0)].\\ &= (-1)^n\frac{(x-a)^{n+1}}{(n+1)!}2E_{n+1}(1)\\ &= \frac{4(-1)^n(x-a)^{n+1}(2^{n+2}-1)B_{n+2}}{(n+2)!}.\end{aligned}$$

Now apply Theorem 3 and Lemma 2(iii). □

THEOREM 4. *Suppose the assumptions of Theorem 3 are satisfied. Additionally, suppose $f^{(n+1)}$ is differentiable and such that*

$$M^{(n+2)}(x):=\sup_{t\in[a,x]}\left|f^{(n+2)}(t)\right|<\infty,\ \text{for } a\leqslant x.$$

Then the remainder $\tilde{G}_n(f;a,x)$ satisfies the estimation

$$\left|\tilde{G}_n(f;a,x)\right|\leqslant\frac{(x-a)^2M^{(n+2)}(x)}{\sqrt{12}}\sqrt{T(P_n,P_n)}$$

for all $x\geqslant a,\ x\in I$.

Proof. If $g,h:[a,x]\to\mathbf{R}$ are absolutely continuous and if g',h' are bounded, then the following Chebyshev's inequality is valid (see [3, p. 207])

$$|T(g,h)|\leqslant\frac{1}{12}(x-a)^2\sup_{t\in[a,x]}|g'(t)|\sup_{t\in[a,x]}|h'(t)|.$$

Setting $h=g$ we get

$$T(g,g)=|T(g,g)|\leqslant\frac{1}{12}(x-a)^2\left(\sup_{t\in[a,x]}|g'(t)|\right)^2.$$

Combining this with (3.1) we have

$$T^2(g,h)\leqslant\frac{1}{12}(x-a)^2\left(\sup_{t\in[a,x]}|g'(t)|\right)^2T(h,h)$$

or

$$|T(g,h)|\leqslant\frac{(x-a)}{\sqrt{12}}\sup_{t\in[a,x]}|g'(t)|\sqrt{T(h,h)}. \tag{3.13}$$

Reviewing the proof of Theorem 3 it is easy to see that

$$\tilde{G}_n(f;a,x)=(-1)^n(x-a)T(f^{(n+1)},P_n). \tag{3.14}$$

So, if we apply (3.13) to $g=f^{(n+1)}$ and $h=P_n$, then we get

$$\left|\tilde{G}_n(f;a,x)\right|=(x-a)\left|T(f^{(n+1)},P_n)\right|\leqslant\frac{(x-a)^2}{\sqrt{12}}M^{(n+2)}(x)\sqrt{T(P_n,P_n)}.$$

□

COROLLARY 4. *Let the assumptions of Theorem 4. be satisfied. Then we have the representation (1.1) and the remainder $G_n(f;a,x)$ satisfies the estimation:*

$$|G_n(f;a,x)|\leqslant\tilde{\Delta}_n:=\frac{n(x-a)^{n+2}M^{(n+2)}(x)}{\sqrt{12}(n+1)!\sqrt{2n+1}}$$

Proof. Set $P_n(t)=\frac{(t-x)^n}{n!}$ and apply Theorem 4. Then use (3.10) to obtain the desired result. □

REMARK 2. In [2, Theorem 2.4] the following estimation was obtained

$$|G_n(f;a,x)|\leqslant\Delta_n:=\frac{(x-a)^{n+2}M^{(n+2)}(x)}{12[(n-1)!]}.$$

We have

$$\tilde{\Delta}_n=\frac{\sqrt{12}}{(n+1)\sqrt{2n+1}}\Delta_n$$

and $\frac{\sqrt{12}}{(n+1)\sqrt{2n+1}}<1$, for $n>1$. So, our estimation established in Corollary 4 is better than one given in [2].

As a corollary to Theorem 4 we give further estimations which depend on the choice of the polynomials $P_n(t)$.

COROLLARY 5. *Under the assumptions of Theorem 4 and with the notation of Corollary 3 we have the following estimations:*

$$\left|G_n^M(f;a,x)\right| \leqslant \frac{(x-a)^{n+2}M^{(n+2)}(x)}{\sqrt{12}n!2^n\sqrt{2n+1}}\left[1-\frac{1+(-1)^n}{2(n+1)}\right],$$

$$\left|G_n^B(f;a,x)\right| \leqslant \frac{(x-a)^{n+2}M^{(n+2)}(x)}{\sqrt{12}}\sqrt{\frac{|B_{2n}|}{(2n)!}}$$

and

$$\left|G_n^E(f;a,x)\right| \leqslant \frac{(x-a)^{n+2}M^{(n+2)}(x)}{\sqrt{3}}\sqrt{\frac{(4^{n+1}-1)\,|B_{2n+2}|}{(2n+2)!}-\left[\frac{2(2^{n+2}-1)B_{n+2}}{(n+2)!}\right]^2}.$$

Proof. We use Theorem 4 and Lemma 2 to obtain the desired estimations. □

THEOREM 5. *Suppose the assumptions of Theorem 3 are satisfied. Also suppose that $f^{(n+1)}$ is locally absolutely continuous on $I_x=(a,x)$ and $f^{(n+2)}\in L_2(I_x)$ for every $x\in I,\ x>a$. Then the remainder $\tilde{G}_n(f;a,x)$ satisfies the estimation*

$$\left|\tilde{G}_n(f;a,x)\right| \leqslant \frac{(x-a)^2}{\pi}N_2(f^{(n+2)};a,x)\sqrt{T(P_nP_n)},$$

where

$$N_2(f^{(n+2)};a,x)=\left(\frac{1}{x-a}\int_a^x\left|f^{(n+2)}(t)\right|^2\right)^{1/2},\ \ \textit{for } x>a.$$

Proof. We recall the result of Lupaş (see [3, p. 210]): if $g,h:(a,x)\to\mathbf{R}$ are locally absolutely continuous on $I=(a,x)$ and $g',h'\in L_2(I)$, then

$$|T(g,h)| \leqslant \frac{(x-a)^2}{\pi^2}\|g'\|_2\,\|h'\|_2\,,$$

where

$$\|f\|_2:=\left(\frac{1}{x-a}\int_a^x|f(t)|^2\,dt\right)^{1/2},\ \ \text{for } f\in L_2(I).$$

Setting $h=g$ we get

$$T(g,g)=|T(g,g)| \leqslant \frac{(x-a)^2}{\pi^2}\|g'\|_2^2\,,$$

and this, in combination with (3.1), implies

$$T^2(g,h) \leqslant \frac{(x-a)^2}{\pi^2} \|g'\|_2^2 \, T(h,h)$$

or

$$|T(g,h)| \leqslant \frac{x-a}{\pi} \|g'\|_2 \sqrt{T(h,h)}. \tag{3.15}$$

Now apply (3.15) to $g = f^{(n+1)}$ and $h = P_n$ and use (3.14) to obtain

$$\begin{aligned} \left|\tilde{G}_n(f;a,x)\right| &\leqslant (x-a)\frac{x-a}{\pi} \left\| \left(f^{(n+1)}\right)' \right\|_2 \sqrt{T(P_n P_n)} \\ &= \frac{(x-a)^2}{\pi} N_2(f^{(n+2)};a,x)\sqrt{T(P_n P_n)}. \end{aligned}$$

□

COROLLARY 6. *Let the assumptions of Theorem 5. be satisfied. Then we have the representation (1.1) and the remainder* $G_n(f;a,x)$ *satisfies the estimation:*

$$|G_n(f;a,x)| \leqslant \frac{n(x-a)^{n+2} N_2(f^{(n+2)};a,x)}{\pi (n+1)! \sqrt{2n+1}}.$$

Proof. Set $P_n(t) = \frac{(t-x)^n}{n!}$ and apply Theorem 5. Then use (3.10) to obtain the desired result. □

COROLLARY 7. *Under the assumptions of Theorem 5 and with the notation of Corollary 3 we have the following estimations:*

$$\left|G_n^M(f;a,x)\right| \leqslant \frac{(x-a)^{n+2} N_2(f^{(n+2)};a,x)}{\pi n! 2^n \sqrt{2n+1}} \left[1 - \frac{1+(-1)^n}{2(n+1)}\right],$$

$$\left|G_n^B(f;a,x)\right| \leqslant \frac{(x-a)^{n+2} N_2(f^{(n+2)};a,x)}{\pi} \sqrt{\frac{|B_{2n}|}{(2n)!}}$$

and

$$\begin{aligned} &\left|G_n^E(f;a,x)\right| \\ \leqslant\ & \frac{2(x-a)^{n+2} N_2(f^{(n+2)};a,x)}{\pi} \sqrt{\frac{(4^{n+1}-1)\,|B_{2n+2}|}{(2n+2)!} - \left[\frac{2(2^{n+2}-1)B_{n+2}}{(n+2)!}\right]^2}. \end{aligned}$$

Proof. The proof is a simple consequence of Theorem 5 and Lemma 2. □

4. Applications to the logarithmic mapping

Consider the logarithmic function $f : (0,\infty) \to \mathbf{R}$, $f(t) = \ln t$. We shall apply the generalized Taylor's perturbed formula to this function. We have

$$f^{(k)}(t) = \frac{(-1)^{k-1}(k-1)!}{t^k}, \text{ for } t > 0 \text{ and } k \in \mathbf{N}.$$

For $n \in \mathbf{N}$ and $x \geqslant a > 0$ we get

$$\left[f^{(n)}; a, x\right] = \frac{f^{(n)}(x) - f^{(n)}(a)}{x-a} = \frac{(-1)^{n-1}(n-1)!}{x-a}\left(\frac{1}{x^n} - \frac{1}{a^n}\right). \tag{4.1}$$

Note that $f^{(n+1)}$ is strictly monotonic on $(0,\infty)$ which implies

$$\gamma(x) = \inf_{t\in[a,x]} f^{(n+1)}(t) = \min\left\{f^{(n+1)}(a), f^{(n+1)}(x)\right\}$$

and

$$\Gamma(x) = \sup_{t\in[a,x]} f^{(n+1)}(t) = \max\left\{f^{(n+1)}(a), f^{(n+1)}(x)\right\}.$$

So

$$\begin{aligned} & \Gamma(x) - \gamma(x) \\ = {} & \max\left\{f^{(n+1)}(a), f^{(n+1)}(x)\right\} - \min\left\{f^{(n+1)}(a), f^{(n+1)}(x)\right\} \\ = {} & \left|f^{(n+1)}(a) - f^{(n+1)}(x)\right| \\ = {} & n!\left(\frac{1}{a^{n+1}} - \frac{1}{x^{n+1}}\right). \end{aligned} \tag{4.2}$$

Now, let us observe four different cases.

Case 1: Let $P_n(t) = \frac{(t-x)^n}{n!}$ for $n \in \mathbf{N}$ and $P_0(t) = 1$. By Corollary 2, equality (1.1) holds. An easy calculation gives

$$T_n(\ln; a, x) = \ln a + \sum_{k=1}^{n} (-1)^{k+1} \frac{(x-a)^k}{ka^k},$$

while (4.1) gives

$$\frac{(x-a)^{n+1}}{(n+1)!}\left[f^{(n)}; a, x\right] = \frac{(a-x)^n}{n(n+1)}\left(\frac{1}{a^n} - \frac{1}{x^n}\right).$$

So, by (1.1) we have

$$\ln x = \ln a + \sum_{k=1}^{n} (-1)^{k+1} \frac{(x-a)^k}{ka^k} + \frac{(a-x)^n}{n(n+1)}\left(\frac{1}{a^n} - \frac{1}{x^n}\right) + G_n(\ln; a, x).$$

Using (3.9) and (4.2) we get the estimation

$$|G_n(\ln; a, x)| \leqslant \tilde{\delta}_n := \frac{n(x-a)^{n+1}}{2(n+1)\sqrt{2n+1}} \left(\frac{1}{a^{n+1}} - \frac{1}{x^{n+1}} \right). \tag{4.3}$$

In [2] the estimation

$$|G_n(\ln; a, x)| \leqslant \delta_n := \frac{(x-a)^{n+1}}{4} \left(\frac{1}{a^{n+1}} - \frac{1}{x^{n+1}} \right) \tag{4.4}$$

was obtained. We have

$$\tilde{\delta}_n = \frac{2n}{(n+1)\sqrt{2n+1}} \delta_n < \delta_n,$$

since $\frac{2n}{(n+1)\sqrt{2n+1}} < 1$ for all $n \in \mathbf{N}$. Moreover, $\frac{2n}{(n+1)\sqrt{2n+1}}$ tends to zero when n tends to ∞. So the estimation (4.3) is much better than the estimation (4.4).

Case 2: Let $P_n(t) = \frac{1}{n!}\left(t - \frac{a+x}{2}\right)^n$ for $n \in \mathbf{N}$ and $P_0(t) = 1$. In this case we can apply Corollary 3 (i). By (2.5) we have

$$\begin{aligned} T_n^M(\ln; a, x) &= \ln a + \sum_{k=1}^{n} \frac{(x-a)^k}{2^k k!} \left[\frac{(-1)^{k-1}(k-1)!}{a^k} - (-1)^k \frac{(-1)^{k-1}(k-1)!}{x^k} \right] \\ &= \ln a + \sum_{k=1}^{n} \frac{(x-a)^k}{k 2^k} \left[\frac{1}{x^k} + \frac{(-1)^{k-1}}{a^k} \right], \end{aligned}$$

while, using (4.1), we get

$$\frac{(x-a)^{n+1}[1+(-1)^n]}{(n+1)!2^{n+1}} \left[f^{(n)}; a, x \right] = \frac{1+(-1)^n}{n(n+1)2^{n+1}} \left(\frac{1}{a^n} - \frac{1}{x^n} \right) (x-a)^n.$$

So

$$\begin{aligned} \ln x &= \ln a + \sum_{k=1}^{n} \frac{(x-a)^k}{k 2^k} \left[\frac{1}{x^k} + \frac{(-1)^{k-1}}{a^k} \right] + \\ & \quad \frac{1+(-1)^n}{n(n+1)2^{n+1}} \left(\frac{1}{a^n} - \frac{1}{x^n} \right) (x-a)^n + G_n^M(\ln; a, x), \end{aligned}$$

where, by Corollary 3 (i) and by (4.2), the remainder $G_n^M(\ln; a, x)$ satisfies the estimation

$$\left| G_n^M(\ln; a, x) \right| \leqslant \frac{(x-a)^{n+1}}{2^{n+1}\sqrt{2n+1}} \left[1 - \frac{1+(-1)^n}{2(n+1)} \right] \left(\frac{1}{a^{n+1}} - \frac{1}{x^{n+1}} \right).$$

Case 3: Let $P_n(t) = \frac{(x-a)^n}{n!} B_n(\frac{t-a}{x-a})$ for $n \in \mathbf{N}$ and $P_0(t) = 1$. Now we apply Corollary 3 (ii). Using (2.10) we easily calculate

$$T_n^B(\ln; a, x) = \ln a + \frac{x^2 - a^2}{2ax} - \frac{1}{2} \sum_{k=1}^{\left[\frac{n}{2}\right]} \frac{B_{2k}}{k} \left(\frac{1}{a^{2k}} - \frac{1}{x^{2k}} \right) (x-a)^{2k}.$$

By Corollary 3 (ii)

$$\ln x = \ln a + \frac{x^2 - a^2}{2ax} - \frac{1}{2}\sum_{k=1}^{\left[\frac{n}{2}\right]} \frac{B_{2k}}{k}\left(\frac{1}{a^{2k}} - \frac{1}{x^{2k}}\right)(x-a)^{2k} + G_n^B(\ln; a, x),$$

where, by Corollary 3 (ii) and by (4.2), the remainder $G_n^B(\ln; a, x)$ satisfies the estimation

$$\left|G_n^B(\ln; a, x)\right| \leqslant \frac{n!}{2}\sqrt{\frac{|B_{2n}|}{(2n)!}}\left(\frac{1}{a^{n+1}} - \frac{1}{x^{n+1}}\right)(x-a)^{n+1}.$$

Case 4: Let $P_n(t) = \frac{(x-a)^n}{n!}E_n(\frac{t-a}{x-a})$ for $n \in \mathbf{N}$ and $P_0(t) = 1$. In this case we apply Corollary 3 (iii). Using (2.14) we easily calculate

$$T_n^E(\ln; a, x) = \ln a + \sum_{k=1}^{\left[\frac{n+1}{2}\right]} \frac{(4^k - 1)B_{2k}}{k(2k-1)}\left(\frac{1}{a^{2k-1}} + \frac{1}{x^{2k-1}}\right)(x-a)^{2k-1},$$

while (4.1) implies

$$\begin{aligned} & \frac{4(-1)^n (x-a)^{n+1}(2^{n+2}-1)B_{n+2}}{(n+2)!}\left[f^{(n)}; a, x\right] \\ = \ & \frac{4(2^{n+2}-1)B_{n+2}}{n(n+1)(n+2)}\left(\frac{1}{a^n} - \frac{1}{x^n}\right)(x-a)^n. \end{aligned}$$

By Corollary 3 (iii)

$$\begin{aligned} \ln x \ = \ & \ln a + \sum_{k=1}^{\left[\frac{n+1}{2}\right]} \frac{(4^k - 1)B_{2k}}{k(2k-1)}\left(\frac{1}{a^{2k-1}} + \frac{1}{x^{2k-1}}\right)(x-a)^{2k-1} + \\ & \frac{4(2^{n+2}-1)B_{n+2}}{n(n+1)(n+2)}\left(\frac{1}{a^n} - \frac{1}{x^n}\right)(x-a)^n + G_n^E(\ln; a, x) \end{aligned}$$

and, by Corollary 3 (iii) and by (4.2), the remainder $G_n^E(\ln; a, x)$ satisfies the estimation

$$\begin{aligned} & \left|G_n^E(\ln; a, x)\right| \\ \leqslant \ & n!\sqrt{\frac{(4^{n+1}-1)\left|B_{2n+2}\right|}{(2n+2)!} - \left[\frac{2(2^{n+2}-1)B_{n+2}}{(n+2)!}\right]^2}\left(\frac{1}{a^{n+1}} - \frac{1}{x^{n+1}}\right)(x-a)^{n+1}. \end{aligned}$$

REMARK 3. In obvious way, similar estimations can be established for the remainders $G_n(\ln; a, x)$, $G_n^M(\ln; a, x)$, $G_n^B(\ln; a, x)$ and $G_n^E(\ln; a, x)$, by application of Corollaries 4, 5, 6 and 7.

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