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RIESZ'S FUNCTIONS AND CARLESON INEQUALITIES

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Abstract. Let μ be a finite positive Borel measure on the open unit disc D and H a set of all analytic functions on D. For each a in D, put

$$r(\mu, a) = \sup |f(a)|^2$$

where $f \in H$ and $\int_D |f|^2 d\mu \leqslant 1$. Unless the support set of μ is a finite set, $\int_D r(\mu, a) d\mu(a) = \infty$. However

$$\sup_{z \in D} \int_{D_t(z)} r(\mu, a) d\mu(a) < \infty$$

may happen where $D_t(z)$ denotes the Bergman disc in D. We study when this is possible. When ν is a discrete measure such that $d\nu = \sum_{a \in A} s(\mu, a)\delta_a$,

$$\sup_{z \in D} \int_{D_t(z)} r(\mu, a) d\nu(a) = \sup_{z \in D} \sum_{a \in A \cap D_t(z)} 1.$$

Under some condition on μ, we show that $\sup_{z \in D} \int_{D_t(z)} r(\mu, a) d\nu(a) < \infty$ for a finite positive Borel measure ν on D if and only if (ν, μ)-Carleson inequality is valid.

1. Introduction

Let D be the open unit disc in $\mathbf{C}$ and H a set of all analytic functions on D. When μ is a finite positive Borel measure on D and $a \in D$, put

$$s(\mu) = s(\mu, a) = \inf\{\int_D |f|^2 d\mu \; ; \; f \in H \text{ and } f(a) = 1\}$$

and

$$r(\mu) = r(\mu, a) = \sup\{|f(a)|^2 \; ; \; f \in H \text{ and } \int_D |f|^2 d\mu \leqslant 1\}.$$

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In the previous paper [2], we noted the following : $r(\mu,a)s(\mu,a) = 1$ for $a \in D$, assuming $\infty \times 0 = 1, r(\mu)$ is lower semicontinuous and $s(\mu)$ is upper semicontinuous on D. r and s are called Riesz's functions. Corollary 2 in [2] shows that

$$\int_D r(\mu,a)d\mu(a) = \infty$$

if supp μ is not a finite set. By Theorem 8 and Lemma 2 in [2], when (supp μ) $\cap D$ is a uniqueness set for H, $L^2_a(\mu) = H \cap L^2(\mu)$ is closed if and only if for all compact sets K in D

$$\int_K r(\mu,a)d\mu(a) < \infty.$$

For any z in D, let ϕ_z be the Möbius function on D and put

$$\beta(z,w) = \frac{1}{2}\log(1+|\phi_z(w)|)(1-|\phi_z(w)|)^{-1} \quad (z,w \in D).$$

For $0 < t < \infty$ and z in D, set

$$D_t(z) = \{w \in D \,;\, \beta(z,w) < t\}$$

which is called the Bergman disc with "center" z and "radius" t. For all compact sets K in D $\int_K r(\mu,a)d\mu(a) < \infty$ if and only if for any z in D, $\int_{D_t(z)} r(\mu,a)d\mu(a) < \infty$. We are interested in when

$$\sup_{z\in D}\int_{D_t(z)} r(\mu,a)d\mu(a) < \infty.$$

In Section 2, we study a finite positive Borel measure μ such that

$$\sup_{z\in D}\int_{D_t(z)} r(\mu,a)d\mu(a) < \infty.$$

If $\sup_{z\in D}\int_{D_t(z)} r(\mu,a)d\mu(a) < \infty$ and ν is a finite positive Borel measure on D with $\nu \leqslant \mu$ then $\sup_{z\in D}\int_{D_t(z)} r(\mu,a)d\nu(a) < \infty$. Even if $\sup_{z\in D}\int_{D_t(z)} r(\mu,a)d\mu(a) = \infty$, $\sup_{z\in D}\int_{D_t(z)} r(\mu,a)d\nu(a) < \infty$ may happen when ν is enough small. In Section 3, we study a finite positive Borel measure ν for each μ such that $\sup_{z\in D}\int_{D_t(z)} r(\mu,a)d\nu(a) < \infty$.

Throughout this paper, the measure m denotes the normalized Lebesgue area measure on D. We define an average of a finite positive Borel measure μ on $D_t(z)$ by $\hat{\mu}_t(z) = \mu(D_t(z))/m(D_t(z))$ $(z \in D)$. We say that ν and μ satisfy the (ν,μ)-Carleson inequality, if there is a constant $C > 0$ such that

$$\int_D |f|^2 d\nu \leqslant C\int_D |f|^2 d\mu$$

for all f in H. Under some condition on μ, the (ν,μ)-Carleson inequality is valid if and only if $\hat{\nu}_t \leqslant \gamma\hat{\mu}_t$ on D for some positive constant $\gamma > 0$ (see [1]). In Section 4, under the same condition above on μ, we show that the (ν,μ)-Carleson inequality is valid if and only if $\sup_{z\in D}\int_{D_t(z)} r(\mu,a)d\nu(a) < \infty$. If $d\nu = \sum_{a\in A} s(\mu,a)\delta_a$ where A is a discrete set in D and δ_a denotes a po int mass measure at a, then note that

$$\sup_{z\in D}\int_{D_t(z)} r(\mu,a)d\nu(a) = \sup_{z\in D}\sum_{a\in A\cap D_t(z)} 1.$$

Hence our result implies a generalization of a theorem of K.Zhu in [4, Theorem 1] when $\mu = m$.

For a finite positive Borel measure μ on D, put

$$\tilde{\mu}(a) = \int_D |k_a(z)|^2 d\mu(z) \quad (a \in D)$$

where $k_a(z) = (1-|a|^2)/(1-\bar{a}z)^2$. When $d\mu = wdm$, we denote the function by $\tilde{w}$ instead of $\tilde{\mu}$. We say that w is in $(A_2)_\partial$ if there exists a finite positive constant γ such that

$$\tilde{w}(a) \times (w^{-1})^{\sim}(a) \leqslant \gamma$$

for all a in D.

2. $r(\mu,a)d\mu(a)$

It is easy to see that $\sup_{z\in D}\int_{D_t(z)} r(m,a)dm(a) < \infty$. In fact,

$$\begin{aligned}
&\sup_{z\in D}\int_{D_t(z)} (1-|a|^2)^{-2}dm(a)\\
&\leqslant \sup_{z\in D}\left\{\sup_{a\in D_t(z)} (1-|a|^2)^{-2}\right\} m(D_t(z))\\
&= \sup_{z\in D}\left\{\sup_{w\in D_t(0)}\left(1-\left|\frac{z-w}{1-\bar{z}w}\right|^2\right)^{-2}\right\}\frac{(1-|z|^2)^2k^2}{(1-|z|^2k^2)^2}\\
&\leqslant \sup_{z\in D}(1-|z|^2k^2)^{-2} \times \frac{16}{k^2}\sup_{w\in D_t(0)}(1-|w|^2)^{-2} < \infty
\end{aligned}$$

because $k = \tanh t \in (0,1)$. When $d\mu = \sum_{a\in A} s(\mu,a)\delta_a$ and A is a set of finitely many separated sequences in D, $\sup_{z\in D}\int_{D_t(z)} r(\mu,a)d\mu(a) < \infty$ because $r(\mu,a)d\mu(a) = \sum_{a\in A}\delta_a$. Put

$$\mathscr{R} = \{\mu\ ;\ \sup_{z\in D}\int_{D_t(z)} r(\mu,a)d\mu(a) < \infty\},$$

then m belongs to $\mathscr{R}$ and $d\mu = \sum_{a\in A} s(\mu,a)\delta_a$ belongs to A if and only if A is a set of finitely many separated sequences in D, that is, the supremum of the number of points in $A\cap D_t(z)$ is finite.

LEMMA 1. *Suppose $d\mu = wdm$ and w is in $(A_2)_\partial$, and fix $t>0$. If $\beta(z,a)<t$ then there exists a constant $C>0$ such that*

$$\frac{1}{C}\leqslant\frac{\mu(D_t(a))}{\mu(D_t(z))}\leqslant C.$$

Proof. *When $\beta(z,a)<t$, $D_t(a)\subset D_{2t}(z)$ and so it is enough to prove that*

$$\sup_{z\in D}\frac{\mu(D_{2t}(z))}{\mu(D_t(z))}<\infty.$$

This is equivalent to that

$$\sup_{z\in D}\frac{\hat{\mu}_{2t}(z)}{\hat{\mu}_t(z)}<\infty.$$

$\hat{\mu}_t\leqslant\tilde{\mu}$ on D for any t by Lemma 4.3.3 in [5]. By hypothesis on w, there exists a finite positive constant γ such that $\tilde{\mu}\leqslant\gamma\hat{\mu}_t$ on D for any t (see [1, p157]). This implies that $\sup_{z\in D}\hat{\mu}_{2t}(z)/\hat{\mu}_t(z)<\infty$.

LEMMA 2. *Suppose $d\mu = wdm$ and w is in $(A_2)_\partial$, and fix $t>0$. Then there exists a constant $C>0$ such that*

$$\frac{1}{C}\leqslant\frac{s(\mu,z)}{\mu(D_t(z))}\leqslant C\quad(z\in D).$$

Proof. Since $w\in(A_2)_\partial$, there exists a positive constant γ

$$\gamma^{-1}\tilde{w}\leqslant(\widetilde{w^{-1}})^{-1}\leqslant\exp(\widetilde{\log w})$$

on D. By Proposition 4 in [2]

$$\gamma^{-1}\tilde{w}(z)\leqslant\frac{s(\mu,z)}{(1-|z|^2)^2}\leqslant\tilde{w}(z).$$

on D. By the proof of Lemma 1, $\tilde{w}(z)$ is equivalent to $\hat{\mu}_t(z)=(1-|z|^2)^{-2}\mu(D_t(z))$ on D. This implies that $s(\mu,z)$ is equivalent to $\mu(D_t(z))$ on D.

THEOREM 1. *Let μ be a finite positive Borel measure on D.*

(1) If μ_j $(j=1,2)$ belong to $\mathscr{R}$, and λ_j $(j=1,2)$ is a positive constant, then $\lambda_1\mu_1+\lambda_2\mu_2$ belongs to $\mathscr{R}$.

(2) Suppose $d\mu=\sum_{a\in A}v(a)\delta_a$ and each point in A is isolated in D. Then μ belongs to $\mathscr{R}$ if and only if A is a set of finitely many separated sequences.

(3) Suppose $\sup_{z\in D}\int_{D_t(z)}(1-|a|^2)^{-9/2}d\mu(a)<\infty$. If $d\mu=wdm$ and $\int_{K^c}w^{-1}dm<\infty$ for some compact set K in D, then μ belongs to $\mathscr{R}$.

(4) If $d\mu = wdm$ and w is in $(A_2)_\partial$, then μ belongs to $\mathscr{R}$.
(5) If $d\mu = wdm$ and $w = |f|^2$ for some f in H, then μ belongs to $\mathscr{R}$.

Proof. (1) Note that for $j = 1, 2, \lambda_j\mu_j \leqslant \lambda_1\mu_1 + \lambda_2\mu_2$ and so $\lambda_j r(\mu_j, a) \geqslant r(\lambda_1\mu_1 + \lambda_2\mu_2, a)$ $(a \in D)$. Then

$$\begin{aligned}&\sup_{z\in D}\int_{D_t(z)} r(\lambda_1\mu_1 + \lambda_2\mu_2, a)d(\lambda_1\mu_1 + \lambda_2\mu_2)(a)\\ &\leqslant \lambda_1^2 \sup_{z\in D}\int_{D_t(z)} r(\mu_1, a)d\mu_1(a) + \lambda_2^2 \sup_{z\in D}\int_{D_t(z)} r(\mu_2, a)d\mu_2(a) < \infty.\end{aligned}$$

(2) Since A is isolated in D, for each $a \in A$ there exists a function f in H such that $f(a) = 1$ and $f = 0$ on $A\backslash\{a\}$ (cf. [3, Theorem 15.11]). This implies that $s(\mu, a) = v(a)$ and so $r(\mu, a) = v(a)^{-1}$. Hence

$$\sup_{z\in D}\int_{D_t(z)} r(\mu, a)d\mu(a) = \sup_{z\in D}\sum_{a\in D_t(z)\cap A} 1.$$

This implies that μ belongs to $\mathscr{R}$ if and only if A is a set of finitely many separated sequences.

(3) By Theorem 5 in [2], if $\int_{K^c} w^{-1}dm < \infty$ for some compact set K in D, then

$$s(\mu, a) \geqslant C(1 - |a|^2)^{9/2} \quad (a \in D)$$

for some positive constant C. Hence by hypothesis on μ,

$$\sup_{z\in D}\int_{D_t(z)} r(\mu, a)d\mu(a) \leqslant C \sup_{z\in D}\int_{D_t(z)} (1 - |a|^2)^{-9/2}d\mu(a) < \infty.$$

Thus μ belongs to $\mathscr{R}$.

(4) By Lemmas 1 and 2,

$$\begin{aligned}&\sup_{z\in D}\int_{D_t(z)} r(\mu, a)d\mu(a)\\ &\leqslant \sup_{z\in D}\left\{\left(\sup_{a\in D_t(z)} r(\mu, a)\right) \times \mu(D_t(z))\right\}\\ &= \sup_{z\in D}\left\{\left(\sup_{a\in D_t(z)} \frac{1}{s(\mu, a)}\right) \times \mu(D_t(z))\right\}\\ &\leqslant C_1 \sup_{z\in D}\left\{\left(\sup_{a\in D_t(z)} \frac{1}{\mu(D_t(a))}\right) \times \mu(D_t(z))\right\}\\ &\leqslant C_2 \sup_{z\in D}\left\{\frac{1}{\mu(D_t(z))} \times \mu(D_t(z))\right\} = C_2\end{aligned}$$

where C_1 and C_2 are finite positive constant.

(5) Since $w = |f|^2$ and $f \in H$, $\exp(\log w)^{\sim}(a) \geqslant w(a)$ $(a \in D)$. For

$$(\log |f|^2)^{\sim}(a) = \int_D \log |f \circ \phi_a(z)|^2 dm(z) \geqslant \log |f \circ \phi_a(0)|^2.$$

Hence by (2) of Proposition 4 in [2]

$$\begin{aligned}
&\sup_{z\in D} \int_{D_t(z)} r(\mu, a) d\mu(a) \\
&= \sup_{z\in D} \int_{D_t(z)} \frac{1}{s(\mu, a)} w(a) dm(a) \\
&\leqslant \sup_{z\in D} \int_{D_t(z)} \frac{w(a)}{(1-|a|^2)^2 \exp(\log w)^{\sim}(a)} dm(a) \\
&\leqslant \sup_{z\in D} \int_{D_t(z)} (1-|a|^2)^{-2} dm(a)
\end{aligned}$$

By the remark in the first line in this section, $m \in \mathscr{R}$ and so μ belongs to $\mathscr{R}$.

3. $r(\mu, a)d\nu(a)$

If v is a Borel function such that $0 \leqslant v \leqslant s(\mu)$ on D, and $d\nu = vdm$, then

$$\sup_{z\in D} \int_{D_t(z)} r(\mu, a) d\nu(a) \leqslant \int_D r(\mu, a) v(a) dm(a) \leqslant 1.$$

Put

$$\mathscr{R}^{\mu} = \{\nu \,;\, \sup_{z\in D} \int_{D_t(z)} r(\mu, a) d\nu(a) < \infty,\ \nu \text{ is a finite positive Borel measure}\}$$

for each finite positive Borel measure μ. Then the above measure vdm belongs to $\mathscr{R}^{\mu}$. In this section, we study the set $\mathscr{R}^{\mu}$.

THEOREM 2. *μ, ν and σ denote finite positive Borel measures on D.*

(1) If μ is in $\mathscr{R}$ and $\nu \leqslant \gamma\mu$ for some positive constant γ then ν belongs to $\mathscr{R}^{\mu}$.

(2) Suppose $d\mu = wdm$ and w is in $(A_2)_\partial$. If ν is in $\mathscr{R}^{\mu}$ and $\hat{\sigma}_t \leqslant \gamma \hat{\nu}_t$ on D for some $\gamma > 0$ and some $t > 0$ then σ belongs to $\mathscr{R}^{\mu}$.

(3) When λ is a (not necessarily finite) positive Borel measure on D and $d\nu = s(\mu, a)d\lambda(a)$, ν belongs to $\mathscr{R}^{\mu}$ if and only if $\sup_{z\in D} \lambda(D_t(z)) < \infty$ for some $t > 0$. In particular, when $d\lambda = \sum_{a\in A} \delta_a$ and A is a discrete set in D, ν belongs to $\mathscr{R}^{\mu}$ if and only if A is a set of finitely many separated sequences.

Proof. (1) is clear.

(2) Suppose $w \in (A_2)_\partial$. By Lemmas 1 and 2, there exists a positive finite constant C such that

$$\frac{1}{C}\sup_{z\in D}\frac{\alpha(D_t(z))}{\mu(D_t(z))} \leqslant \sup_{z\in D}\int_{D_t(z)} r(\mu,a)d\alpha(a) \leqslant C\sup_{z\in D}\frac{\alpha(D_t(z))}{\mu(D_t(z))}$$

because $r(\mu,a)\ s(\mu,a) \equiv 1$. These inequalities imply (2) by applying it for $\alpha = \nu$ and $\alpha = \sigma$. In fact, as $\alpha = \nu$ in the above inequalities

$$\sup_{z\in D}\frac{\nu(D_t(z))}{\mu(D_t(z))} < \infty$$

because $\nu \in \mathscr{R}^\mu$. As $\alpha = \sigma$,

$$\sup_{z\in D}\frac{\sigma(D_t(z))}{\mu(D_t(z))} \leqslant \sup_{z\in D}\frac{\nu(D_t(z))}{\mu(D_t(z))}\sup_{z\in D}\frac{\sigma(D_t(z))}{\nu(D_t(z))} < \infty$$

because $\hat{\sigma}_t \leqslant \gamma\nu_t$.

(3) If $d\nu = s(\mu,a)d\lambda(a)$, then

$$\sup_{z\in D}\int_{D_t(z)} r(\mu,a)d\nu(a) = \sup_{z\in D}\lambda(D_t(z)).$$

This implies (3).

4. (ν,μ)-Carleson inequality

If $\int_D r(\mu,a)d\nu(a) < \infty$ then (ν,μ)-Carleson inequality is valid (see [2, Theorem 7]). The following question is natural. Is (ν,μ)-Carleson inequality valid for ν in $\mathscr{R}^\mu$? Theorems 2 and 3 answer for it positively when $d\mu = wdm$ and w is in $(A_2)_\partial$. Corollary 1 is a generalization of a result of K.Zhu [4, Theorem 1].

THEOREM 3. *Let μ be a finite positive Borel measure and λ a (not necessarily finite) positive Borel measure on D. Suppose $d\mu = wdm$ and w is in $(A_2)_\partial$, and $d\nu = s(\mu,a)d\lambda(a)$. $\sup_{z\in D}\lambda(D_t(z)) < \infty$ if and only if (ν,μ)-Carleson inequality is valid.*

Proof. By Theorem 3 in [1], in order to prove this theorem, it is sufficient to show that $\sup_{z\in D}\lambda(D_t(z)) < \infty$ if and only if

$$\sup_{z\in D}\frac{\hat{\nu}_t(z)}{\hat{\mu}_t(z)} < \infty.$$

The proof of Theorem 2 shows this.

COROLLARY 1. *Suppose $d\mu = wdm$ is a finite positive Borel measure with w in $(A_2)_\partial$ and $d\nu = \sum_{a\in A} s(\mu,a)\delta_a$ with a discrete set A in D. A is a set of finitely many separated sequences in D if and only if (ν,μ)-Carleson inequality is valid.*

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