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A BOUNDEDNESS THEOREM ON HIGHER DIMENSIONAL HILL EQUATIONS

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Abstract. We present in this note some sufficient conditions on boundedness of solutions to higher dimensional Hill equations.

Consider the well known Hill equation

$$\ddot{y} + a(t)y = 0 \tag{1}$$

There are many results [1], [2], [4] on the asymptotic behavior of solutions to (1). One of them is the boundedness of all solutions to (1) under certain conditions imposed on coefficient function $a(t)$.

We consider in this note the higher dimensional Hill equation system

$$\ddot{X} + A(t)X = 0, \tag{2}$$

where $X \in \mathbf{R}^n$, $A(t) = (a_{ij})$ is a symmetric $n \times n$ matrix function of time t. We are interested in knowing under what condition all solutions to (2) are bounded, becuse it is significant in investigation of stability and instability of geodesic on Riemannian manifolds where Jacobi fields can be expressed in form of Hill equation system [3]. This fact has been used by some physicists to study dynamics in Hamiltonian systems [5].

Certain sufficient conditions for boundedness of solutions to (2) are given in the main theorem, which may be regarded as a generalization of a corresponding theorem concerning (1) in [1].

THEOREM. *Suppose that there exist two positive constants K and $\overline{K}$ such that*

$$K\|x\|^2 \leqslant X^T A(t)X \leqslant \overline{K}\|X\|^2. \tag{3}$$

Furthermore, assume that

$$(i) \qquad X^T \dot{A}(t)X \geqslant 0, \quad or \quad (ii) \qquad X^T \dot{A}(t)X \leqslant 0. \tag{4}$$

Then every solution to (2) is bounded.

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Proof. Case (i), $X^T\dot{A}(t)X \geqslant 0$. From (2) one has

$$\langle \dot{X}, \ddot{X}\rangle + \langle \dot{X}, A(t)X\rangle = 0.$$

Integration by parts from 0 to t yields

$$\frac{1}{2}\|\dot{X}\|^2 + \frac{1}{2}X^TA(t)X - \frac{1}{2}\int_0^t X^T\dot{A}(t)Xdt = C_1.$$

Accordingly, one has

$$\begin{aligned}
X^TA(t)X &\leqslant |C_1| + \frac{1}{2}\int_0^t X^T\dot{A}(t)Xdt \qquad (A(t) \text{ is symmetric})\\
&= |C_1| + \int_0^t X^TA(t)X\cdot\frac{X^T\dot{A}(t)X}{X^TA(t)X}dt\\
&\leqslant |C_1| + \int_0^t X^TA(t)X\cdot\frac{(\operatorname{trace}\dot{A}(t))\|X\|^2}{K\|X\|^2}dt, \qquad (\text{by } (i) \text{ in } (4))\\
&= |C_1| + \frac{1}{K}\int_0^t X^TA(t)X\Big(\sum_{i=1}^n \dot{a}_{ii}(t)\Big)dt.
\end{aligned}$$

By Gronwal inequality, one has

$$\begin{aligned}
X^TAX &\leqslant |C_1|\exp\Big(\frac{1}{K}\int_0^t \sum_{i=1}^n \dot{a}_{ii}(t)\Big)dt\\
&= |C_1|\exp\Big(\frac{1}{K}\sum_{i=1}^n (a_{ii} - a_{ii}(0)\Big)\\
&\leqslant |C_1|\exp\Big(\frac{1}{K}\sum_{i=1}^n a_{ii}(t)\Big).
\end{aligned}$$

Because of inequality (3), we have

$$|a_{ii}(t)| \leqslant \overline{K}.$$

Therefore

$$X^TA(t)X \leqslant |C_1|\exp\frac{n\overline{K}}{K},$$

which implies that $X(t)$ is bounded.

Case (ii), $X^T\dot{A}X \leqslant 0$. Construct a Lyapunov function

$$V(t, X, \dot{X}) = X^TA(t)X + \|\dot{X}\|^2.$$

Derivation of V along the trajectories of (2) yields

$$\begin{aligned}\dot{V} &= \dot{X}A(t)X + X^T\dot{A}(t)X + X^TA\dot{X} + 2\langle \dot{X}, \ddot{X}\rangle \\ &= 2\dot{X}^TA(t)X + X^T\dot{A}X + 2\dot{X}^T(-AX) \\ &= X^TAX \\ &= 0\end{aligned}$$

It follows that $X(t)$ is bounded.

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