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# ON GENERALIZED LORENTZ—ZYGMUND SPACES

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(*communicated by J. Pečarić*)

*Abstract.* We study generalized Lorentz–Zygmund spaces with broken logarithmic functions. We derive necessary and sufficient conditions for embeddings between them. We give a complete characterization of their associate spaces. We establish necessary and sufficient conditions for a generalized Lorentz–Zygmund space to be a Banach function space and to have absolutely continuous (quasi-)norm. We describe completely relations between these spaces and Orlicz spaces.

## Contents



## 1. Introduction

In 1980, Bennett and Rudnick [BR] introduced the three-parameter scale of the so-called *Lorentz–Zygmund spaces*. The Lorentz–Zygmund space $L_{p,q;\alpha}$, where $0 < p, q \leqslant \infty$ and $\alpha \in \mathbb{R}$, is the set of all functions $f$ on an appropriate measure space $(\mathscr{R}, \mu)$, whose *non-increasing rearrangement* $f^*$, defined by

$$f^*(t) = \inf\{\lambda > 0; \mu(\{x \in \mathscr{R}; |f(x)| > \lambda\}) \leqslant t\}, \quad t \in [0, \infty),$$

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satisfies

$$\|t^{\frac{1}{p}-\frac{1}{q}}(1+|\log t|)^{\alpha}f^*(t)\|_{q,(0,\mu(\mathscr{R}))}<\infty.$$

Bennett and Rudnick successfully applied Lorentz–Zygmund spaces to the development of a powerful interpolation theory involving operators satisfying certain a-priori rearrangement inequality. This way they considerably improved many results describing the behaviour of operators, especially in limiting cases. The class of Lorentz–Zygmund spaces is very important as it contains such classes as Lebesgue spaces, Lorentz spaces or Zygmund classes, and at the same time it is a class of quite easily tractable function spaces. Another very important example of a Lorentz–Zygmund space is the one normed by

$$\|f\|=\|t^{-\frac{1}{n}}(1+|\log t|)^{-1}f^*(t)\|_{n,(0,1)}$$

(in the above notation, $L_{\infty,n;-1}$). This space was discovered independently by Hansson ([H]) and by Brézis and Wainger ([BW]) as the appropriate target for the limiting case of the Sobolev-type embedding of the space $W^{1,n}$, where $n$ is the dimension of the underlying domain. The significance of this space was recently approved by Edmunds, Kerman and Pick ([EKP]) who showed that it cannot be replaced by any essentially smaller rearrangement invariant space, and by Cwikel and Pustylnik [CP] who proved the same fact in a stronger sense.

Recently, an investigation of double-exponential integrability of convolution operators was carried out by Edmunds, Gurka and Opic ([EGO1]). The authors extended the theory of Lorentz–Zygmund spaces by introducing a second tier of logarithms, calling the outcoming structure *generalized Lorentz–Zygmund* (GLZ) *spaces*. In [EOP1], a variety of sharp interpolation theorems in the sense of Bennett and Rudnick was obtained by simple techniques; the results of [BR] were extended to the context of GLZ spaces, many of them were improved, and their sharpness was shown. In particular, the important *scaling property* of GLZ spaces was discovered, and the *cross-case interpolation* was treated. For technical reasons dictated by various limiting versions of Hardy's inequality behind the proofs, the results of [EOP1] were restricted throughout to the case when the underlying measure space is of finite measure, similarly as in [BR] or [GM]. This difficulty was removed later in [EOP2], where the so-called *broken-logarithmic functions* were introduced. This enabled us to carry out a comprehensive interpolation theory for functions defined on a non-atomic $\sigma$-finite measure space. Using the abbreviations $\ell(t)=1+|\log t|$ and $\ell\ell(t)=1+\log\big(\ell(t)\big)$, $t\in(0,\infty)$, we define broken-logarithmic functions by

$$\ell^{\mathbb{A}}(t)=\begin{cases}\ell^{\alpha_0}(t), & 0<t\leqslant 1;\\ \ell^{\alpha_\infty}(t), & 1<t<\infty,\end{cases}$$

where $\mathbb{A}=(\alpha_0,\alpha_\infty)\in\mathbb{R}^2$; $\ell\ell^{\mathbb{A}}(t)$ is defined analogously. Similarly as $\mathbb{A}$ we shall use few other symbols for two-dimensional vectors, namely $\mathbb{B}=(\beta_0,\beta_\infty)$, $\mathbb{D}=(\delta_0,\delta_\infty)$, $\mathbb{L}=(\lambda_0,\lambda_\infty)$, $\mathbb{E}=(\varepsilon_0,\varepsilon_\infty)$, $\mathbb{S}=(\sigma_0,\sigma_\infty)$, and $\mathbb{W}=(\omega_0,\omega_\infty)$. These symbols should not be confused with the usual letters $\mathbb{N}$ and $\mathbb{R}$ which traditionally denote the set of all natural numbers and the set of all real numbers, respectively.

The present paper is devoted to a detailed study of two types of GLZ spaces with broken-logarithmic terms, namely

$$L_{p,q;\mathbb{A},\mathbb{B}} = \left\{f \in \mathscr{M}(\mathscr{R},\mu);\, \|f\|_{p,q;\mathbb{A},\mathbb{B}} = \|t^{\frac{1}{p}-\frac{1}{q}}\ell^{\mathbb{A}}(t)\ell\ell^{\mathbb{B}}(t)f^*(t)\|_{q,(0,\mu(\mathscr{R}))} < \infty\right\}$$

and

$$L_{(p,q;\mathbb{A},\mathbb{B})} = \left\{f \in \mathscr{M}(\mathscr{R},\mu);\, \|f\|_{(p,q;\mathbb{A},\mathbb{B})} = \|t^{\frac{1}{p}-\frac{1}{q}}\ell^{\mathbb{A}}(t)\ell\ell^{\mathbb{B}}(t)f^{**}(t)\|_{q,(0,\mu(\mathscr{R}))} < \infty\right\},$$

where $f^{**}(t) = t^{-1}\int_0^t f^*(s)\,\mathrm{d}s$. An extension to the cases involving more tiers of logarithms is just a technical matter (cf. e.g. [EGO4]).

In several directions of our recent research (let us name, for example, the investigation of limiting cases of convolution inequalities ([EGO1]), the study of embeddings of Bessel potential spaces based upon GLZ spaces ([EGO2], [EGO3], [EGO4], [GO]), the development of real interpolation theory with broken logarithmic functors ([EOP1], [EOP2], [EO]), or the investigation of embedding theorems for Bessel potential spaces with logarithmic smoothness ([OT])), we found a reasonably complete information on basic properties of GLZ spaces indispensable. Thus, guided by requirements coming directly from applications, we have been collecting for several years pieces of information until we reached a point of being able to write up a self-contained comprehensive "primer" on GLZ spaces. Such a primer is presented in this paper. Since the information is complete and exhaustive, we believe that the list of results can be found handy by many authors (by those interested in the area of limiting behaviour of operators and also by those seeking non-trivial examples or counterexamples). Therefore, we are convinced that our primer is worth publishing although some of the results are not strictly "new" as they might be obtained (usually via a tedious and time-consuming calculation) from more general criteria, scattered in existing literature. Needless to say, when we first started this work, some of the papers that appeared recently were not available (for example, [GHS], [So], [CS3], [CPSS] etc.).

A typical more general context is that of classical Lorentz spaces. The spaces $\Lambda^q(w)$, determined by the quantity

$$\|f\|_{\Lambda^q(w)} = \left(\int_0^\infty (f^*(t))^q w(t)\,\mathrm{d}t\right)^{1/q},$$

which were introduced in [Lo1] and later studied by many authors, contain all the spaces $L_{p,q;\mathbb{A},\mathbb{B}}$ as long as $q \in (0,\infty)$. Similarly, for $q \in (0,\infty)$, the spaces $L_{(p,q;\mathbb{A},\mathbb{B})}$ are particular examples of the spaces $\Gamma^q(w)$, introduced by Sawyer in [Sa], where

$$\|f\|_{\Gamma^q(w)} = \left(\int_0^\infty (f^{**}(t))^q w(t)\,\mathrm{d}t\right)^{1/q}.$$

The situation is not so straightforward when $q = \infty$; in such case many (but not all) of the GLZ spaces are covered by weak modifications of classical Lorentz spaces

$\Lambda^{q,\infty}(w)$, $\Gamma^{q,\infty}(w)$, respectively (cf. [CS1]), where

$$\|f\|_{\Lambda^{q,\infty}(w)} = \sup_{0<t<\infty} f^*(t)\left(\int_0^t w(s)\,\mathrm{d}s\right)^{1/q},$$

and

$$\|f\|_{\Gamma^{q,\infty}(w)} = \sup_{0<t<\infty} f^{**}(t)\left(\int_0^t w(s)\,\mathrm{d}s\right)^{1/q}.$$

Now we shall give a detailed outline of the paper with the discussion of its relations to the existing literature.

Section 2 contains preliminaries and notation. In Section 3 we collect very basic facts about GLZ spaces (frequently needed in subsequent sections), such as the characterization of those parameters $p, q, \mathbb{A}, \mathbb{B}$, for which the corresponding GLZ space is non-trivial, that is, not equal to $\{0\}$, the list of the fundamental functions of GLZ spaces, and some inclusion relations between both types of GLZ spaces. All these facts follow simply from definitions, apart perhaps from the inclusion relations in Theorems 3.8 and 3.16. For more general spaces, inclusion relations can be found, e.g., in [Sa, Theorem 2], [CS2, Theorem 3.2], and [So, Proposition 2.7, Theorem 4.1 (i), and Theorem 4.2 (ii)]. For the questions of non-triviality and fundamental functions, cf. also [CPSS, Lemma 3.6].

In Sections 4 and 5 we review embedding relations between the spaces $L_{p,q;\mathbb{A},\mathbb{B}}$ and $L_{(p,q;\mathbb{A},\mathbb{B})}$, respectively. To be more precise, in Section 4 we give a complete characterization of the parameters involved for which the embedding

$$L_{P_1,Q;\mathbb{L},\mathbb{E}} \hookrightarrow L_{P_2,R;\mathbb{S},\mathbb{W}}$$

holds. Almost all the results can be obtained from more general criteria provided by [Sa, Remark, p. 148] ($1 < Q, R < \infty$), [St, Proposition 1] ($0 < Q, R < \infty$) and [So, Proposition 2.7] (some particular cases when $Q = \infty$ and/or $R = \infty$); we thus omit the proofs. Analogously, in Section 5 we deal with the embeddings

$$L_{(P_1,Q;\mathbb{L},\mathbb{E})} \hookrightarrow L_{(P_2,R;\mathbb{S},\mathbb{W})}.$$

General criteria for this type of embedding were obtained by a discretization method in [GHS]. However, in some cases the conditions are very implicit and hard to verify. For this reason, we present detailed and self-contained proofs in the Appendix.

In Section 6 we give a complete characterization of associate spaces of GLZ spaces. We present elementary proofs based on rearrangement techniques in the spirit of [BS]. Again, for some cases, certain more general results are known. In [Sa, Remark, p. 147], the associate space of $\Lambda^q(w)$ is characterized provided that $1 < q < \infty$ and $\int_0^\infty w(s)\,\mathrm{d}s = \infty$. Similarly, in [GHS, Theorem 3.1], the associate space of $\Gamma^q(w)$ is described, but, again, in the case $1 < q < \infty$, the criteria are given in rather implicit terms involving discretization. For some further results and references see [CPSS, Section 9].

Our next concern is the question when a GLZ space satisfies all the axioms of the so-called Banach function space in the sense of Luxemburg (cf. [BS, Chapters 1 and 2]). Clearly, neither of the quantities $\|\cdot\|_{p,q;\mathbb{A},\mathbb{B}}$ or $\|\cdot\|_{(p,q;\mathbb{A},\mathbb{B})}$ is necessarily a norm; consider, for example, the cases when $q \in (0,1)$. In Section 7 we give a full characterization of those GLZ spaces which are rearrangement-invariant Banach function spaces. These results are, as far as we know, new.

In Section 8 we present a comprehensive analysis of the problem when a GLZ space $L_{p,q;\mathbb{A},\mathbb{B}}$ or $L_{(p,q;\mathbb{A},\mathbb{B})}$ coincides with an appropriate Orlicz space. For $\mu(\mathscr{R}) < \infty$, related particular results can be found, for example, in [BR], cf. also [EGO1, Lemma 3.10], [EGO2, Lemma 4.2] or [EOP1, Lemma 2.2]. Since the situation is in general rather complicated and most of the results are new, we include detailed proofs.

Finally, in Section 9 we characterize all GLZ spaces whose norm is absolutely continuous (cf. [BS, Chapter 1, Section 3]).

## 2. Preliminaries

The symbol $C$ will denote various constants independent of appropriate quantities. We write $A \lesssim B$ whenever $A \leqslant CB$, and $A \approx B$ whenever both $A \lesssim B$ and $B \lesssim A$. For a set $E$ we denote by $\chi_E$ the characteristic function of $E$. We shall use the convention $1/\infty = 0$ and $\infty/\infty = 0$, and for $0 < q \leqslant \infty$ we define $q'$ by $\frac{1}{q} + \frac{1}{q'} = 1$ when $q \neq 1$, and $q = +\infty$ when $q = 1$ (note that $q' < 0$ when $0 < q < 1$).

Throughout the paper, $(\mathscr{R}, \mu)$ denotes a totally $\sigma$-finite measure space with a *non-atomic measure* $\mu$, and $\mathscr{M}(\mathscr{R}, \mu)$ is the set of all extended complex-valued $\mu$-measurable functions on $\mathscr{R}$. By $\mathscr{M}^+(\mathscr{R}, \mu)$ we denote the set of all non-negative functions from $\mathscr{M}(\mathscr{R}, \mu)$. In the case when $\mathscr{R} = (0,\infty)$ and $\mu$ is the Lebesgue measure on $(0,\infty)$, we simply write $\mathscr{M}^+(0,\infty)$ instead of $\mathscr{M}^+(\mathscr{R}, \mu)$. By $\mathscr{M}^+(0,\infty;\downarrow)$ we mean the subset of $\mathscr{M}^+(0,\infty)$, consisting of all non-increasing functions on $(0,\infty)$.

Let $X, Y$ be two (quasi-)normed linear spaces of functions from $\mathscr{M}(\mathscr{R}, \mu)$. We say that $X$ *coincides* with $Y$ (and write $X = Y$) if $X$ and $Y$ are equal in the algebraic and the topological sense (their (quasi-)norms are equivalent).

We shall use the symbol $\hookrightarrow$ for the continuous embedding of (quasi-)normed linear spaces.

Following Luxemburg ([Lu], cf. also [BS]), we say that a Banach space $X$ of extended complex-valued $\mu$-measurable functions defined on $\mathscr{R}$ is a *Banach function space* (BFS), if the following axioms hold:

(P1) the norm $\|\cdot\|_X$ is defined for every $f \in \mathscr{M}(\mathscr{R}, \mu)$, and such $f$ belongs to $X$ if and only if $\|f\|_X < \infty$;
(P2) $\|f\|_X = 0$ if and only if $f = 0$ $\mu$-a.e.;
(P3) $\|f\|_X = \|\,|f|\,\|_X$ whenever $f \in \mathscr{M}(\mathscr{R}, \mu)$;
(P4) $0 \leqslant g \leqslant f$ $\mu$-a.e. implies $\|g\|_X \leqslant \|f\|_X$;
(P5) $0 \leqslant f_n \nearrow f$ $\mu$-a.e. implies $\|f_n\|_X \nearrow \|f\|_X$;
(P6) $\|\chi_E\|_X < \infty$ whenever $\mu(E) < \infty$;
(P7) If $\mu(E) < \infty$, then there is a constant $C_E$ such that $\int_E |f| \leqslant C_E \|f\|_X$ for all $f \in X$.

We shall use the following

CONVENTION. Let $X = (X, \|\cdot\|_X)$ be a (quasi-)normed linear space of functions from $\mathscr{M}(\mathscr{R},\mu)$. By saying "$X$ is a BFS" we mean that there is a norm $\|\cdot\|$ on $X$, equivalent to $\|\cdot\|_X$, such that the space $(X,\|\cdot\|)$ is a BFS.

Let $X$ be a quasi-normed linear space of functions $f \in \mathscr{M}(\mathscr{R},\mu)$ satisfying (P1)–(P6), modified in the sense that $\|\cdot\|_X$ may be a quasi-norm. The space $X$ is said to have *absolutely continuous* (*quasi-*)*norm* if every $f \in X$ satisfies the axiom

(ACN) $\|f\chi_{E_n}\|_X \to 0$ for every sequence $\{E_n\} \subset \mathscr{R}$ such that $E_n \searrow \emptyset$ $\mu$-a.e.

(Recall that $E_n \searrow \emptyset$ $\mu$-a.e. if $\chi_{E_n} \searrow 0$ $\mu$-a.e.) Moreover, the set $X'$, given by

$$X' = \Big\{f \in \mathscr{M}(\mathscr{R},\mu);\ \int_{\mathscr{R}} |fg|\,\mathrm{d}\mu < \infty \text{ for all } g \in X\Big\},$$

and endowed with the norm

$$\|f\|_{X'} = \sup\Big\{\int_{\mathscr{R}} |fg|\,\mathrm{d}\mu;\ \|g\|_X \leqslant 1\Big\},$$

is called the *associate space* of $X$. The *Hölder inequality*

$$\int_{\mathscr{R}} |fg|\,\mathrm{d}\mu \leqslant \|f\|_X\|g\|_{X'}$$

holds for every $f \in X$, $g \in X'$, and moreover

$$\|f\|_X = \sup\Big\{\int_{\mathscr{R}} |fg|\,\mathrm{d}\mu; \|g\|_{X'} \leqslant 1\Big\}.$$

By [BS, Chapter 1, Theorem 2.7], $X'' = (X')' = X$ provided that $X$ is a BFS.

Let $f \in \mathscr{M}(\mathscr{R},\mu)$. The *distribution function* $\mu_f$ of $f$ is defined by

$$\mu_f(\lambda) = \mu\big(\{x \in \mathscr{R}; |f(x)| > \lambda\}\big), \qquad \lambda \in [0,\infty).$$

The *non-increasing rearrangement* $f^*$ of $f$ is given by

$$f^*(t) = \inf\{\lambda > 0; \mu_f(\lambda) \leqslant t\}, \quad t \in [0,\infty),$$

and the *maximal function* $f^{**}$ of $f^*$ by

$$f^{**}(t) = \frac{1}{t}\int_0^t f^*(s)\,\mathrm{d}s, \quad 0 < t < \infty.$$

Recall that $\operatorname{supp} f^* \subset [0,\mu(\mathscr{R})]$ and $f^*(t) \leqslant f^{**}(t)$ for every $f$ and $t$.

Let $X$ be a quasi-normed linear space of functions $f \in \mathscr{M}(\mathscr{R},\mu)$, satisfying the axioms (P1)–(P6) (modified again in the sense that the norm of $X$ may be a quasi-norm) and such that $\|f\|_X = \|g\|_X$ whenever $f^* = g^*$. Then $X$ is called a *rearrangement-invariant* (r.i.) space.

Let $X$ be an r.i. space. For each finite $t \in [0, \mu(\mathscr{R})]$, let $E$ be any subset of $\mathscr{R}$ with $\mu(E) = t$, and

$$\varphi_X(t) = \|\chi_E\|_X.$$

The function $\varphi_X$ so defined is called the *fundamental function of* $X$. Let us note that for two r.i. spaces $X$ and $Y$ such that $X \hookrightarrow Y$, we necessarily have $\varphi_Y(t) \lesssim \varphi_X(t)$ for all finite $t \in [0, \mu(\mathscr{R}))$.

Since $\mu$ is non-atomic, for every finite $t \in (0, \mu(\mathscr{R})]$ there is a $\mu$–measurable subset $E$ of $\mathscr{R}$ such that $\mu(\mathscr{R}) = t$, and therefore $(\chi_E)^* = \chi_{(0,t)}$. Hence (cf. [BS]), for every $f \in \mathscr{M}(\mathscr{R}, \mu)$ and every finite $t \in (0, \mu(\mathscr{R})]$,

$$\int_0^t f^*(s)\, \mathrm{d}s = \int_0^\infty f^*(s)\chi_E^*(s)\, \mathrm{d}s \leqslant \|\chi_E\|_X \|f\|_{X'},$$

which yields

$$\int_0^t f^*(s)\, \mathrm{d}s \leqslant \varphi_X(t)\, \|f\|_{X'}. \tag{2.1}$$

We write $\varphi \in \mathcal{F}$ provided that

(i) $\varphi(t) = 0$ if and only if $t = 0$,

(ii) $\varphi$ is continuous except perhaps at 0,

(iii) $\varphi$ is equivalent to a non-decreasing concave function on $(0, \mu(\mathscr{R}))$.

If $X$ is an r.i. BFS, then $\varphi_X \in \mathcal{F}$ (cf. [BS]). Moreover, the fundamental function $\varphi_{X'}$ of $X'$ satisfies

$$\varphi_{X'}(t) = \frac{t}{\varphi_X(t)}, \qquad t \in (0, \mu(\mathscr{R})), \qquad \varphi_{X'}(0) = 0. \tag{2.2}$$

Conversely, if $\varphi \in \mathcal{F}$, then $\varphi$ is a fundamental function of some r.i. Banach function space(s). Among all such spaces, two are of an extraordinary importance. Namely, the spaces $\Lambda_\varphi$ and $M_\varphi$, defined as the families of all functions in $\mathscr{M}(\mathscr{R}, \mu)$ for which the functionals

$$\|f\|_{\Lambda_\varphi} = \int_0^{\mu(\mathscr{R})} f^*(t)\, \mathrm{d}\,\varphi(t), \tag{2.3}$$

and

$$\|f\|_{M_\varphi} = \sup_{0<t<\mu(\mathscr{R})} f^{**}(t)\varphi(t), \tag{2.4}$$

respectively, are finite. Both $\Lambda_\varphi$ and $M_\varphi$ are rearrangement-invariant Banach function spaces with fundamental function $\varphi$, and, in fact, $\Lambda_\varphi$ is the smallest and $M_\varphi$ is the largest such a space. In particular, for any r.i. BFS $X$,

$$\Lambda_{\varphi_X} \hookrightarrow X \hookrightarrow M_{\varphi_X}. \tag{2.5}$$

It follows that (cf. [Sh] or [BS, Chapter 4, Exercise 21 (d)])

$$(\Lambda_\varphi)' = M_{\tilde\varphi}, \qquad (M_\varphi)' = \Lambda_{\tilde\varphi}, \tag{2.6}$$

where $\tilde\varphi(t) = t/\varphi(t)$.

A typical example of an r.i. space is the *Lebesgue space* $L^p = L^p(\mathscr{R}, \mu)$ with $0 < p \leqslant \infty$, whose (quasi-)norm is defined by

$$\|f\|_p = \begin{cases} \Big(\int_{\mathscr{R}} |f(x)|^p \,\mathrm{d}\mu\Big)^{1/p} & \text{if} \quad 0 < p < \infty; \\ \operatorname{ess\,sup}_{x\in\mathscr{R}} |f(x)| & \text{if} \quad p = \infty. \end{cases} \tag{2.7}$$

When $\mathscr{R} = (a,b)$, $-\infty \leqslant a < b \leqslant \infty$, and $\mu$ is the Lebesgue measure, we sometimes write $\|\cdot\|_{q,(a,b)}$ for the (quasi-)norm (2.7). We recall that $L^p$ is a BFS if and only if $1 \leqslant p \leqslant \infty$. Moreover, for $t \in [0, \mu(\mathscr{R}))$, $\varphi_{L^p}(t) = t^{1/p}$ when $0 < p < \infty$, and $\varphi_{L^\infty}(t) = \chi_{(0,\mu(\mathscr{R}))}$.

Another important example of an r.i. BFS is the *Orlicz space* $L_\Phi = L_\Phi(\mathscr{R}, \mu)$, generated by a Young function $\Phi\colon [0,\infty) \to [0,\infty)$, which is an increasing convex function satisfying

$$\lim_{t\to 0_+} \frac{\Phi(t)}{t} = \lim_{t\to\infty} \frac{t}{\Phi(t)} = 0. \tag{2.8}$$

The space $L_\Phi$ is the collection of functions $f \in \mathscr{M}(\mathscr{R}, \mu)$ for which there exists a $\lambda > 0$ such that $\int_{\mathscr{R}} \Phi(|f(x)|/\lambda)\,\mathrm{d}\mu < \infty$. If $\Phi$ satisfies the $\Delta_2$*–condition*, that is, $\Phi(2t) \lesssim \Phi(t)$ for all $t \geqslant 0$, then $f \in L_\Phi$ if and only if $\int_{\mathscr{R}} \Phi(|f|)\,\mathrm{d}\mu < \infty$. The (Luxemburg) norm in $L_\Phi$ is given by

$$\|f\|_{L_\Phi} = \inf \Big\{ \lambda > 0; \int_{\mathscr{R}} \Phi\Big(\frac{|(x)|}{\lambda}\Big)\,\mathrm{d}\mu \leqslant 1 \Big\},$$

and $(L_\Phi)' = L_{\tilde\Phi}$, where $\tilde\Phi$ is the *complementary function* of $\Phi$,

$$\tilde\Phi(t) = \sup_{s\geqslant 0} \big(st - \Phi(s)\big), \quad 0 \leqslant t < \infty.$$

Let us also recall the *Young inequality*

$$st \leqslant \Phi(s) + \tilde\Phi(t), \quad s,t \in [0,\infty).$$

The fundamental function of the space $L_\Phi$ endowed with the Luxemburg norm reads (cf. [KR, (9.23)])

$$\varphi_{L_\Phi}(t) = \frac{1}{\Phi^{-1}(1/t)}, \qquad t \in [0, \mu(\mathscr{R})). \tag{2.9}$$

Let $0 < q \leqslant \infty$ and let $w$ be a weight function (an a.e. positive measurable function on $(0,\infty)$). If $\mu(\mathscr{R}) = \infty$, we define the *classical Lorentz spaces* $\Lambda^q(w)$ and $\Gamma^q(w)$ (cf. [Lo1] and [Sa]) as the sets of all functions $f \in \mathscr{M}(\mathscr{R}, \mu)$ such that

$$\|f^* w\|_{q,(0,\infty)} < \infty \qquad \text{and} \qquad \|f^{**} w\|_{q,(0,\infty)} < \infty,$$

respectively. It is worth noting that, for $1 \leqslant q \leqslant \infty$, $\Gamma^q(w)$ is a rearrangement-invariant BFS if and only if $w$ satisfies, for every $t \in (0,\infty)$,

$$\|w(s)\|_{q,(0,t)} < \infty \qquad \text{and} \qquad \|s^{-1}w(s)\|_{q,(t,\infty)} < \infty. \tag{2.10}$$

Indeed, the axioms (P1)–(P5) readily follow from elementary properties of rearrangements (cf. [BS, Chapter 2]). The axiom (P6) is a consequence of (2.10) and the identity

$$\chi^{**}_{(0,t)}(s) = \chi_{(0,t)}(s) + ts^{-1}\chi_{(t,\infty)}(s). \tag{2.11}$$

As for (P7), note that for a set $E$ with $\mu(E) = t$,

$$\int_E |f|\,d\mu \leqslant \int_0^t f^*(s)\,\mathrm{d}\,s = C_t\,\|s^{-1}w(s)\|_{q,(t,\infty)} \int_0^t f^*(s)\,\mathrm{d}\,s \leqslant C_t\,\|f\|_{\Gamma^q(w)},$$

where $C_t = (\|s^{-1}w(s)\|_{q,(t,\infty)})^{-1} < \infty$. If, conversely, (2.10) is not satisfied, then evidently (P6) does not hold and therefore $\Gamma^q(w)$ is not a BFS.

Let $1 < q < \infty$. By [Sa, Theorem 4], $\Lambda^q(w)$ is a Banach space (that is, there is a norm $\|\cdot\|$ on $\Lambda^q(w)$ equivalent to the original one) if and only if $\Lambda^q(w) = \Gamma^q(w)$. If, additionally, $\int_0^\infty w(t)\,\mathrm{d}\,t = \infty$, then $\big[\Lambda^q(w)\big]' = \Gamma^{q'}(\tilde{w})$, where $\frac{1}{q} + \frac{1}{q'} = 1$, and $\tilde{w}(t) = t^{q'}w(t)\Big(\int_0^t w(s)\ \mathrm{d}\,s\Big)^{-q'}$.

Important examples of classical Lorentz spaces are the generalized Lorentz–Zygmund spaces, defined in Section 3 below.

## 3. The GLZ spaces – definitions and basic properties

As usual, the symbols $\mathbb{R}$ and $\mathbb{N}$ stand for the set of all real numbers and the set of all natural numbers, respectively. Moreover, we shall use the letters $\mathbb{A}, \mathbb{B}, \mathbb{D}, \mathbb{L}, \mathbb{E}, \mathbb{S}$ and $\mathbb{W}$ for two-dimensional real vectors, that is, $\mathbb{A} = (\alpha_0, \alpha_\infty)$, $\mathbb{B} = (\beta_0, \beta_\infty)$, $\mathbb{D} = (\delta_0, \delta_\infty)$, $\mathbb{L} = (\lambda_0, \lambda_\infty)$, $\mathbb{E} = (\varepsilon_0, \varepsilon_\infty)$, $\mathbb{S} = (\sigma_0, \sigma_\infty)$, and $\mathbb{W} = (\omega_0, \omega_\infty) \in \mathbb{R}^2$. Given $\sigma \in \mathbb{R}$, we shall use the convention $\mathbb{A} + \sigma = (\alpha_0 + \sigma, \alpha_\infty + \sigma)$ and $\sigma\mathbb{A} = (\sigma\alpha_0, \sigma\alpha_\infty)$. We also write $\mathbb{A} < \mathbb{B}$ and $\mathbb{A} \leqslant \mathbb{B}$ when $\alpha_i < \beta_i$ and $\alpha_i \leqslant \beta_i$, respectively, $i = 0, \infty$. If $\mathbb{A} = (\alpha_0, \alpha_\infty) \in \mathbb{R}^2$, we put $\tilde{\mathbb{A}} = (\alpha_\infty, \alpha_0)$.

We shall use the abbreviations

$$\ell(t) = 1 + |\log t|, \quad \ell\ell(t) = 1 + \log\big(\ell(t)\big), \quad \ell\ell\ell(t) = 1 + \log\big(\ell\ell(t)\big), \qquad t > 0.$$

If $\mathbb{A} = (\alpha_0, \alpha_\infty) \in \mathbb{R}^2$, we define

$$\ell^{\mathbb{A}}(t) = \begin{cases} \ell^{\alpha_0}(t), & 0 < t \leqslant 1 \\ \ell^{\alpha_\infty}(t), & 1 < t < \infty, \end{cases}$$

and analogously for $\ell\ell^{\mathbb{A}}(t)$ and $\ell\ell\ell^{\mathbb{A}}(t)$.

We are in a position to define generalized Lorentz–Zygmund spaces with broken logarithmic functions.

3.1. DEFINITION. Let $0 < p, q \leqslant \infty$ and $\mathbb{A}, \mathbb{B} \in \mathbb{R}^2$. The *generalized Lorentz–Zygmund* (GLZ) space $L_{p,q;\mathbb{A},\mathbb{B}}$ is given by

$$L_{p,q;\mathbb{A},\mathbb{B}} = \left\{ f \in \mathscr{M}(\mathscr{R},\mu);\, \|f\|_{p,q;\mathbb{A},\mathbb{B}} = \|t^{\frac{1}{p}-\frac{1}{q}} \ell^{\mathbb{A}}(t) \ell\ell^{\mathbb{B}}(t) f^*(t)\|_{q,(0,\infty)} < \infty \right\}.$$

3.2. REMARKS. (i) Generalized Lorentz–Zygmund spaces $L_{p,q;\mathbb{A},\mathbb{B}}$ include many familiar ones: When $\mathbb{A} = \mathbb{B} = (0,0)$, we obtain just the Lorentz space $L^{p,q}$. If, moreover, $p = q$, then $L_{p,q;\mathbb{A},\mathbb{B}} = L^p$ is the classical Lebesgue space, and the (quasi-) norms coincide. If $\mu(\mathscr{R}) < \infty$, $\alpha \in \mathbb{R}$, and $\mathbb{A} = (\alpha,\alpha)$, $\mathbb{B} = (0,0)$, then $L_{p,q;\mathbb{A},\mathbb{B}}$ is the Lorentz–Zygmund space $L^{p,q}(\log L)^{\alpha}$, considered by Bennett and Rudnick ([BR]), which coincides with the Zygmund class $L^p(\log L)^{\alpha}$ when $p = q$.

(ii) Note that the use of different powers near 0 and near $\infty$ is reasonable only if $\mu(\mathscr{R}) = \infty$.

When $\mu(\mathscr{R}) < \infty$, the space $L_{p,q;\mathbb{A},\mathbb{B}}$ coincides with $L_{p,q;\alpha_0,\beta_0}$ introduced in [EGO1]. Below we shall use the following slight modification of these spaces.

Let $0 < p, q \leqslant \infty$, $\alpha, \beta \in \mathbb{R}$, and $T \in (0, \mu(\mathscr{R})]$. Then we put

$$L_{p,q;\alpha,\beta}(0,T) = \left\{ f \in \mathscr{M}(\mathscr{R},\mu);\, \|f\|_{p,q;\alpha,\beta,(0,T)} < \infty \right\},$$

where, for $0 \leqslant t < T \leqslant \mu(\mathscr{R})$,

$$\|f\|_{p,q;\alpha,\beta,(t,T)} = \|s^{\frac{1}{p}-\frac{1}{q}} \ell^{\alpha}(s) \ell\ell^{\beta}(s) f^*(s)\|_{q,(t,T)}.$$

If $0 < T < \infty$, then it is easy to see that

$$f \in L_{p,q;\alpha,\beta}(0,T) \qquad \text{if and only if} \qquad f \in L_{p,q;\alpha,\beta}(0,1),$$

and for all $f \in \mathscr{M}(\mathscr{R},\mu)$,

$$\|f\|_{p,q;\alpha,\beta,(0,T)} \approx \|f\|_{p,q;\alpha,\beta,(0,1)}.$$

We put

$$L_{p,q;\alpha,\beta} := L_{p,q;\alpha,\beta}(0,\mu(\mathscr{R})) = L_{p,q;(\alpha,\alpha),(\beta,\beta)}$$

and

$$\|\cdot\|_{p,q;\alpha,\beta} := \|\cdot\|_{p,q;(\alpha,\alpha),(\beta,\beta)}.$$

In addition to the above notation we write for $g \in \mathscr{M}(\mathscr{R},\mu)$, $p, q \in (0,\infty]$, $\mathbb{A}, \mathbb{B} \in \mathbb{R}^2$, and $0 \leqslant t < T \leqslant \mu(\mathscr{R})$,

$$\|g\|_{p,q;\mathbb{A},\mathbb{B},(t,T)} = \|s^{\frac{1}{p}-\frac{1}{q}} \ell^{\mathbb{A}}(s) \ell\ell^{\mathbb{B}}(s) g^*(s)\|_{q,(t,T)}.$$

Besides the spaces $L_{p,q;\mathbb{A},\mathbb{B}}$ we also introduce their analogues $L_{(p,q;\mathbb{A},\mathbb{B})}$ by replacing the non-increasing rearrangement $f^*$ by the maximal function $f^{**}$. Let us be more precise.

3.3. DEFINITION. Let $0 < p, q \leqslant \infty$ and $\mathbb{A}, \mathbb{B} \in \mathbb{R}^2$. The *generalized Lorentz–Zygmund* (GLZ) space $L_{(p,q;\mathbb{A},\mathbb{B})}$ is given by

$$L_{(p,q;\mathbb{A},\mathbb{B})} = \left\{ f \in \mathscr{M}(\mathscr{R},\mu);\, \|f\|_{(p,q;\mathbb{A},\mathbb{B})} = \|t^{\frac{1}{p}-\frac{1}{q}} \ell^{\mathbb{A}}(t) \ell\ell^{\mathbb{B}}(t) f^{**}(t)\|_{q,(0,\mu(\mathscr{R}))} < \infty \right\}.$$

3.4. REMARKS. (i) Let $0 < p,q \leqslant \infty$, $\alpha,\beta \in \mathscr{R}$, $\mathbb{A},\mathbb{B} \in \mathbb{R}^2$, and $0 \leqslant t < T \leqslant \infty$. The spaces $L_{(p,q;\alpha,\beta)}(0,T)$, $L_{(p,q;\alpha,\beta)}$, and the quantities $\|f\|_{(p,q;\alpha,\beta)(t,T)}$, $\|f\|_{(p,q;\alpha,\beta)}$, and $\|f\|_{(p,q;\mathbb{A},\mathbb{B})(t,T)}$, are defined in an obvious way (cf. Remark 3.2 (ii)).

(ii) Occasionally we shall use a third tier of logarithms (cf. e.g. Section 6). In such cases we work with the spaces

$$L_{p,q;\mathbb{A},\mathbb{B},\mathbb{D}} = \left\{f \in \mathscr{M}(\mathscr{R},\mu); \|f\|_{p,q;\mathbb{A},\mathbb{B},\mathbb{D}} < \infty\right\},$$

and

$$L_{(p,q;\mathbb{A},\mathbb{B},\mathbb{D})} = \left\{f \in \mathscr{M}(\mathscr{R},\mu); \|f\|_{(p,q;\mathbb{A},\mathbb{B},\mathbb{D})} < \infty\right\},$$

where

$$\|f\|_{p,q;\mathbb{A},\mathbb{B},\mathbb{D}} = \|t^{\frac{1}{p}-\frac{1}{q}}\ell^{\mathbb{A}}(t)\ell\ell^{\mathbb{B}}(t)\ell\ell\ell^{\mathbb{D}}(t)f^*(t)\|_{q,(0,\mu(\mathscr{R}))},$$

and

$$\|f\|_{(p,q;\mathbb{A},\mathbb{B},\mathbb{D})} = \|t^{\frac{1}{p}-\frac{1}{q}}\ell^{\mathbb{A}}(t)\ell\ell^{\mathbb{B}}(t)\ell\ell\ell^{\mathbb{D}}(t)f^{**}(t)\|_{q,(0,\mu(\mathscr{R}))}.$$

As we have already mentioned (cf. Remark 3.2 (ii)), the use of different powers near 0 and near $\infty$ is reasonable only if $\mu(\mathscr{R}) = \infty$. We thus adopt the following

CONVENTION. Throughout the paper we assume that $\mu(\mathscr{R}) = \infty$ unless it is explicitly said that $\mu(\mathscr{R}) < \infty$.

Since $f^* \leqslant f^{**}$, we have

$$L_{(p,q;\mathbb{A},\mathbb{B})} \hookrightarrow L_{p,q;\mathbb{A},\mathbb{B}}. \tag{3.1}$$

Let us first clarify when the spaces $L_{p,q,\mathbb{A},\mathbb{B}}$ and $L_{(p,q,\mathbb{A},\mathbb{B})}$ are non-trivial.

3.5. LEMMA. *Let* $0 < p,q \leqslant \infty$, $\mathbb{A} = (\alpha_0,\alpha_\infty)$, *and* $\mathbb{B} = (\beta_0,\beta_\infty)$.

(i) *The space* $L_{p,q;\mathbb{A},\mathbb{B}}$ *is not trivial, that is, not equal to* $\{0\}$, *if and only if one of the following conditions holds:*

$$\begin{cases} p < \infty; \\ p = \infty,\ \alpha_0 + \frac{1}{q} < 0; \\ p = \infty,\ \alpha_0 + \frac{1}{q} = 0,\ \beta_0 + \frac{1}{q} < 0; \\ p = \infty,\ q = \infty,\ \alpha_0 = 0,\ \beta_0 = 0. \end{cases} \tag{3.2}$$

(ii) *The space* $L_{(p,q;\mathbb{A},\mathbb{B})}$ *is not trivial, that is, not equal to* $\{0\}$, *if and only if one of the following conditions holds:*

$$\begin{cases} 1 < p < \infty; \\ p = \infty,\ \alpha_0 + \frac{1}{q} < 0; \\ p = \infty,\ \alpha_0 + \frac{1}{q} = 0,\ \beta_0 + \frac{1}{q} < 0; \\ p = \infty,\ q = \infty,\ \alpha_0 = 0,\ \beta_0 = 0; \\ p = 1,\ \alpha_\infty + \frac{1}{q} < 0; \\ p = 1,\ \alpha_\infty + \frac{1}{q} = 0,\ \beta_\infty + \frac{1}{q} < 0; \\ p = 1,\ q = \infty,\ \alpha_\infty = 0,\ \beta_\infty = 0. \end{cases} \tag{3.3}$$

*Proof.* The proof of (3.3) is an easy modification of that of Corollary 2.3 in [EOP2]. For (3.2), cf. also [EOP1, Lemma 6.1]. □

3.6. REMARK. Let $0 < p \leqslant \infty$, $1 \leqslant q \leqslant \infty$, $\mathbb{A} = (\alpha_0, \alpha_\infty)$, and $\mathbb{B} = (\beta_0, \beta_\infty) \in \mathbb{R}^2$. Then $X = L_{(p,q;\mathbb{A},\mathbb{B})}$ is a rearrangement-invariant BFS if and only if $X \neq \{0\}$. This follows from the fact that the function $w(t) = t^{\frac{1}{p}-\frac{1}{q}}\ell^{\mathbb{A}}(t)\ell\ell^{\mathbb{B}}(t)$ obeys (2.10) if and only if one of the conditions in (3.3) holds.

We shall now list the fundamental functions of GLZ spaces.

3.7. LEMMA. *Let* $0 < p, q \leqslant \infty$, $\mathbb{A} = (\alpha_0, \alpha_\infty)$, *and* $\mathbb{B} = (\beta_0, \beta_\infty)$.

(i) *Assume that the space* $X = L_{p,q;\mathbb{A},\mathbb{B}}$ *is not trivial* (*cf.* Lemma 3.5). *Then, for* $0 < t \leqslant 1$,

$$\varphi_X(t) \approx \begin{cases} t^{\frac{1}{p}}\ell^{\alpha_0}(t)\ell\ell^{\beta_0}(t) & \text{if} \quad 0 < p < \infty; \\ \ell^{\alpha_0+\frac{1}{q}}(t)\ell\ell^{\beta_0}(t) & \text{if} \quad p = \infty,\ \alpha_0 + \frac{1}{q} < 0; \\ \ell\ell^{\beta_0+\frac{1}{q}}(t) & \text{if} \quad p = \infty,\ \alpha_0 + \frac{1}{q} = 0,\ \beta_0 + \frac{1}{q} < 0; \\ 1 & \text{if} \quad p = \infty,\ q = \infty,\ \alpha_0 = 0,\ \beta_0 = 0; \end{cases}$$

*whereas, for* $1 < t < \infty$,

$$\varphi_X(t) \approx \begin{cases} t^{\frac{1}{p}}\ell^{\alpha_\infty}(t)\ell\ell^{\beta_\infty}(t) & \text{if} \quad 0 < p < \infty; \\ 1 & \text{if} \quad p = \infty,\ \text{either } \alpha_\infty + \frac{1}{q} < 0, \\ & \qquad \text{or} \quad \alpha_\infty + \frac{1}{q} = 0,\ \beta_\infty + \frac{1}{q} < 0, \\ & \qquad \text{or} \quad q = \infty,\ \alpha_\infty = 0,\ \beta_\infty = 0; \\ \ell^{\alpha_\infty+\frac{1}{q}}(t)\ell\ell^{\beta_\infty}(t) & \text{if} \quad p = \infty,\ \alpha_\infty + \frac{1}{q} > 0; \\ \ell\ell^{\beta_\infty+\frac{1}{q}}(t) & \text{if} \quad p = \infty,\ \alpha_\infty + \frac{1}{q} = 0,\ \beta_\infty + \frac{1}{q} > 0; \\ \ell\ell\ell^{\frac{1}{q}}(t) & \text{if} \quad p = \infty,\ \alpha_\infty + \frac{1}{q} = 0,\ \beta_\infty + \frac{1}{q} = 0. \end{cases}$$

(ii) *Assume that the space* $Y = L_{(p,q;\mathbb{A},\mathbb{B})}$ *is not trivial. Then, for* $0 < t \leqslant 1$,

$$\varphi_Y(t) \approx \begin{cases} t^{\frac{1}{p}}\ell^{\alpha_0}(t)\ell\ell^{\beta_0}(t) & \text{if} \quad 1 < p < \infty; \\ \ell^{\alpha_0+\frac{1}{q}}(t)\ell\ell^{\beta_0}(t) & \text{if} \quad p = \infty,\ \alpha_0 + \frac{1}{q} < 0; \\ \ell\ell^{\beta_0+\frac{1}{q}}(t) & \text{if} \quad p = \infty,\ \alpha_0 + \frac{1}{q} = 0,\ \beta_0 + \frac{1}{q} < 0; \\ 1 & \text{if} \quad p = \infty,\ q = \infty,\ \alpha_0 = 0,\ \beta_0 = 0; \\ t\ell^{\alpha_0+\frac{1}{q}}(t)\ell\ell^{\beta_0}(t) & \text{if} \quad p = 1,\ \alpha_0 + \frac{1}{q} > 0; \\ t\ell\ell^{\beta_0+\frac{1}{q}}(t) & \text{if} \quad p = 1,\ \alpha_0 + \frac{1}{q} = 0,\ \beta_0 + \frac{1}{q} > 0; \\ t\ell\ell\ell^{\frac{1}{q}}(t) & \text{if} \quad p = 1,\ \alpha_0 + \frac{1}{q} = 0,\ \beta_0 + \frac{1}{q} = 0; \\ t & \text{if} \quad p = 1,\ \text{either } \alpha_0 + \frac{1}{q} < 0, \\ & \qquad \text{or} \quad \alpha_0 + \frac{1}{q} = 0,\ \beta_0 + \frac{1}{q} < 0; \end{cases}$$

*whereas, for* $1 < t < \infty$,

$$\varphi_Y(t) \approx \begin{cases} t^{\frac{1}{p}}\ell^{\alpha_\infty}(t)\ell\ell^{\beta_\infty}(t) & \text{if } \ 1 < p < \infty; \\ 1 & \text{if } \ p = \infty, \text{ either } \alpha_\infty + \frac{1}{q} < 0, \\ & \qquad \text{or } \ \alpha_\infty + \frac{1}{q} = 0, \ \beta_\infty + \frac{1}{q} < 0, \\ & \qquad \text{or } \ q = \infty, \ \alpha_\infty = 0, \ \beta_\infty = 0; \\ \ell^{\alpha_\infty + \frac{1}{q}}(t)\ell\ell^{\beta_\infty}(t) & \text{if } \ p = \infty, \ \alpha_\infty + \frac{1}{q} > 0; \\ \ell\ell^{\beta_\infty + \frac{1}{q}}(t) & \text{if } \ p = \infty, \ \alpha_\infty + \frac{1}{q} = 0, \ \beta_\infty + \frac{1}{q} > 0; \\ \ell\ell\ell^{\frac{1}{q}}(t) & \text{if } \ p = \infty, \ \alpha_\infty + \frac{1}{q} = 0, \ \beta_\infty + \frac{1}{q} = 0; \\ t\ell^{\alpha_\infty + \frac{1}{q}}(t)\ell\ell^{\beta_\infty}(t) & \text{if } \ p = 1, \ \alpha_\infty + \frac{1}{q} < 0; \\ t\ell\ell^{\beta_\infty + \frac{1}{q}}(t) & \text{if } \ p = 1, \text{ either } \alpha_\infty + \frac{1}{q} = 0, \ \beta_\infty + \frac{1}{q} < 0, \\ & \qquad \text{or } \ q = \infty, \ \alpha_\infty = 0, \ \beta_\infty = 0. \end{cases}$$

*Proof.* This is just an elementary calculation, using (2.11) in the case (ii). □

Our next aim is to study relations between the spaces $L_{p,q,\mathbb{A},\mathbb{B}}$ and $L_{(p,q,\mathbb{A},\mathbb{B})}$.

3.8. THEOREM. *Let* $1 \leqslant p \leqslant \infty$, $0 < q \leqslant \infty$, $\mathbb{A} = (\alpha_0, \alpha_\infty)$, $\mathbb{B} = (\beta_0, \beta_\infty) \in \mathbb{R}^2$, *and assume that one of the conditions in* (3.3) *is satisfied.*

(i) *If* $1 < p \leqslant \infty$, *then*

$$L_{(p,q;\mathbb{A},\mathbb{B})} = L_{p,q;\mathbb{A},\mathbb{B}}. \tag{3.4}$$

(ii) *The space* $L_{(1,1;\mathbb{A},\mathbb{B})}$ *coincides with the space*

$$\begin{array}{lll} L_{1,1;(0,\alpha_\infty+1),(0,\beta_\infty)} & \text{if} & \alpha_\infty + 1 < 0, \text{ either } \alpha_0 + 1 < 0, \\ & & \qquad \text{or } \ \alpha_0 + 1 = 0, \ \beta_0 + 1 < 0; \\ L_{1,1;\mathbb{A}+1,\mathbb{B}} & \text{if} & \alpha_\infty + 1 < 0, \ \alpha_0 + 1 > 0; \\ L_{1,1;(0,\alpha_\infty+1),(\beta_0+1,\beta_\infty)} & \text{if} & \alpha_\infty + 1 < 0, \ \alpha_0 + 1 = 0, \ \beta_0 + 1 > 0; \\ L_{1,1;(0,\alpha_\infty+1),(0,\beta_\infty),(1,0)} & \text{if} & \alpha_\infty + 1 < 0, \ \alpha_0 + 1 = 0, \ \beta_0 + 1 = 0; \\ L_{1,1;(0,0),(0,\beta_\infty+1)} & \text{if} & \alpha_\infty + 1 = 0, \ \beta_\infty + 1 < 0, \\ & & \text{and either } \alpha_0 + 1 < 0, \text{ or } \ \alpha_0 + 1 = 0, \ \beta_0 + 1 < 0; \\ L_{1,1;(\alpha_0+1,0),(\beta_0,\beta_\infty+1)} & \text{if} & \alpha_\infty + 1 = 0, \ \beta_\infty + 1 < 0, \ \alpha_0 + 1 > 0; \\ L_{1,1;(0,0),\mathbb{B}+1} & \text{if} & \alpha_\infty + 1 = 0, \ \beta_\infty + 1 < 0, \ \alpha_0 + 1 = 0, \ \beta_0 + 1 > 0; \\ L_{1,1;(0,0),(0,\beta_\infty+1),(1,0)} & \text{if} & \alpha_\infty + 1 = 0, \ \beta_\infty + 1 < 0, \ \alpha_0 + 1 = 0, \ \beta_0 + 1 = 0. \end{array}$$

(iii) *Let* $1 < q \leqslant \infty$. *Then*

$$L_{1,q;\mathbb{A}+1,\mathbb{B}} \subsetneqq L_{(1,q;\mathbb{A},\mathbb{B})} \quad \text{if} \quad \alpha_0 + \frac{1}{q} > 0, \ \alpha_\infty + \frac{1}{q} < 0, \tag{3.5}$$

*and*

$$L_{1,q;(\frac{1}{q'},\frac{1}{q'}),\mathbb{B}+1} \subsetneqq L_{(1,q;(-\frac{1}{q},-\frac{1}{q}),\mathbb{B})} \quad \text{if} \quad \beta_0+\frac{1}{q}>0,\ \beta_\infty+\frac{1}{q}<0. \tag{3.6}$$

(iv) Let $0<q<1$. Then

$$L_{1,q;\mathbb{A}+\frac{1}{q},\mathbb{B}} \subsetneqq L_{(1,q;\mathbb{A},\mathbb{B})} \quad \text{if} \quad \alpha_0+\frac{1}{q}>0,\ \alpha_\infty+\frac{1}{q}<0$$

and

$$L_{1,q;(0,0),\mathbb{B}+\frac{1}{q}} \subset L_{(1,q;(-\frac{1}{q},-\frac{1}{q}),\mathbb{B})} \quad \text{if} \quad \beta_0+\frac{1}{q}>0,\ \beta_\infty+\frac{1}{q}<0.$$

*Proof.* (i) Assume first that $1 \leqslant q \leqslant \infty$. Since $p>1$, the Hardy inequality

$$\Big\| t^{\frac{1}{p}-\frac{1}{q}} \ell^{\mathbb{A}}(t)\ell\ell^{\mathbb{B}}(t)\, t^{-1} \int_0^t g(s)\, \mathrm{d}s \Big\|_q \lesssim \Big\| t^{\frac{1}{p}-\frac{1}{q}} \ell^{\mathbb{A}}(t)\ell\ell^{\mathbb{B}}(t) g(t) \Big\|_q \tag{3.7}$$

holds for every $g \in \mathscr{M}^+(0,\infty)$ (cf. [OK, Theorem 5.9]). Applied to $g=f^*$, (3.7) implies $L_{p,q;\mathbb{A},\mathbb{B}} \hookrightarrow L_{(p,q;\mathbb{A},\mathbb{B})}$. Combined with (3.1), this yields (3.4). If $0<q<1$, we use an analogous argument, applying [La, Theorem 2.2].

(ii) By the Fubini theorem,

$$\|f\|_{(1,1;\mathbb{A},\mathbb{B})} = \int_0^\infty f^*(s) \Big( \int_s^\infty t^{-1} \ell^{\mathbb{A}}(t)\ell\ell^{\mathbb{B}}(t)\, \mathrm{d}t \Big)\, \mathrm{d}s.$$

Calculating the inner integral, we obtain the assertion.

(iii) Both embeddings in (3.5) and (3.6) follow from the corresponding Hardy inequality (cf. [EOP2, Lemmas 4.2 and 4.3]). The distinction of the spaces follows by comparing their fundamental functions (cf. Lemma 3.7).

(iv) This follows from [La, Theorem 2.2]. □

Now we shall prove some auxiliary results which will be needed later.

3.9. LEMMA. *Let $0<q\leqslant\infty$, and $\mathbb{A},\mathbb{B}\in\mathbb{R}^2$. Assume that for each $i\in\{0,\infty\}$ one of the following conditions holds:*

$$\alpha_i+\frac{1}{q}<0;$$
$$\alpha_i+\frac{1}{q}=0,\ \beta_i+\frac{1}{q}<0;$$
$$q=\infty,\ \alpha_i=0,\ \beta_i=0.$$

*Then for all $f\in L_{\infty,q;\mathbb{A},\mathbb{B}}$,*

$$\|t^{-\frac{1}{q}}\ell^{\alpha_\infty}(t)\ell\ell^{\beta_\infty}(t)f^*(t)\|_{q,(1,\infty)} \lesssim \|t^{-\frac{1}{q}}\ell^{\alpha_0}(t)\ell\ell^{\beta_0}(t)f^*(t)\|_{q,(0,1)}.$$

*Proof.* Our assumptions imply that

$$\|t^{-\frac{1}{q}}\ell^{\alpha_\infty}(t)\ell\ell^{\beta_\infty}(t)\|_{q,(1,\infty)} \approx 1 \approx \|t^{-\frac{1}{q}}\ell^{\alpha_0}(t)\ell\ell^{\beta_0}(t)\|_{q,(0,1)}. \tag{3.8}$$

Consequently, for all $f \in L_{\infty,q;\mathbb{A},\mathbb{B}}$,

$$\begin{aligned}&\|t^{-\frac{1}{q}}\ell^{\alpha_\infty}(t)\ell\ell^{\beta_\infty}(t)f^*(t)\|_{q,(1,\infty)} \leqslant f^*(1)\,\|t^{-\frac{1}{q}}\ell^{\alpha_\infty}(t)\ell\ell^{\beta_\infty}(t)\|_{q,(1,\infty)}\\ &\approx f^*(1)\,\|t^{-\frac{1}{q}}\ell^{\alpha_0}(t)\ell\ell^{\beta_0}(t)\|_{q,(0,1)} \leqslant \|t^{-\frac{1}{q}}\ell^{\alpha_0}(t)\ell\ell^{\beta_0}(t)f^*(t)\|_{q,(0,1)}. \quad \square\end{aligned}$$

3.10. COROLLARY. *Let all the assumptions of* Lemma 3.9 *be satisfied. Then*

$$L_{\infty,q;\mathbb{A},\mathbb{B}} = L_{\infty,q;\alpha_0,\beta_0}(0,1).$$

*If moreover* $q = \infty$, *then*

$$L_{\infty,\infty;\mathbb{A},\mathbb{B}} = L_{\infty,\infty;\alpha_0,\beta_0}(0,1) = L_{\infty,\infty;(\alpha_0,0),(\beta_0,0)}. \tag{3.9}$$

The following result is a dual version of Lemma 3.9.

3.11. LEMMA. *Let* $0 < q \leqslant \infty$, *and* $\mathbb{A},\mathbb{B} \in \mathbb{R}^2$. *Assume that for each* $i \in \{0,\infty\}$ *one of the following conditions holds:*

$$\begin{aligned}&\alpha_i + \frac{1}{q} < 0;\\ &a_i + \frac{1}{q} = 0,\ \beta_i + \frac{1}{q} < 0;\\ &q = \infty,\ \alpha_i = 0,\ \beta_i = 0.\end{aligned}$$

*Then for all* $f \in L_{(1,q;\mathbb{A},\mathbb{B})}$,

$$\|t^{1-\frac{1}{q}}\ell^{\alpha_0}(t)\ell\ell^{\beta_0}(t)f^{**}(t)\|_{q,(0,1)} \lesssim \|t^{1-\frac{1}{q}}\ell^{\alpha_\infty}(t)\ell\ell^{\beta_\infty}(t)f^{**}(t)\|_{q,(1,\infty)}.$$

*Proof.* Using (3.8), we have for all $f \in L_{(1,q;\mathbb{A},\mathbb{B})}$,

$$\|t^{1-\frac{1}{q}}\ell^{\alpha_0}(t)\ell\ell^{\beta_0}(t)f^{**}(t)\|_{q,(0,1)} = \|t^{-\frac{1}{q}}\ell^{\alpha_0}(t)\ell\ell^{\beta_0}(t)\int_0^t f^*(s)\,\mathrm{d}s\|_{q,(0,1)}$$

$$\leqslant \|t^{-\frac{1}{q}}\ell^{\alpha_0}(t)\ell\ell^{\beta_0}(t)\|_{q,(0,1)}\int_0^1 f^*(s)\,\mathrm{d}s \approx \|t^{-\frac{1}{q}}\ell^{\alpha_\infty}(t)\ell\ell^{\beta_\infty}(t)\|_{q,(1,\infty)}\int_0^1 f^*(s)\,\mathrm{d}s$$

$$\leqslant \|t^{-\frac{1}{q}}\ell^{\alpha_\infty}(t)\ell\ell^{\beta_\infty}(t)\int_0^t f^*(s)\,\mathrm{d}s\|_{q,(1,\infty)} = \|t^{1-\frac{1}{q}}\ell^{\alpha_\infty}(t)\ell\ell^{\beta_\infty}(t)f^{**}(t)\|_{q,(1,\infty)}. \quad \square$$

3.12. COROLLARY. *Let the assumptions of* Lemma 3.11 *with* $q=\infty$ *be satisfied. Then*

$$L_{(1,\infty;\mathbb{A},\mathbb{B})} = L_{(1,\infty;(0,\alpha_\infty),(0,\beta_\infty))}.$$

We conclude this section with analogues of Lemmas 3.5, 3.7, and Theorem 3.8 for the case when $\mu(\mathscr{R})<\infty$. Proofs are analogous to the corresponding ones above and therefore omitted.

3.13. LEMMA. *Let* $\mu(\mathscr{R})<\infty$, $0<p,q\leqslant\infty$, *and* $\alpha,\beta\in\mathbb{R}$. *Let* $X$ *be one of the spaces* $L_{p,q;\alpha,\beta}$, $L_{(p,q;\alpha,\beta)}$. *Then* $X$ *is not trivial if and only if one of the following conditions holds:*

$$\begin{cases} p<\infty; \\ p=\infty,\ \alpha+\frac{1}{q}<0; \\ p=\infty,\ \alpha+\frac{1}{q}=0,\ \beta+\frac{1}{q}<0; \\ p=\infty,\ q=\infty,\ \alpha=0,\ \beta=0. \end{cases} \tag{3.10}$$

3.14. LEMMA. *Let* $R=\mu(\mathscr{R})<\infty$, $0<p,q\leqslant\infty$, *and* $\alpha,\beta\in\mathbb{R}$. *Assume that one of the conditions in* (3.10) *is satisfied.*

(i) *Let* $X=L_{p,q;\alpha,\beta}$. *Then, for all* $t\in(0,R]$,

$$\varphi_X(t)\approx\begin{cases} t^{\frac{1}{p}}\ell^{\alpha}(t)\ell\ell^{\beta}(t) & \textit{if}\quad 0<p<\infty; \\ \ell^{\alpha+\frac{1}{q}}(t)\ell\ell^{\beta}(t) & \textit{if}\quad p=\infty,\ \alpha+\frac{1}{q}<0; \\ \ell\ell^{\beta+\frac{1}{q}}(t) & \textit{if}\quad p=\infty,\ \alpha+\frac{1}{q}=0,\ \beta+\frac{1}{q}<0; \\ 1 & \textit{if}\quad p=\infty,\ q=\infty,\ \alpha=0,\ \beta=0. \end{cases}$$

(ii) *Let* $Y=L_{(p,q;\alpha,\beta)}$. *Then, for all* $t\in(0,R]$,

$$\varphi_Y(t)\approx\begin{cases} t^{\frac{1}{p}}\ell^{\alpha}(t)\ell\ell^{\beta}(t) & \textit{if}\quad 1<p<\infty; \\ \ell^{\alpha+\frac{1}{q}}(t)\ell\ell^{\beta}(t) & \textit{if}\quad p=\infty,\ \alpha+\frac{1}{q}<0; \\ \ell\ell^{\beta+\frac{1}{q}}(t) & \textit{if}\quad p=\infty,\ \alpha+\frac{1}{q}=0,\ \beta+\frac{1}{q}<0; \\ 1 & \textit{if}\quad p=\infty,\ q=\infty,\ \alpha=0,\ \beta=0; \\ t\ell^{\alpha+\frac{1}{q}}(t)\ell\ell^{\beta}(t) & \textit{if}\quad p=1,\ \alpha+\frac{1}{q}>0; \\ t\ell\ell^{\beta+\frac{1}{q}}(t) & \textit{if}\quad p=1,\ \alpha+\frac{1}{q}=0,\ \beta+\frac{1}{q}>0; \\ t\ell\ell\ell^{\frac{1}{q}}(t) & \textit{if}\quad p=1,\ \alpha+\frac{1}{q}=0,\ \beta+\frac{1}{q}=0; \\ t & \textit{if either}\ 0<p<1, \\ & \qquad \textit{or}\ \ p=1,\ \alpha+\frac{1}{q}<0, \\ & \qquad \textit{or}\ \ p=1,\ \alpha+\frac{1}{q}=0,\ \beta+\frac{1}{q}<0. \end{cases}$$

If $\mu(\mathscr{R})<\infty$, we see from Lemma 3.14 that, in certain cases, $L_{(p,q;\alpha,\beta)}$ has the fundamental function as $L^1$. The next assertion states that in fact in such cases these spaces coincide.

3.15. LEMMA. *Let* $R=\mu(\mathscr{R})<\infty$, $0<q\leqslant\infty$, *and* $\alpha,\beta\in\mathbb{R}$, *and let one of the following conditions be satisfied:*

$$\begin{cases} 0<p<1; \\ p=1,\ \alpha+\frac{1}{q}<0; \\ p=1,\ \alpha+\frac{1}{q}=0,\ \beta+\frac{1}{q}<0; \\ p=1,\ q=\infty,\ \alpha=0,\ \beta=0. \end{cases} \tag{3.11}$$

*Then*

$$L_{(p,q;\alpha,\beta)}=L^1.$$

*Proof.* Let $R=\mu(\mathscr{R})$. Our assumptions imply that

$$\|t^{\frac{1}{p}-1-\frac{1}{q}}\ell^{\alpha}(t)\ell\ell^{\beta}(t)\|_{q,(0,R)}\approx 1\approx\|t^{\frac{1}{p}-\frac{1}{q}}\ell^{\alpha}(t)\ell\ell^{\beta}(t)\|_{q,(0,R)}.$$

Consequently, for all $f\in\mathscr{M}(\mathscr{R},\mu)$,

$$\begin{aligned}\|f\|_{(p,q;\alpha,\beta)}&\leqslant\Big(\int_0^R f^*(s)\,\mathrm{d}s\Big)\|t^{\frac{1}{p}-1-\frac{1}{q}}\ell^{\alpha}(t)\ell\ell^{\beta}(t)\|_{q,(0,R)}\\ &\approx\int_0^R f^*(s)\,\mathrm{d}s\approx f^{**}(R)\,\|t^{\frac{1}{p}-\frac{1}{q}}\ell^{\alpha}(t)\ell\ell^{\beta}(t)\|_{q,(0,R)}\leqslant\|f\|_{(p,q;\alpha,\beta)},\end{aligned}$$

and the result follows. □

3.16. THEOREM. *Let* $\mu(\mathscr{R})<\infty$, $1\leqslant p\leqslant\infty$, $0<q\leqslant\infty$, *and* $\alpha,\beta\in\mathbb{R}$.

(i) *Let* $1<p\leqslant\infty$ *and let one of the conditions in* (3.10) *be satisfied. Then*

$$L_{(p,q;\alpha,\beta)}=L_{p,q;\alpha,\beta}.$$

(ii) *The space* $L_{(1,1;\alpha,\beta)}$ *coincides with the space*

| | | |
|---|---|---|
| $L^1$ | *if* | *either* $\alpha+1<0$, |
| | | *or* $\alpha+1=0,\ \beta+1<0$; |
| $L_{1,1;\alpha+1,\beta}$ | *if* | $\alpha+1>0$; |
| $L_{1,1;0,\beta+1}$ | *if* | $\alpha+1=0,\ \beta+1>0$; |
| $L_{1,1;0,0,1}$ | *if* | $\alpha+1=0,\ \beta+1=0$. |

(iii) *Let* $1<q\leqslant\infty$. *Then*

$$L_{1,q;\alpha+1,\beta}\subsetneqq L_{(1,q;\alpha,\beta)}\quad\textit{if}\quad\alpha+\frac{1}{q}>0,$$

*and*

$$L_{1,q;\frac{1}{q'},\beta+1}\subsetneqq L_{(1,q;-\frac{1}{q},\beta)}\quad\textit{if}\quad\beta+\frac{1}{q}>0.$$

(iv) *Let* $0 < q < 1$. *Then*

$$L_{1,q;\alpha+\frac{1}{q},\beta} \subset L_{(1,q;\alpha,\beta)} \quad \textit{if} \quad \alpha + \frac{1}{q} > 0,$$

*and*

$$L_{1,q;0,\beta+\frac{1}{q}} \subset L_{(1,q;-\frac{1}{q},\beta)} \quad \textit{if} \quad \beta + \frac{1}{q} > 0.$$

## 4. Embeddings among $L_{p,q;\mathbb{A},\mathbb{B}}$-spaces

Our objective here is to characterize the embedding

$$L_{P_1,Q;\mathbb{L},\mathbb{E}} \hookrightarrow L_{P_2,R;\mathbb{S},\mathbb{W}} \tag{4.1}$$

with $0 < P_1, P_2, Q, R \leqslant \infty$ and $\mathbb{L} = (\lambda_0, \lambda_\infty)$, $\mathbb{E} = (\varepsilon_0, \varepsilon_\infty)$, $\mathbb{S} = (\sigma_0, \sigma_\infty)$, $\mathbb{W} = (\omega_0, \omega_\infty) \in \mathbb{R}^2$.

First we shall investigate the embedding (4.1) with $P_1 = P_2 = P$. In the case when $\mu(\mathscr{R}) < \infty$ such an embedding is completely characterized in terms of inequalities involving the first components of vector exponents of logarithmic functions (cf. [EOP1]). If $\mu(\mathscr{R}) = \infty$, the second components of these exponents will take place in the corresponding conditions as well.

For the case of brevity, we present only statements of the main results. If $0 < P_1, P_2 < \infty$, these follow from the more general theorems in [St]. If $P_1 = P_2 = \infty$, one can use the recent results of [So, Proposition 2.7] and [CPSS, Section 3] to prove them under certain additional assumptions on weights involved. Proofs in the remaining cases are left to the reader. Our original proofs were different and analogous to those of [EOP1] (cf. also the proofs of the results of Section 5 in the Appendix).

Our first theorem characterizes the embedding $L_{P,Q;\mathbb{L},\mathbb{E}} \hookrightarrow L_{P,R;\mathbb{S},\mathbb{W}}$ provided that $0 < Q \leqslant R \leqslant \infty$ and $0 < P < \infty$.

4.1. THEOREM. *Let* $0 < Q \leqslant R \leqslant \infty$, $0 < P < \infty$, $\mu(\mathscr{R}) = \infty$, *and* $L_{P,Q;\mathbb{L},\mathbb{E}} \neq \{0\}$. *Then*

$$L_{P,Q;\mathbb{L},\mathbb{E}} \hookrightarrow L_{P,R;\mathbb{S},\mathbb{W}}$$

*if and only if*

$$\mathbb{L} \geqslant \mathbb{S}$$

*and*

$$\textit{if} \quad \lambda_i = \sigma_i \quad \textit{for some} \quad i \in \{0, \infty\}, \quad \textit{then} \quad \varepsilon_i \geqslant \omega_i.$$

Next, we characterize the embedding

$$L_{P,Q;\mathbb{L},\mathbb{E}} \hookrightarrow L_{P,R;\mathbb{S},\mathbb{W}}. \tag{4.2}$$

provided that $0 < Q \leqslant R \leqslant \infty$ and $P = \infty$.

4.2. THEOREM. *Let* $0 < Q \leqslant R \leqslant \infty$, $P = \infty$, $\mu(\mathscr{R}) = \infty$, *and* $L_{P,Q;\mathbb{L},\mathbb{E}} \neq \{0\}$. *Then*

$$L_{P,Q;\mathbb{L},\mathbb{E}} \hookrightarrow L_{P,R;\mathbb{S},\mathbb{W}}$$

*if and only if*

$$\lambda_0 + \frac{1}{Q} > \sigma_0 + \frac{1}{R};$$

*or*

$$0 = \lambda_0 + \frac{1}{Q} = \sigma_0 + \frac{1}{R}, \qquad \varepsilon_0 + \frac{1}{Q} \geqslant \omega_0 + \frac{1}{R};$$

*or*

$$0 > \lambda_0 + \frac{1}{Q} = \sigma_0 + \frac{1}{R}, \qquad \varepsilon_0 \geqslant \omega_0$$

*and simultaneously one of the following conditions is satisfied:*

$$\begin{aligned}
&\lambda_\infty + \frac{1}{Q} < 0, && \sigma_\infty + \frac{1}{R} < 0;\\
&\lambda_\infty + \frac{1}{Q} < 0, && \sigma_\infty + \frac{1}{R} = 0, \qquad \omega_\infty + \frac{1}{R} < 0;\\
&\lambda_\infty + \frac{1}{Q} < 0, && R = \infty, \qquad \sigma_\infty = 0, \qquad \omega_\infty = 0;\\
0 \leqslant{} &\lambda_\infty + \frac{1}{Q}, && \lambda_\infty + \frac{1}{Q} > \sigma_\infty + \frac{1}{R};\\
0 <{} &\lambda_\infty + \frac{1}{Q} = \sigma_\infty + \frac{1}{R}, && \varepsilon_\infty \geqslant \omega_\infty;\\
0 ={} &\lambda_\infty + \frac{1}{Q} = \sigma_\infty + \frac{1}{R}, && \varepsilon_\infty + \frac{1}{Q} \geqslant 0, \qquad \varepsilon_\infty + \frac{1}{Q} \geqslant \omega_\infty + \frac{1}{R};\\
0 ={} &\lambda_\infty + \frac{1}{Q} = \sigma_\infty + \frac{1}{R}, && \varepsilon_\infty + \frac{1}{Q} < 0, \qquad \omega_\infty + \frac{1}{R} < 0.
\end{aligned}$$

Now, we shall characterize the embedding (4.2) provided that $0 < R < Q \leqslant \infty$ and $0 < P \leqslant \infty$. We shall start with the case $0 < P < \infty$.

4.3. THEOREM. *Let* $0 < R < Q \leqslant \infty$, $0 < P < \infty$, $\mu(\mathscr{R}) = \infty$, *and* $L_{P,Q;\mathbb{L},\mathbb{E}} \neq \{0\}$. *Then*

$$L_{P,Q;\mathbb{L},\mathbb{E}} \hookrightarrow L_{P,R;\mathbb{S},\mathbb{W}}$$

*if and only if*

$$\mathbb{L} + \frac{1}{Q} \geqslant \mathbb{S} + \frac{1}{R}$$

*and*

$$\text{if} \quad \lambda_i + \frac{1}{Q} = \sigma_i + \frac{1}{R} \text{ for some } i \in \{0,\infty\}, \ \text{ then } \varepsilon_i + \frac{1}{Q} > \omega_i + \frac{1}{R}.$$

In the next theorem we consider the embedding (4.2) in the case $0 < R < Q \leqslant \infty$ and $P = \infty$.

4.4. THEOREM. *Let* $0 < R < Q \leqslant \infty$, $P = \infty$, $\mu(\mathscr{R}) = \infty$, *and* $L_{P,Q;\mathbb{L},\mathbb{E}} \neq \{0\}$. *Then*

$$L_{P,Q;\mathbb{L},\mathbb{E}} \hookrightarrow L_{P,R;\mathbb{S},\mathbb{W}}$$

*if and only if either*

$$\lambda_0 + \frac{1}{Q} > \sigma_0 + \frac{1}{R};$$

*or*

$$\lambda_0 + \frac{1}{Q} = \sigma_0 + \frac{1}{R}, \qquad \varepsilon_0 + \frac{1}{Q} > \omega_0 + \frac{1}{R}$$

*and simultaneously one of the following conditions is satisfied:*

$$\lambda_\infty + \frac{1}{Q} < 0, \qquad \sigma_\infty + \frac{1}{R} < 0;$$

$$\lambda_\infty + \frac{1}{Q} < 0, \qquad \sigma_\infty + \frac{1}{R} = 0, \qquad \omega_\infty + \frac{1}{R} < 0;$$

$$0 \leqslant \lambda_\infty + \frac{1}{Q}, \qquad \lambda_\infty + \frac{1}{Q} > \sigma_\infty + \frac{1}{R};$$

$$0 < \lambda_\infty + \frac{1}{Q} = \sigma_\infty + \frac{1}{R}, \qquad \varepsilon_\infty + \frac{1}{Q} > \omega_\infty + \frac{1}{R};$$

$$0 = \lambda_\infty + \frac{1}{Q} = \sigma_\infty + \frac{1}{R}, \qquad \varepsilon_\infty + \frac{1}{Q} \geqslant 0, \qquad \varepsilon_\infty + \frac{1}{Q} > \omega_\infty + \frac{1}{R};$$

$$0 = \lambda_\infty + \frac{1}{Q} = \sigma_\infty + \frac{1}{R}, \qquad \varepsilon_\infty + \frac{1}{Q} < 0, \qquad \omega_\infty + \frac{1}{R} < 0.$$

In all the preceding theorems we have assumed that $\mu(\mathscr{R}) = \infty$. When $\mu(\mathscr{R}) < \infty$, then the results remain valid if we omit all the assumptions on the second components of vectors $\mathbb{L}, \mathbb{E}, \mathbb{S}$, and $\mathbb{W}$ (cf. [EOP1, Theorem 6.3] and remarks on GLZ-spaces with $\mu(\mathscr{R}) < \infty$ in Section 3). We thus have the following result.

4.5. THEOREM. *Assume that* $\mu(\mathscr{R}) < \infty$ *and* $L_{P,Q;\mathbb{L},\mathbb{E}} \neq \{0\}$. *Then*

$$L_{P,Q;\mathbb{L},\mathbb{E}} \hookrightarrow L_{P,R;\mathbb{S},\mathbb{W}}$$

*if and only if one of the following conditions is satisfied:*

(i) $0 < Q \leqslant R \leqslant \infty$, $0 < P < \infty$, $\lambda_0 > \sigma_0$;
(ii) $0 < Q \leqslant R \leqslant \infty$, $0 < P < \infty$, $\lambda_0 = \sigma_0$, $\varepsilon_0 \geqslant \omega_0$;
(iii) $0 < Q \leqslant R \leqslant \infty$, $P = \infty$, $\lambda_0 + \frac{1}{Q} > \sigma_0 + \frac{1}{R}$;
(iv) $0 < Q \leqslant R \leqslant \infty$, $P = \infty$, $\lambda_0 + \frac{1}{Q} = \sigma_0 + \frac{1}{R} = 0$, $\varepsilon_0 + \frac{1}{Q} \geqslant \omega_0 + \frac{1}{R}$;
(v) $0 < Q \leqslant R \leqslant \infty$, $P = \infty$, $\lambda_0 + \frac{1}{Q} = \sigma_0 + \frac{1}{R} < 0$, $\varepsilon_0 \geqslant \omega_0$;
(vi) $0 < R < Q \leqslant \infty$, $0 < P \leqslant \infty$, $\lambda_0 + \frac{1}{Q} > \sigma_0 + \frac{1}{R}$;
(vii) $0 < R < Q \leqslant \infty$, $0 < P \leqslant \infty$, $\lambda_0 + \frac{1}{Q} = \sigma_0 + \frac{1}{R}$, $\varepsilon_0 + \frac{1}{Q} > \omega_0 + \frac{1}{R}$.

So far we have investigated embeddings among $L_{p,q;\mathbb{A},\mathbb{B}}$ spaces provided that the first index $p$ was fixed. Embeddings with $p$ varying are similar to those for Lebesgue spaces $L^p$.

4.6. THEOREM. *Let $0 < P_1, P_2, Q, R \leqslant \infty$, $P_1 \neq P_2$, and $L_{P_1,Q;\mathbb{L},\mathbb{E}} \neq \{0\}$. Then*

$$L_{P_1,Q;\mathbb{L},\mathbb{E}} \hookrightarrow L_{P_2,R;\mathbb{S},\mathbb{W}}$$

*if and only if $\mu(\mathscr{R}) < \infty$ and $P_1 > P_2$.*

## 5. Embeddings among $L_{(p,q;\mathbb{A},\mathbb{B})}$-spaces

The aim of this section is to characterize the embedding

$$L_{(P_1,Q;\mathbb{L},\mathbb{E})} \hookrightarrow L_{(P_2,R;\mathbb{S},\mathbb{W})} \tag{5.1}$$

with $0 < P_1, P_2, Q, R \leqslant \infty$, and $\mathbb{L} = (\lambda_0, \lambda_\infty)$, $\mathbb{E} = (\varepsilon_0, \varepsilon_\infty)$, $\mathbb{S} = (\sigma_0, \sigma_\infty)$, $\mathbb{W} = (\omega_0, \omega_\infty) \in \mathbb{R}^2$.

To this end one can use the approach of [GHS, Theorem 5.2] where embeddings among Lorentz spaces $\Gamma^p(v)$ are characterized. However, the characterization is described in terms of discretizing sequences and thus it is not explicit. We shall point out a simple characterization of (5.1). As in the Section 4, we present only the statements of results. Detailed proofs can be found in the Appendix.

First, we consider the embedding (5.1) with $P_1 = P_2 = P$, that is

$$L_{(P,Q;\mathbb{L},\mathbb{E})} \hookrightarrow L_{(P,R;\mathbb{S},\mathbb{W})}. \tag{5.2}$$

We already know (see Section 4) necessary and sufficient conditions for the embedding

$$L_{P,Q;\mathbb{L},\mathbb{E}} \hookrightarrow L_{P,R;\mathbb{S},\mathbb{W}}.$$

Because (cf. Section 3), for $0 < q \leqslant \infty$ and $\mathbb{A}, \mathbb{B} \in \mathbb{R}^2$,

$$\begin{aligned}
&L_{(p,q;\mathbb{A},\mathbb{B})} = L_{p,q;\mathbb{A},\mathbb{B}} && \text{if} \quad 1 < p \leqslant \infty,\\
&L_{(p,q;\mathbb{A},\mathbb{B})} = \{0\} && \text{if} \quad 0 < p < 1 \text{ and } \mu(\mathscr{R}) = \infty,\\
&L_{(p,q;\mathbb{A},\mathbb{B})} = L^1 && \text{if} \quad 0 < p < 1 \text{ and } \mu(\mathscr{R}) < \infty,
\end{aligned} \tag{5.3}$$

it remains to characterize the embedding (5.2) for $P = 1$. Such a characterization is given in Theorems 5.1–5.4 while Theorem 5.5 characterizes the embedding (5.1) for $P_1 \neq P_2$.

First, we consider the case when $0 < Q \leqslant R \leqslant \infty$.

5.1. THEOREM. *Let $0 < Q \leqslant R \leqslant \infty$, $\mu(\mathscr{R}) = \infty$, and $L_{(1,Q;\mathbb{L},\mathbb{E})} \neq \{0\}$. Then*

$$L_{(1,Q;\mathbb{L},\mathbb{E})} \hookrightarrow L_{(1,R;\mathbb{S},\mathbb{W})}$$

*if and only if either*

$$\lambda_\infty + \frac{1}{Q} > \sigma_\infty + \frac{1}{R};$$

*or*

$$0 > \lambda_\infty + \frac{1}{Q} = \sigma_\infty + \frac{1}{R}, \qquad \varepsilon_\infty \geqslant \omega_\infty;$$

*or*

$$0 = \lambda_\infty + \frac{1}{Q} = \sigma_\infty + \frac{1}{R}, \qquad \varepsilon_\infty + \frac{1}{Q} \geqslant \omega_\infty + \frac{1}{R}$$

*and simultaneously one of the following conditions is satisfied:*

$$\begin{aligned}
&\lambda_0 + \frac{1}{Q} < 0, && \sigma_0 + \frac{1}{R} < 0;\\
&\lambda_0 + \frac{1}{Q} < 0, && \sigma_0 + \frac{1}{R} = 0, \qquad \omega_0 + \frac{1}{R} < 0;\\
&\lambda_0 + \frac{1}{Q} < 0, && R = \infty, \qquad \sigma_0 = \omega_0 = 0;\\
0 \leqslant &\lambda_0 + \frac{1}{Q}, && \lambda_0 + \frac{1}{Q} > \sigma_0 + \frac{1}{R};\\
0 < &\lambda_0 + \frac{1}{Q} = \sigma_0 + \frac{1}{R}, && \varepsilon_0 \geqslant \omega_0;\\
0 = &\lambda_0 + \frac{1}{Q} = \sigma_0 + \frac{1}{R}, && \varepsilon_0 + \frac{1}{Q} \geqslant 0, \qquad \varepsilon_0 + \frac{1}{Q} \geqslant \omega_0 + \frac{1}{R};\\
0 = &\lambda_0 + \frac{1}{Q} = \sigma_0 + \frac{1}{R}, && \varepsilon_0 + \frac{1}{Q} < 0, \qquad \omega_0 + \frac{1}{R} < 0.
\end{aligned}$$

The following theorem characterizes the embedding

$$L_{(1,Q;\mathbb{L},\mathbb{E})} \hookrightarrow L_{(1,R;\mathbb{S},\mathbb{W})}$$

in the case when $0 < R < Q \leqslant \infty$.

5.2. THEOREM. *Let* $0 < R < Q \leqslant \infty$, $\mu(\mathscr{R}) = \infty$, *and* $L_{(1,Q;\mathbb{L},\mathbb{E})} \neq \{0\}$. *Then*

$$L_{(1,Q;\mathbb{L},\mathbb{E})} \hookrightarrow L_{(1,R;\mathbb{S},\mathbb{W})}$$

*if and only if either*

$$\lambda_\infty + \frac{1}{Q} > \sigma_\infty + \frac{1}{R};$$

*or*

$$\lambda_\infty + \frac{1}{Q} = \sigma_\infty + \frac{1}{R}, \qquad \varepsilon_\infty + \frac{1}{Q} > \omega_\infty + \frac{1}{R}$$

*and simultaneously one of the following conditions is satisfied:*

$$\lambda_0 + \frac{1}{Q} < 0, \qquad \sigma_0 + \frac{1}{R} < 0;$$
$$\lambda_0 + \frac{1}{Q} < 0, \qquad \sigma_0 + \frac{1}{R} = 0, \qquad \omega_0 + \frac{1}{R} < 0;$$
$$0 \leqslant \lambda_0 + \frac{1}{Q}, \qquad \lambda_0 + \frac{1}{Q} > \sigma_0 + \frac{1}{R};$$
$$0 < \lambda_0 + \frac{1}{Q} = \sigma_0 + \frac{1}{R}, \qquad \varepsilon_0 + \frac{1}{Q} > \omega_0 + \frac{1}{R};$$
$$0 = \lambda_0 + \frac{1}{Q} = \sigma_0 + \frac{1}{R}, \qquad \varepsilon_0 + \frac{1}{Q} \geqslant 0, \qquad \varepsilon_0 + \frac{1}{Q} > \omega_0 + \frac{1}{R};$$
$$0 = \lambda_0 + \frac{1}{Q} = \sigma_0 + \frac{1}{R}, \qquad \varepsilon_0 + \frac{1}{Q} < 0, \qquad \omega_0 + \frac{1}{R} < 0.$$

In Theorems 5.1 and 5.2 we have assumed that $\mu(\mathscr{R}) = \infty$. Now, we shall characterize the embedding (5.2) provided that $P = 1$ and $\mu(\mathscr{R}) < \infty$. In this case Theorems 5.1 and 5.2 remain true if we omit all the assumptions on the second components of vectors $\mathbb{L}, \mathbb{E}, \mathbb{S}$ and $\mathbb{W}$ (cf. [EOP1, Theorem 6.3] and remarks on the GLZ spaces with $\mu(\mathscr{R}) < \infty$ in Section 3). Since the condition $\mu(\mathscr{R}) < \infty$ implies that the spaces $L_{(1,q;\mathbb{A},\mathbb{B})}$ (with $0 < q \leqslant \infty$, $\mathbb{A} = (\alpha_0, \alpha_\infty)$, $\mathbb{B} = (\beta_0, \beta_\infty) \in \mathbb{R}^2$) and $L_{(1,q;\alpha_0,\beta_0)}$ coincide, we can consider, instead of (5.2), the embedding

$$L_{(1,Q;\lambda,\varepsilon)} \hookrightarrow L_{(1,R;\sigma,\omega)},$$

where $0 < Q, R \leqslant \infty$ and $\lambda, \varepsilon, \sigma, \omega \in \mathbb{R}$. Using the same method as in the case $P = 1$ and $\mu(\mathscr{R}) = \infty$, one can prove the following two theorems.

5.3. THEOREM. *Let* $0 < \mu(\mathscr{R}) < \infty$, $0 < Q \leqslant R \leqslant \infty$, *and* $\lambda, \varepsilon, \sigma, \omega \in \mathbb{R}$. *Then*

$$L_{(1,Q;\lambda,\varepsilon)} \hookrightarrow L_{(1,R;\sigma,\omega)}$$

*if and only if one of the following conditions is satisfied:*

$$\lambda + \frac{1}{Q} < 0, \qquad \sigma + \frac{1}{R} < 0;$$
$$\lambda + \frac{1}{Q} < 0, \qquad \sigma + \frac{1}{R} = 0, \qquad \omega + \frac{1}{R} < 0;$$
$$\lambda + \frac{1}{Q} < 0, \qquad R = \infty, \qquad \sigma = \omega = 0;$$
$$0 \leqslant \lambda + \frac{1}{Q}, \qquad \lambda + \frac{1}{Q} > \sigma + \frac{1}{R};$$
$$0 < \lambda + \frac{1}{Q} = \sigma + \frac{1}{R}, \qquad \varepsilon \geqslant \omega;$$
$$0 = \lambda + \frac{1}{Q} = \sigma + \frac{1}{R}, \qquad \varepsilon + \frac{1}{Q} \geqslant 0, \qquad \varepsilon + \frac{1}{Q} \geqslant \omega + \frac{1}{R};$$
$$0 = \lambda + \frac{1}{Q} = \sigma + \frac{1}{R}, \qquad \varepsilon + \frac{1}{Q} < 0, \qquad \omega + \frac{1}{R} < 0.$$

5.4. THEOREM. *Let* $0 < \mu(\mathscr{R}) < \infty$, $0 < R < Q \leqslant \infty$, *and* $\lambda, \varepsilon, \sigma, \omega \in \mathbb{R}$. *Then*

$$L_{(1,Q;\lambda,\varepsilon)} \hookrightarrow L_{(1,R;\sigma,\omega)}$$

*if and only if one of the following conditions is satisfied:*

$$\begin{aligned}
&\lambda + \frac{1}{Q} < 0, && \sigma + \frac{1}{R} < 0; \\
&\lambda + \frac{1}{Q} < 0, && \sigma + \frac{1}{R} = 0, \quad \omega + \frac{1}{R} < 0; \\
0 \leqslant{}&\lambda + \frac{1}{Q}, && \lambda + \frac{1}{Q} > \sigma + \frac{1}{R}; \\
0 <{}&\lambda + \frac{1}{Q} = \sigma + \frac{1}{R}, && \varepsilon + \frac{1}{Q} > \omega + \frac{1}{R}; \\
0 ={}&\lambda + \frac{1}{Q} = \sigma + \frac{1}{R}, && \varepsilon + \frac{1}{Q} \geqslant 0, \quad \varepsilon + \frac{1}{Q} > \omega + \frac{1}{R}; \\
0 ={}&\lambda + \frac{1}{Q} = \sigma + \frac{1}{R}, && \varepsilon + \frac{1}{Q} < 0, \quad \omega + \frac{1}{R} < 0.
\end{aligned}$$

The next theorem describes embedding among $L_{(p,q;\mathbb{A},\mathbb{B})}$-spaces with $p$ varying.

5.5. THEOREM. *Let* $0 < P_1, P_2, Q, R \leqslant \infty$, $P_1 \neq P_2$, $\mathbb{L}, \mathbb{E}, \mathbb{S}, \mathbb{W} \in \mathbb{R}^2$ *and*

$$L_{(P_1,Q;\mathbb{L},\mathbb{E})} \neq \{0\}. \tag{5.4}$$

*Then*

$$L_{(P_1,Q;\mathbb{L},\mathbb{E})} \hookrightarrow L_{(P_2,R;\mathbb{S},\mathbb{W})} \tag{5.5}$$

*if and only if* $\mu(\mathscr{R}) < \infty$ *and one of the following conditions is satisfied:*

$$1 \leqslant P_2 < P_1 \leqslant \infty; \tag{5.6}$$

$$0 < P_2 < 1 \leqslant P_1 \leqslant \infty; \tag{5.7}$$

$$0 < P_1, P_2 < 1; \tag{5.8}$$

$$0 < P_1 < 1, \qquad P_2 = 1, \qquad \sigma_0 + \frac{1}{R} < 0; \tag{5.9}$$

$$0 < P_1 < 1, \qquad P_2 = 1, \qquad \sigma_0 + \frac{1}{R} = 0, \qquad \omega_0 + \frac{1}{R} < 0; \tag{5.10}$$

$$0 < P_1 < 1, \qquad P_2 = 1, \qquad R = \infty, \qquad \sigma_0 = \omega_0 = 0. \tag{5.11}$$

## 6. Associate spaces of GLZ spaces

In this section we give a complete description of the associate space of a non-trivial GLZ space. To begin, we single out the GLZ spaces whose associate space is trivial.

6.1. THEOREM. *Let* $0 < p, q \leqslant \infty$ *and* $\mathbb{A} = (\alpha_0, \alpha_\infty)$, $\mathbb{B} = (\beta_0, \beta_\infty) \in \mathbb{R}^2$. *Put* $X = L_{p,q;\mathbb{A},\mathbb{B}}$.

(i) *Assume that one of the following conditions holds:*

$$0 < p < 1;$$
$$p = 1, \quad 0 < q \leqslant 1, \quad \alpha_0 < 0;$$
$$p = 1, \quad 0 < q \leqslant 1, \quad \alpha_0 = 0, \quad \beta_0 < 0.$$

*Then*

$$X' = \{0\}.$$

(ii) *Assume that one of the following conditions holds:*

$$p = 1, \quad 1 < q \leqslant \infty, \quad \alpha_0 < \frac{1}{q'};$$
$$p = 1, \quad 1 < q \leqslant \infty, \quad \alpha_0 = \frac{1}{q'}, \quad \beta_0 \leqslant \frac{1}{q'};$$

*Then*

$$X' = \{0\}.$$

*Proof.* (i) By [BS, Chapter 2, Corollary 7.8], for every $t \in (0,1)$ there exists a function $g_t \in \mathscr{M}(\mathscr{R}, \mu)$ such that $g_t^* = \chi_{(0,t)}$. By Lemma 3.7 (i),

$$\|g_t\|_X \approx t^{1/p} \ell^{\alpha_0}(t) \ell\ell^{\beta_0}(t), \qquad 0 < t < 1.$$

Assume that $f \in \mathscr{M}(\mathscr{R}, \mu)$ and $f \not\equiv 0$. Then there exist two positive constants, $\varepsilon, \delta$, such that $f^*(s) \geqslant \delta$ for $s \in (0, \varepsilon)$. We claim that $f \notin X'$, that is, $\|f\|_{X'} = \infty$. Indeed,

$$\|f\|_{X'} = \sup_{\|g\|_X \leqslant 1} \int_0^\infty f^*(s) g^*(s) \, \mathrm{d}s \geqslant \sup_{0<t<\varepsilon} \int_0^\infty f^*(s) \frac{g_t^*(s)}{\|g_t\|_X} \, \mathrm{d}s$$
$$\gtrsim \delta \sup_{0<t<\varepsilon} t^{-1/p} \ell^{-\alpha_0}(t) \ell\ell^{-\beta_0}(t) \int_0^t \mathrm{d}s = \infty.$$

(ii) The function $h(t) = t^{-1} \ell^{-1}(t) \ell\ell^{-1}(t) \chi_{(0,\varepsilon)}(t)$ is, for some $\varepsilon$ small enough, non-increasing on $(0, \infty)$. Moreover,

$$\|t^{\frac{1}{p}-\frac{1}{q}} \ell^{\mathbb{A}}(t) \ell\ell^{\mathbb{B}}(t) h(t)\|_{q,(0,\infty)} < \infty, \tag{6.1}$$

but

$$\int_0^\delta h(t) \, \mathrm{d}t = \infty \quad \text{for every} \quad \delta > 0. \tag{6.2}$$

By [BS, Chapter 2, Corollary 7.8], there is a $g \in \mathscr{M}(\mathscr{R}, \mu)$ such that $g^* = h$, and, by (6.1), $g \in X$. Now, let $f \in X'$. Then $\int_0^\infty f^*(t) g^*(t) \, \mathrm{d}t < \infty$. However, by (6.2), that is possible only if $f \equiv 0$. The proof is complete. $\square$

In the next theorem we describe associated spaces of $X = L_{p,q;\mathbb{A},\mathbb{B}}$ provided that $1 \leqslant p \leqslant \infty$ and $1 < q \leqslant \infty$. In accordance with our definition of the associate space (cf. Section 2 above), we restrict ourselves to the cases when $X \neq \{0\}$.

6.2. THEOREM. *Let* $1 \leqslant p \leqslant \infty$, $1 < q \leqslant \infty$, $\mathbb{A} = (\alpha_0, \alpha_\infty)$, $\mathbb{B} = (\beta_0, \beta_\infty) \in \mathbb{R}^2$, *and assume that the space* $L_{p,q;\mathbb{A},\mathbb{B}}$ *is not trivial* (*cf.* (3.2)). *Then* $(L_{p,q;\mathbb{A},\mathbb{B}})' = \mathscr{L}$, *the space described below:*

(i) *Let* $1 \leqslant p < \infty$, $1 < q \leqslant \infty$. *Then*

$$\mathscr{L} = L_{(p',q';-\mathbb{A},-\mathbb{B})} = L_{p',q';-\mathbb{A},-\mathbb{B}}. \tag{6.3}$$

(ii) *Let* $p = \infty$, $1 < q < \infty$, $\alpha_0 + \frac{1}{q} < 0$, *and either* $\alpha_\infty + \frac{1}{q} > 0$ *or* $\alpha_\infty + \frac{1}{q} = 0$, $\beta_\infty + \frac{1}{q} \geqslant 0$. *Then*

$$\mathscr{L} = \begin{cases} L_{(1,q';-\mathbb{A}-1,-\mathbb{B})} & \textit{if} \quad \alpha_\infty + \frac{1}{q} > 0; \\ L_{(1,q';(-\alpha_0-1,-\frac{1}{q'}),(-\beta_0,-\beta_\infty-1))} & \textit{if} \quad \alpha_\infty + \frac{1}{q} = 0,\ \beta_\infty + \frac{1}{q} > 0; \\ L_{(1,q';(-\alpha_0-1,-\frac{1}{q'}),(-\beta_0,-\frac{1}{q'}),(0,-1))} & \textit{if} \quad \alpha_\infty + \frac{1}{q} = 0,\ \beta_\infty + \frac{1}{q} = 0. \end{cases}$$

(iii) *Let* $p = \infty$, $1 < q < \infty$, $\alpha_0 + \frac{1}{q} = 0$, $\beta_0 + \frac{1}{q} < 0$, *and either* $\alpha_\infty + \frac{1}{q} > 0$ *or* $\alpha_\infty + \frac{1}{q} = 0$, $\beta_\infty + \frac{1}{q} \geqslant 0$. *Then*

$$\mathscr{L} = \begin{cases} L_{(1,q';(-\frac{1}{q'},-\alpha_\infty-1),(-\beta_0-1,-\beta_\infty))} & \textit{if} \quad \alpha_\infty + \frac{1}{q} > 0; \\ L_{(1,q';(-\frac{1}{q'},-\frac{1}{q'}),-\mathbb{B}-1)} & \textit{if} \quad \alpha_\infty + \frac{1}{q} = 0,\ \beta_\infty + \frac{1}{q} > 0; \\ L_{(1,q';(-\frac{1}{q'},-\frac{1}{q'}),(-\beta_0-1,-\frac{1}{q'}),(0,-1))} & \textit{if} \quad \alpha_\infty + \frac{1}{q} = 0,\ \beta_\infty + \frac{1}{q} = 0. \end{cases}$$

(iv) *Let* $p = \infty$, $q = \infty$, *and either* $\alpha_\infty > 0$ *or* $\alpha_\infty = 0$, $\beta_\infty \geqslant 0$. *Then*

$$\mathscr{L} = L_{1,1;-\mathbb{A},-\mathbb{B}}.$$

(v) *Let* $p = \infty$, $1 < q \leqslant \infty$, *and either* $\alpha_\infty + \frac{1}{q} < 0$ *or* $\alpha_\infty + \frac{1}{q} = 0$ *and* $\beta_\infty + \frac{1}{q} < 0$. *Then*

$$\mathscr{L} = \left\{f \in \mathscr{M}(\mathscr{R},\mu);\ \|f\|_{\mathscr{L}} := \|f\|_{X(0,1)} + \int_0^\infty f^*(t)\,\mathrm{d}t < \infty\right\},$$

*where*

$$X(0,1) = \begin{cases} L_{(1,q';-\alpha_0-1,-\beta_0)}(0,1) & \textit{if} \quad 1 < q < \infty,\ \alpha_0 + \frac{1}{q} < 0; \\ L_{(1,q';-\frac{1}{q'},-\beta_0-1)}(0,1) & \textit{if} \quad 1 < q < \infty,\ \alpha_0 + \frac{1}{q} = 0,\ \beta_0 + \frac{1}{q} < 0; \\ L_{1,1;-\alpha_0,-\beta_0}(0,1) & \textit{if} \quad q = \infty. \end{cases}$$

6.3. REMARK. Since $L_{(1,r;\sigma,\omega)}(0,1) \hookrightarrow L^1(0,1)$ for every $1 \leqslant r \leqslant \infty$, $\sigma, \omega \in \mathbb{R}$, and $L_{1,1;-\alpha_0,-\beta_0}(0,1) \hookrightarrow L^1(0,1)$ if either $\alpha_0 < 0$, or $\alpha_0 = 0$ and $\beta_0 \leqslant 0$, we can write in Theorem 6.2 (v),

$$\mathscr{L} = \left\{f \in \mathscr{M}(\mathscr{R},\mu);\ \|f\|_{\mathscr{L}} := \|f\|_{X(0,1)} + \int_1^\infty f^*(t)\,\mathrm{d}t < \infty\right\}.$$

*Proof of Theorem 6.2.* We first prove the assertion in the cases (i)–(iii) for $1 < q < \infty$, since the technique of the proof is common. We shall start with proving the inclusion

$$(L_{p,q;\mathbb{A},\mathbb{B}})' \hookrightarrow \mathscr{L}. \tag{6.4}$$

For this purpose it is enough to verify the inequality

$$\|f\|_{\mathscr{L}} \lesssim \|f\|_{(L_{p,q;\mathbb{A},\mathbb{B}})'} \tag{6.5}$$

for all step functions $f$. For convenience, let us denote by $b(t)$ the function defined by

$$\|f\|_{\mathscr{L}} = \|t^{\frac{1}{p'}-\frac{1}{q'}} b(t) f^{**}(t)\|_{q'}. \tag{6.6}$$

We further put

$$\varrho(t) = \big(f^{**}(t)\big)^{q'-1} t^{\frac{q'}{p'}-1} b^{q'}(t), \tag{6.7}$$

and

$$g(t) = \int_t^\infty \frac{\varrho(s)}{s}\, \mathrm{d}s. \tag{6.8}$$

Then $g^* = g$ and there exists $\tilde{g} \in \mathscr{M}(\mathscr{R},\mu)$ such that $\tilde{g}^* = g$. By the Fubini theorem and Hölder's inequality,

$$\|f\|_{\mathscr{L}}^{q'} = \int_0^\infty \varrho(t) f^{**}(t)\, \mathrm{d}t = \int_0^\infty g(t) f^*(t)\, \mathrm{d}t \leqslant \|\tilde{g}\|_{p,q;\mathbb{A},\mathbb{B}} \|f\|_{(L_{p,q;\mathbb{A},\mathbb{B}})'}.$$

Now, in order to obtain (6.5), it is enough to show that

$$\|\tilde{g}\|_{p,q;\mathbb{A},\mathbb{B}} \lesssim \|f\|_{\mathscr{L}}^{q'-1}. \tag{6.9}$$

Rewriting (6.9) with the help of Definition 3.1 and (6.6) we get, using also (6.8) and (6.7),

$$\Big\|t^{\frac{1}{p}-\frac{1}{q}} \ell^{\mathbb{A}}(t) \ell\ell^{\mathbb{B}}(t) \int_t^\infty \big(f^{**}(s)\big)^{q'-1} s^{\frac{q'}{p'}-2} b^{q'}(s)\, \mathrm{d}s\Big\|_q \lesssim \Big\|t^{\frac{q'-1}{p'}-\frac{1}{q}} b^{q'-1}(t) \big(f^{**}(t)\big)^{q'-1}\Big\|_q.$$

Using an appropriate substitution, this amounts to the Hardy inequality

$$\Big\|t^{\frac{1}{p}-\frac{1}{q}} \ell^{\mathbb{A}}(t) \ell\ell^{\mathbb{B}}(t) \int_t^\infty h(s)\, \mathrm{d}s\Big\|_q \lesssim \Big\|t^{\frac{1}{p}+\frac{1}{q'}} \big(b(t)\big)^{-1} h(t)\Big\|_q. \tag{6.10}$$

A sufficient condition for (6.10) is given by (cf. [OK])

$$\sup_{0<x<\infty} \|t^{\frac{1}{p}-\frac{1}{q}} \ell^{\mathbb{A}}(t) \ell\ell^{\mathbb{B}}(t)\|_{q,(0,x)} \|t^{-\frac{1}{p}-\frac{1}{q'}} b(t)\|_{q',(x,\infty)} < \infty. \tag{6.11}$$

Now it is a matter of a tedious but elementary calculation to verify (6.11) for appropriate $b(t)$, given by (6.6), in all the cases of $\mathscr{L}$. This yields the inclusion (6.4) for the cases (i)–(iii) restricted to $1 < q < \infty$.

Now let us prove the converse inclusion $\mathscr{L} \hookrightarrow {L_{p,q;\mathbb{A},\mathbb{B}}}'$, that is,

$$\|f\|_{(L_{p,q;\mathbb{A},\mathbb{B}})'} \lesssim \|f\|_{\mathscr{L}} \qquad \text{for all} \quad f \in \mathscr{L} \tag{6.12}$$

in the cases (i)–(iii) with $1 < q < \infty$. Since for every $f \in \mathscr{M}(\mathscr{R}, \mu)$,

$$\|f\|_{(L_{p,q;\mathbb{A},\mathbb{B}})'} = \sup_{\|g\|_{p,q;\mathbb{A},\mathbb{B}} \leqslant 1} \int_{\mathscr{R}} f(x)g(x)\, \mathrm{d}\mu \tag{6.13}$$

and (cf. [BS, Chapter 2, Theorem 2.2])

$$\int_{\mathscr{R}} f(x)g(x)\, \mathrm{d}\mu \leqslant \int_0^\infty f^*(t)g^*(t)\, \mathrm{d}t, \tag{6.14}$$

we see that in order to prove (6.12) it is enough to show that

$$\int_0^\infty f^*(t)g^*(t)\, \mathrm{d}t \lesssim \|f\|_{\mathscr{L}}\ \|g\|_{p,q;\mathbb{A},\mathbb{B}} \tag{6.15}$$

for all $f \in \mathscr{L}$ and $g \in L_{p,q;\mathbb{A},\mathbb{B}}$. We use (6.6) and rewrite (6.15) as

$$\int_0^\infty f^*(t)g^*(t)\, \mathrm{d}t \leqslant \|t^{\frac{1}{p'}-\frac{1}{q'}} b(t) f^{**}(t)\|_{q'}\ \|t^{\frac{1}{p}-\frac{1}{q}} \ell^{\mathbb{A}}(t)\ell\ell^{\mathbb{B}}(t) g^*(t)\|_q. \tag{6.16}$$

To get (6.16) we shall use the result of Sawyer ([Sa, (1.7)]):

$$\int_0^\infty f^*(t)g^*(t)\, \mathrm{d}t \leqslant \|g^*\|_{q(v)} \Big( \|f^{**}\|_{q'(\tilde{v})} + \frac{1}{v(0,\infty)^{1/q}} \int_0^\infty f^*(t)\, \mathrm{d}t \Big), \tag{6.17}$$

where $1 < q < \infty$, $v$ is a positive weight, $v(a,b) = \int_a^b v(t)\,\mathrm{d}t$ if $0 \leqslant a < b \leqslant \infty$, $\tilde{v}$ is given by

$$\tilde{v}(t) = \frac{t^{q'} v(t)}{(v(0,t))^{q'}}, \qquad t \in (0,\infty), \tag{6.18}$$

and $\|h\|_{q(v)} = \big(\int_0^\infty h^q v\big)^{1/q}$. The suitable choice of $v$ in our situation reads as

$$v(t) = t^{\frac{q}{p}-1} \ell^{\mathbb{A}q}(t) \ell\ell^{\mathbb{B}q}(t), t \in (0,\infty). \tag{6.19}$$

Now, in the cases (i)–(iii) with $1 < q < \infty$, we have $v(0,\infty) = \infty$. Together with the convention $\infty/\infty = 0$ (used also in [Sa]) this implies that the second summand at the right hand side of (6.17) disappears, and (6.15) will follows once we show that

$$t^{\frac{q'}{p'}-1} b^{q'}(t) = \tilde{v}(t), \tag{6.20}$$

where $b$ is the function from (6.6) that corresponds to the space $\mathscr{L}$ from (i)–(iii), and $v$, $\tilde{v}$ are from (6.19), (6.18). Since (6.20) follows by a calculation, we have proved (i)–(iii) for $1 < q < \infty$.

Our next step will be to prove (v) for $1 < q < \infty$. Assume that $\alpha_0 + \frac{1}{q} < 0$. (The proof in the case $\alpha_0 + \frac{1}{q} = 0$ and $\beta_0 + \frac{1}{q} < 0$ is entirely analogous and therefore omitted.) By our definition of $\mathscr{L}$,

$$\|f\|_{\mathscr{L}} = \|t^{\frac{1}{q}} \ell^{-\alpha_0-1}(t)\ell\ell^{-\beta_0}(t) f^{**}(t)\|_{q',(0,1)} + \int_0^\infty f^*(t)\,\mathrm{d}t.$$

Exactly in the same way as above we can show that

$$\|t^{\frac{1}{q}} \ell^{-\alpha_0-1}(t)\ell\ell^{-\beta_0}(t) f^{**}(t)\|_{q',(0,1)} \lesssim \|f\|_{(L_{\infty,q;\mathbb{A},\mathbb{B}})'}.$$

Further, by the Hölder inequality,

$$\int_0^\infty f^*(t)\,\mathrm{d}t \leqslant \|f\|_{(L_{\infty,q;\mathbb{A},\mathbb{B}})'}\,\|1\|_{\infty,q;\mathbb{A},\mathbb{B}},$$

and, since $1 \in L_{\infty,q;\mathbb{A},\mathbb{B}}$, this yields $(L_{\infty,q;\mathbb{A},\mathbb{B}})' \hookrightarrow \mathscr{L}$.

To prove the converse embedding, we shall use the Sawyer's inequality (6.17) again, but this time the last summand does not disappear, as $v(0,\infty) < \infty$, where $v$ is from (6.19), that is (recall $p = \infty$),

$$v(t) = t^{-1}\ell^{\mathbb{A}q}(t)\ell\ell^{\mathbb{B}q}(t), \qquad t \in (0,\infty).$$

We get

$$v(0,t) \approx \ell^{\alpha_0 q+1}(t)\ell\ell^{\beta_0 q}(t)\chi_{(0,1)}(t) + \chi_{(1,\infty)}(t),$$

and, by (6.18),

$$\tilde{v}(t) \approx t^{q'-1}\ell^{(-\alpha_0-1)q'}(t)\ell\ell^{-\beta_0 q'}(t)\chi_{(0,1)}(t) + t^{q'-1}\ell^{\alpha_\infty q}(t)\ell\ell^{\beta_\infty q}(t)\chi_{(1,\infty)}(t).$$

Therefore, by (6.17),

$$\int_0^\infty f^*(t)g^*(t)\,\mathrm{d}t \lesssim \|g\|_{\infty,q;\mathbb{A},\mathbb{B}}\Big(\Big(\int_0^\infty \big(f^{**}(t)\big)^{q'}\tilde{v}(t)\,\mathrm{d}t\Big)^{1/q'} + \int_0^\infty f^*(t)\,\mathrm{d}t\Big).$$

Moreover,

$$\Big(\int_0^\infty \big(f^{**}(t)\big)^{q'}\tilde{v}(t)\,\mathrm{d}t\Big)^{1/q'} = \|t^{\frac{1}{q}}\ell^{-\alpha_0-1}(t)\ell\ell^{-\beta_0}(t)f^{**}(t)\|_{q',(0,1)}$$

$$+\|t^{\frac{1}{q}}\ell^{\alpha_\infty(q-1)}(t)\ell\ell^{\beta_\infty(q-1)}(t)f^{**}(t)\|_{q',(1,\infty)} = I_1 + I_2,$$

say. It is clear that

$$I_1 = \|f^*\|_{X(0,1)}.$$

To estimate $I_2$, we insert for $f^{**}$ and obtain thereby

$$I_2 \approx \int_0^1 f^*(t)\,\mathrm{d}\,t\ \|t^{-\frac{1}{q'}}\ell^{\alpha_\infty(q-1)}(t)\ell\ell^{\beta_\infty(q-1)}(t)\|_{q',(1,\infty)}$$

$$+\left\|t^{-\frac{1}{q'}}\ell^{\alpha_\infty(q-1)}(t)\ell\ell^{\beta_\infty(q-1)}(t)\int_1^t f^*\right\|_{q',(1,\infty)} = I_3 + I_4,$$

say. Observe that

$$\|t^{-\frac{1}{q'}}\ell^{\alpha_\infty(q-1)}(t)\ell\ell^{\beta_\infty(q-1)}(t)\|_{q',(1,\infty)} < \infty,$$

and this yields

$$I_3 \lesssim \int_0^1 f^*(t)\ \mathrm{d}\,t.$$

Further, by the Bradley condition for the Hardy inequality (cf. [OK]),

$$I_4 \lesssim \int_1^\infty f^*(t)\ \mathrm{d}\,t.$$

Consequently,

$$\int_0^\infty f^*(t)\,g^*(t)\ \mathrm{d}\,t \lesssim \|g\|_{\infty,q;\mathbb{A},\mathbb{B}}\Big(\|f\|_{X(0,1)} + \int_0^\infty f^*(t)\ \mathrm{d}\,t\Big).$$

Together with (6.13) and (6.14) this yields the embedding $\mathscr{L} \hookrightarrow (L_{\infty,q;\mathbb{A},\mathbb{B}})'$, which completes the proof of (v) for $1 < q < \infty$.

Finally, let $q = \infty$. Then we have

$$\|f\|_{p,\infty;\mathbb{A},\mathbb{B}} = \sup_{0<t<\infty} t^{\frac{1}{p}}\ell^{\mathbb{A}}(t)\ell\ell^{\mathbb{B}}(t)f^*(t).$$

Assume that $1 < p \leqslant \infty$. Then, by Theorem 3.8 (i), also

$$\|f\|_{p,\infty;\mathbb{A},\mathbb{B}} \approx \sup_{0<t<\infty} t^{\frac{1}{p}}\ell^{\mathbb{A}}(t)\ell\ell^{\mathbb{B}}(t)f^{**}(t).$$

Moreover, our assumptions on $p, \mathbb{A}, \mathbb{B}$ imply

$$\|f\|_{p,\infty;\mathbb{A},\mathbb{B}} \approx \sup_{0<t<\infty} \varphi(t)f^{**}(t),$$

where

$$\varphi(t) \approx \begin{cases} t^{\frac{1}{p}}\ell^{\mathbb{A}}(t)\ell\ell^{\mathbb{B}}(t), & \text{if } 1<p<\infty, \\ & \quad \text{or } p=\infty,\ \alpha_\infty>0, \\ & \quad \text{or } p=\infty,\ \alpha_\infty=0,\ \beta_\infty \geqslant 0; \\ t^{\frac{1}{p}}\ell^{\alpha_0}(t)\ell\ell^{\beta_0}(t)\chi_{(0,1]}(t)+\chi_{(1,\infty)}(t) & \text{if } p=\infty,\ \alpha_\infty<0, \\ & \quad \text{or } p=\infty,\ \alpha_\infty=0,\ \beta_\infty \leqslant 0. \end{cases}$$

By (3.2), $\varphi \in \mathcal{F}$ in all cases. Hence (cf. (2.4)), $L_{p,\infty;\mathbb{A},\mathbb{B}} = M_\varphi$. Applying (2.6), calculating $\frac{\mathrm{d}\tilde{\varphi}}{\mathrm{d}t}$, where $\tilde{\varphi}(t) = \frac{t}{\varphi(t)}$, and using (2.3), we get (i) for $q = \infty$ and $p > 1$, (iv), and (v) for $q = \infty$.

To finish the proof, we have only to verify (i) for $p = 1$ and $q = \infty$. The inequality

$$\int_0^\infty f^*(t)g^*(t)\ \mathrm{d}t \leqslant \|g\|_{1,\infty;\mathbb{A},\mathbb{B}}\,\|f\|_{\infty,1;-\mathbb{A},-\mathbb{B}}$$

yields $L_{\infty,1;-\mathbb{A},-\mathbb{B}} \hookrightarrow (L_{(1,\infty;\mathbb{A},\mathbb{B})})'$. For the converse, set

$$\varrho(t) = t^{-1}\ell^{-\mathbb{A}}(t)\ell\ell^{-\mathbb{B}}(t).$$

Then $\|\varrho\|_{1,\infty;\mathbb{A},\mathbb{B}} \approx 1$, whence, by the Hölder inequality,

$$\|f\|_{\infty,1;-\mathbb{A},-\mathbb{B}} = \int_0^\infty \varrho(t)f^*(t)\ \mathrm{d}t \leqslant \|\varrho\|_{1,\infty;\mathbb{A},\mathbb{B}}\,\|f\|_{(L_{1,\infty;\mathbb{A},\mathbb{B}})'}.$$

The proof is complete. (Note that the second equality in (6.3) follows from Theorem 3.8 (i).) □

It remains to describe associate spaces of $X = L_{p,q;\mathbb{A},\mathbb{B}}$ provided that $X \neq \{0\}$, $0 < q \leqslant 1$ and either $1 < p \leqslant \infty$, or $p = 1$, $\alpha_0 > 0$, or $p = 1$, $\alpha_0 = 0$, and $\beta_0 \geqslant 0$. We shall use the following lemma, of independent interest.

6.4. LEMMA. *Let $X$ be a rearrangement-invariant quasi-Banach space. Let $\varphi_X \in \mathcal{F}$. Assume that $X \hookrightarrow \Lambda_{\varphi_X}$. Then $X' = M_{\widetilde{\varphi}_X}$, where $\widetilde{\varphi}_X(t) = t/\varphi_X(t)$.*

*Proof.* The inclusion $M_{\widetilde{\varphi}_X} \hookrightarrow X'$ is evident (cf. (2.6)). For the converse, by (2.4) and (2.1),

$$\|f\|_{M_{\widetilde{\varphi}_X}} = \sup_{0<t<\mu(\mathscr{R})} \frac{\widetilde{\varphi}_X(t)}{t}\int_0^t f^*(s)\,\mathrm{d}s \leqslant \sup_{0<t<\mu(\mathscr{R})} \frac{\widetilde{\varphi}_X(t)}{t}\,\varphi_X(t)\,\|f\|_{X'} = \|f\|_{X'}.$$ □

To check the condition $\varphi_X \in \mathcal{F}$ of Lemma 6.4, the next lemma will be useful.

6.5. LEMMA. *Let $0 < p \leqslant \infty$ and $\mathbb{L} = (\lambda_0, \lambda_\infty)$, $\mathbb{E} = (\varepsilon_0, \varepsilon_\infty) \in \mathbb{R}^2$. Then the function $\varphi$, given by $\varphi(0) = 0$ and $\varphi(t) = t^{1/p}\ell^{\mathbb{L}}(t)\ell\ell^{\mathbb{E}}(t)$ for $t \in (0,\infty)$, is equivalent on $(0,\infty)$ to a non-decreasing concave function if and only if one of the* *following conditions holds:*

$$(6.21)\qquad \begin{cases} 1<p<\infty; \\ p=1,\ \lambda_0>0,\ \lambda_\infty<0; \\ p=1,\ \lambda_0>0,\ \lambda_\infty=0,\ \varepsilon_\infty\leqslant 0; \\ p=1,\ \lambda_0=0,\ \varepsilon_0\geqslant 0,\ \lambda_\infty<0; \\ p=1,\ \lambda_0=0,\ \varepsilon_0\geqslant 0,\ \lambda_\infty=0,\ \varepsilon_\infty\leqslant 0; \\ p=\infty,\ \lambda_0<0,\ \lambda_\infty>0; \\ p=\infty,\ \lambda_0<0,\ \lambda_\infty=0,\ \varepsilon_\infty\geqslant 0; \\ p=\infty,\ \lambda_0=0,\ \varepsilon_0\leqslant 0,\ \lambda_\infty>0; \\ p=\infty,\ \lambda_0=0,\ \varepsilon_0\leqslant 0,\ \lambda_\infty=0,\ \varepsilon_\infty\geqslant 0. \end{cases}$$

*Proof* follows by a simple calculation. □

6.6. THEOREM. *Let* $0<q\leqslant 1$, $\mathbb{A}=(\alpha_0,\alpha_\infty)$, $\mathbb{B}=(\beta_0,\beta_\infty)\in\mathbb{R}^2$, *and assume that the space* $X=L_{p,q;\mathbb{A},\mathbb{B}}$ *is not trivial* (*cf.* (3.2)). *Let one of the following conditions hold:*

$$1<p\leqslant\infty;$$

$$(6.22)\qquad p=1,\quad \alpha_0>0;$$

$$(6.23)\qquad p=1,\quad \alpha_0=0,\quad \beta_0>0;$$

$$(6.24)\qquad p=1,\quad q=1,\quad \alpha_0=0,\quad \beta_0=0.$$

*Then* $X'=\mathscr{L}$, *a space described below:*

(i) *Let* $1\leqslant p<\infty$. *Then*

$$\mathscr{L}=L_{(p',\infty;-\mathbb{A},-\mathbb{B})}=L_{p',\infty;-\mathbb{A},-\mathbb{B}}.$$

(ii) *Let* $p=\infty$, $\alpha_0+\frac{1}{q}<0$, *and either* $\alpha_\infty+\frac{1}{q}>0$ *or* $\alpha_\infty+\frac{1}{q}=0$, $\beta_\infty+\frac{1}{q}\geqslant 0$. *Then*

$$\mathscr{L}=\begin{cases} L_{(1,\infty;-\mathbb{A}-\frac{1}{q},-\mathbb{B})} & \text{if}\quad \alpha_\infty+\frac{1}{q}>0; \\ L_{(1,\infty;(-\alpha_0-\frac{1}{q},0),(-\beta_0,-\beta_\infty-\frac{1}{q}))} & \text{if}\quad \alpha_\infty+\frac{1}{q}=0,\ \beta_\infty+\frac{1}{q}>0; \\ L_{(1,\infty;(-\alpha_0-\frac{1}{q},0),(-\beta_0,0),(0,-\frac{1}{q}))} & \text{if}\quad \alpha_\infty+\frac{1}{q}=0,\ \beta_\infty+\frac{1}{q}=0. \end{cases}$$

(iii) *Let* $p=\infty$, $\alpha_0+\frac{1}{q}=0$, $\beta_0+\frac{1}{q}<0$, *and either* $\alpha_\infty+\frac{1}{q}>0$ *or* $\alpha_\infty+\frac{1}{q}=0$, $\beta_\infty+\frac{1}{q}\geqslant 0$. *Then*

$$\mathscr{L}=\begin{cases} L_{(1,\infty;(0,-\alpha_\infty-\frac{1}{q}),(-\beta_0-\frac{1}{q},-\beta_\infty))} & \text{if}\quad \alpha_\infty+\frac{1}{q}>0; \\ L_{(1,\infty;(0,0),-\mathbb{B}-\frac{1}{q})} & \text{if}\quad \alpha_\infty+\frac{1}{q}=0,\ \beta_\infty+\frac{1}{q}>0; \\ L_{(1,\infty;(0,0),(-\beta_0-\frac{1}{q},0),(0,-\frac{1}{q}))} & \text{if}\quad \alpha_\infty+\frac{1}{q}=0,\ \beta_\infty+\frac{1}{q}=0. \end{cases}$$

(iv) *Let* $p=\infty$, *and either* $\alpha_\infty+\frac{1}{q}<0$ *or* $\alpha_\infty+\frac{1}{q}=0$ *and* $\beta_\infty+\frac{1}{q}<0$. *Then*

$$\mathscr{L}=\Big\{f\in\mathscr{M}(\mathscr{R},\mu);\ \|f\|_{\mathscr{L}}:=\|f\|_{Y(0,1)}+\int_0^\infty f^*(t)\,\mathrm{d}\,t<\infty\Big\},$$

*where*

$$Y(0,1)=\begin{cases} L_{(1,\infty;-\alpha_0-\frac{1}{q},-\beta_0)}(0,1) & \text{if}\quad \alpha_0+\frac{1}{q}<0\\ L_{(1,\infty;0,-\beta_0-\frac{1}{q})}(0,1) & \text{if}\quad \alpha_0+\frac{1}{q}=0,\ \beta_0+\frac{1}{q}<0.\end{cases}$$

*Proof.* I. Assume first that $1<p<\infty$. By Lemma 3.7 (i),

$$\varphi_X(t)\approx t^{\frac{1}{p}}\ell^{\mathbb{A}}(t)\ell\ell^{\mathbb{B}}(t),\qquad t\in(0,\infty). \tag{6.25}$$

Together with Lemma 6.5 this implies that $\varphi_X\in\mathcal{F}$. Moreover, by (6.25), $\Lambda_{\varphi_X}=L_{p,1;\mathbb{A},\mathbb{B}}$, and

$$\widetilde{\varphi}_X(t)\approx t^{\frac{1}{p'}}\ell^{-\mathbb{A}}(t)\ell\ell^{-\mathbb{B}}(t),\qquad t\in(0,\infty).$$

Consequently,

$$M_{\widetilde{\varphi}_X}=L_{(p',\infty;-\mathbb{A},-\mathbb{B})}=L_{p',\infty;-\mathbb{A},-\mathbb{B}}.$$

Thus, by Lemma 6.4,

$$X'=M_{\widetilde{\varphi}_X}=L_{p',\infty;-\mathbb{A},-\mathbb{B}},$$

provided that

$$X=L_{p,q;\mathbb{A},\mathbb{B}}\hookrightarrow L_{p,1;\mathbb{A},\mathbb{B}}=\Lambda_{\varphi_X}.$$

This however follows from Theorem 4.1.

II. In the case when $p=\infty$ the proof is quite analogous to the one above (instead of Theorem 4.1 we use Theorem 4.2]).

III. Assume that either (6.22) or (6.23) or (6.24) holds. Then (cf. (6.21)), $\varphi_X\in\mathcal{F}$ if and only if

$$\text{either}\quad \alpha_\infty<0\quad\text{or}\quad\alpha_\infty=0\quad\text{and}\quad\beta_\infty\leqslant 0. \tag{6.26}$$

In each of these cases we can again apply the argument from part I to get the result.

Assume now that (6.26) is not satisfied, that is, either $\alpha_\infty>0$ or $\alpha_\infty=0$ and $\beta_\infty>0$. We put $Z=L_{1,q;(\alpha_0,0),(\beta_0,0)}$. Then clearly $X\hookrightarrow Z$, whence

$$Z'\hookrightarrow X'. \tag{6.27}$$

Now the parameters of the space $Z'$ fit in the situation described by (6.26), and we thus obtain $Z'=L_{\infty,\infty;(-\alpha_0,0),(-\beta_0,0)}$. Moreover, by Corollary 3.10, $Z'=L_{\infty,\infty;-\mathbb{A},-\mathbb{B}}$. Consequently, $Z'=\mathscr{L}$. Together with (6.27), this yields $\mathscr{L}\hookrightarrow X'$.

Using (2.1) and Lemma 3.7 (i), we get

$$\begin{aligned}\|f\|_{\mathscr{L}}&=\|\ell^{-\mathbb{A}}(t)\ell\ell^{-\mathbb{B}}(t)t^{-1}\int_0^t f^*(s)\,\mathrm{d}\,s\|_{\infty,(0,\infty)}\\ &\leqslant\|f\|_{X'}\|t^{-1}\ell^{-\mathbb{A}}(t)\ell\ell^{-\mathbb{B}}(t)\varphi_X(t)\|_{\infty,(0,\infty)}=\|f\|_{X'},\end{aligned}$$

which proves the converse embedding $X' \hookrightarrow \mathscr{L}$. The proof is complete. $\square$

Next, we are going to describe associated spaces of $L_{(p,q;\mathbb{A},\mathbb{B})}$ with $0 < p, q \leqslant \infty$ and $\mathbb{A}, \mathbb{B} \in \mathbb{R}^2$. However, according to (5.3) and our previous results on associated spaces to $L_{p,q;\mathbb{A},\mathbb{B}}$, it is enough to consider the case when $0 < p \leqslant 1$, $0 < q \leqslant \infty$ and $\mathbb{A}, \mathbb{B} \in \mathbb{R}^2$. If $p = 1$, $1 \leqslant q \leqslant \infty$ and $\mathbb{A}, \mathbb{B} \in \mathbb{R}^2$, the desired description of $(L_{(p,q;\mathbb{A},\mathbb{B})})'$ is given in the following theorem.

6.7. THEOREM. *Let* $1 < q \leqslant \infty$, $\mathbb{A} = (\alpha_0, \alpha_\infty)$, $\mathbb{B} = (\beta_0, \beta_\infty) \in \mathbb{R}^2$, *and assume that the space* $X = L_{(1,q;\mathbb{A},\mathbb{B})}$ *is not trivial* (cf. (3.3)). *Then* $X' = \mathscr{L}$, *a space described below:*

(i) *Let* $1 < q \leqslant \infty$, *let either* $\alpha_0 + \frac{1}{q} > 0$ *or* $\alpha_0 + \frac{1}{q} = 0$ *and* $\beta_0 + \frac{1}{q} \geqslant 0$, *and let either* $\alpha_\infty + \frac{1}{q} < 0$ *or* $\alpha_\infty + \frac{1}{q} = 0$ *and* $\beta_\infty + \frac{1}{q} < 0$. *Then* $\mathscr{L}$ *reads as*

$$L_{\infty,q';-\mathbb{A}-1,-\mathbb{B}} \quad \text{if} \quad \alpha_0 + \frac{1}{q} > 0,\ \alpha_\infty + \frac{1}{q} < 0;$$

$$L_{\infty,q';(-\alpha_0-1,-\frac{1}{q'}),(-\beta_0,-\beta_\infty-1)} \quad \text{if} \quad \alpha_0 + \frac{1}{q} > 0,\ \alpha_\infty + \frac{1}{q} = 0,\ \beta_\infty + \frac{1}{q} < 0;$$

$$L_{\infty,q';(-\frac{1}{q'},-\alpha_\infty-1),(-\beta_0-1,-\beta_\infty)} \quad \text{if} \quad \alpha_0 + \frac{1}{q} = 0,\ \beta_0 + \frac{1}{q} > 0,\ \alpha_\infty + \frac{1}{q} < 0;$$

$$L_{\infty,q';(-\frac{1}{q'},-\frac{1}{q'}),-\mathbb{B}-1} \quad \text{if} \quad \alpha_0 + \frac{1}{q} = 0,\ \beta_0 + \frac{1}{q} > 0,\ \alpha_\infty + \frac{1}{q} = 0,\ \beta_\infty + \frac{1}{q} < 0;$$

$$L_{\infty,q';(-\frac{1}{q'},-\alpha_\infty-1),(-\frac{1}{q'},-\beta_\infty),(-1,0)} \quad \text{if} \quad \alpha_0 + \frac{1}{q} = 0,\ \beta_0 + \frac{1}{q} = 0,\ \alpha_\infty + \frac{1}{q} < 0;$$

$$L_{\infty,q';(-\frac{1}{q'},-\frac{1}{q'}),(-\frac{1}{q'},-\beta_\infty-1),(-1,0)} \quad \text{if} \quad \alpha_0 + \frac{1}{q} = 0,\ \beta_0 + \frac{1}{q} = 0,\ \alpha_\infty + \frac{1}{q} = 0,\ \beta_\infty + \frac{1}{q} < 0.$$

(ii) *Let* $1 < q \leqslant \infty$, *let either* $\alpha_0 + \frac{1}{q} < 0$ *or* $\alpha_0 + \frac{1}{q} = 0$ *and* $\beta_0 + \frac{1}{q} < 0$, *and let either* $\alpha_\infty + \frac{1}{q} < 0$ *or* $\alpha_\infty + \frac{1}{q} = 0$ *and* $\beta_\infty + \frac{1}{q} < 0$. *Then*

$$\mathscr{L} = \{f \in \mathscr{M}(\mathscr{R}, \mu);\ \|f\|_{\mathscr{L}} = \|f\|_\infty + N(f) < \infty\}, \tag{6.28}$$

*where*

$$N(f) = \begin{cases} \|t^{-\frac{1}{q'}} \ell^{-\alpha_\infty - 1}(t) \ell\ell^{-\beta_\infty}(t) f^*(t)\|_{q',(1,\infty)} & \text{if } \alpha_\infty + \frac{1}{q} < 0 \\ \|t^{-\frac{1}{q'}} \ell^{-\frac{1}{q'}}(t) \ell\ell^{-\beta_\infty - 1}(t) f^*(t)\|_{q',(1,\infty)} & \text{if } \alpha_\infty + \frac{1}{q} = 0,\ \beta_\infty + \frac{1}{q} < 0. \end{cases} \tag{6.29}$$

(iii) *Let* $q = \infty$, $\alpha_\infty = 0$, *and* $\beta_\infty = 0$. *Then*

$$\mathscr{L} = \{f \in \mathscr{M}(\mathscr{R}, \mu);\quad \|f\|_{\mathscr{L}} < \infty\},$$

*where*

$$\|f\|_{\mathscr{L}} = \begin{cases} \int_0^1 t^{-1}\ell^{-\alpha_0-1}(t)\ell\ell^{-\beta_0}(t)f^*(t)dt & \text{if} \quad \alpha_0 > 0 \\ \int_0^1 t^{-1}\ell^{-1}(t)\ell\ell^{-\beta_0-1}(t)f^*(t)dt & \text{if} \quad \alpha_0 = 0,\ \beta_0 > 0 \\ \|f\|_\infty & \text{if either} \quad \alpha_0 < 0, \\ & \quad \text{or} \quad \alpha_0 = 0,\ \beta_0 \leqslant 0. \end{cases} \tag{6.30}$$

6.8. REMARK. We see from (6.30) that the space $\mathscr{L}$ of Theorem 6.7 (iii) is given by

$$\mathscr{L} = \begin{cases} L_{\infty,1;\alpha_0-1,-\beta_0}(0,1) & \text{if} \quad \alpha_0 > 0 \\ L_{\infty,1;-1,-\beta_0-1}(0,1) & \text{if} \quad \alpha_0 = 0,\ \beta_0 > 0. \end{cases}$$

*Proof of Theorem 6.7.* (i) By Theorem 3.8 (i), any space in part (i) coincides with its analogue in the norm of which $f^*$ is replaced by $f^{**}$. Thus, by Remark 3.6, any such space is a BFS, i.e., the space $\mathscr{L}$ is a BFS. Consequently, $\mathscr{L} = \mathscr{L}''$. Further, by Theorem 6.2 (ii), (iii), and (v), and by Theorem 6.6 (ii)–(iv) (supplemented by their analogues for spaces with three tiers of logarithms), $\mathscr{L}' = L_{(1,q;\mathbb{A},\mathbb{B})}$. Hence, $\mathscr{L} = \mathscr{L}'' = (L_{(1,q;\mathbb{A},\mathbb{B})})' = X'$.

(ii) In this case we have

$$\|t^{-\frac{1}{q}}\ell^{\alpha_0}(t)\ell\ell^{\beta_0}(t)\|_{q,(\frac{1}{2},1)} \approx 1 \approx \|t^{-\frac{1}{q}}\ell^{\alpha_0}(t)\ell\ell^{\beta_0}(t)\|_{q,(0,1)}.$$

Consequently, we get for every $g \in X$,

$$\begin{aligned} \int_0^1 g^*(t)\,\mathrm{d}\,t &\lesssim \int_0^{\frac{1}{2}} g^*(t)\,\mathrm{d}\,t \approx \|t^{-\frac{1}{q}}\ell^{\alpha_0}(t)\ell\ell^{\beta_0}(t)\|_{q,(\frac{1}{2},1)} \int_0^{\frac{1}{2}} g^*(t)\,\mathrm{d}\,t \\ &\lesssim \|t^{-\frac{1}{q}}\ell^{\alpha_0}(t)\ell\ell^{\beta_0}(t) \int_0^t g^*(s)\,\mathrm{d}\,s\|_{q,(\frac{1}{2},1)} \leqslant \|t^{\frac{1}{q'}}\ell^{\alpha_0}(t)\ell\ell^{\beta_0}(t)g^{**}(t)\|_{q,(0,1)} \\ &\leqslant \|t^{-\frac{1}{q}}\ell^{\alpha_0}(t)\ell\ell^{\beta_0}(t)\|_{q,(0,1)} \int_0^1 g^*(s)\,\mathrm{d}\,s \approx \int_0^1 g^*(t)\,\mathrm{d}\,t, \end{aligned}$$

which implies

$$\|g\|_X \approx \int_0^1 g^*(t)\,\mathrm{d}\,t + \|t^{\frac{1}{q'}}\ell^{\alpha_\infty}(t)\ell\ell^{\beta_\infty}(t)g^{**}(t)\|_{q,(1,\infty)}. \tag{6.31}$$

Now we shall prove the embedding

$$X' \hookrightarrow \mathscr{L}. \tag{6.32}$$

For this purpose it is enough to verify for all step functions $f \in \mathscr{M}(\mathscr{R},\mu)$ the inequality

$$\|f\|_{\mathscr{L}} \lesssim \|f\|_{X'}. \tag{6.33}$$

For a step function $f \in \mathscr{M}(\mathscr{R},\mu)$, we define a function $\varrho$ by

$$\varrho(t) = [f^*(1)]^{q'-1}, \qquad 0 < t \leqslant 1, \tag{6.34}$$

and, for $1 < t \leqslant \infty$, by

$$(6.35) \qquad \varrho(t) = \begin{cases} [f^*(t)]^{q'-1} t^{-1} \ell^{(-\alpha_\infty - 1)q'}(t) \ell\ell^{-\beta_\infty q'}(t) & \text{if} \quad \alpha_\infty + \frac{1}{q} < 0 \\ [f^*(t)]^{q'-1} t^{-1} \ell^{-1}(t) \ell\ell^{(-\beta_\infty - 1)q'}(t) & \text{if} \quad \alpha_\infty + \frac{1}{q} = 0, \\ & \qquad \beta_\infty + \frac{1}{q} < 0. \end{cases}$$

Then $\varrho$ is equivalent to a non-increasing function on $(0,\infty)$ and, by [BS, Chapter 2, Corollary 7.8], there is a $\overline{\varrho} \in \mathscr{M}(\mathscr{R},\mu)$ such that $\overline{\varrho}^* \approx \varrho$. Moreover, we have from (6.28) and (6.29) that

$$(6.36) \qquad \|f\|_{\mathscr{L}}^{q'} \approx \|f\|_{\infty}^{q'} + \int_1^\infty \varrho(t) f^*(t)\,\mathrm{d}\,t.$$

Our assumptions on $\mathbb{A}$ and $\mathbb{B}$ and Lemma 3.7 (ii) imply

$$\varphi_X(t) \approx t \qquad \text{for all} \quad t \in (0,1).$$

Together with (2.1), this yields

$$(6.37) \qquad \|f\|_\infty = \lim_{t\to 0_+} \frac{1}{t} \int_0^t f^*(s)\,\mathrm{d}\,s \leqslant \|f\|_{X'} \lim_{t\to 0_+} \frac{1}{t} \varphi_X(t) \approx \|f\|_{X'}.$$

Moreover, by Hölder's inequality,

$$(6.38) \qquad \int_1^\infty \varrho(t) f^*(t)\,\mathrm{d}\,t \leqslant \|\overline{\varrho}\|_X \|f\|_{X'}.$$

If we prove that

$$(6.39) \qquad \|\overline{\varrho}\|_X \lesssim \|f\|_{\mathscr{L}}^{q'-1},$$

we would obtain from (6.36), (6.28), (6.38), (6.39), and (6.37) that

$$\|f\|_{\mathscr{L}}^{q'} \lesssim \|f\|_{\mathscr{L}}^{q'-1} \|f\|_\infty + \|\overline{\varrho}\|_X \|f\|_{X'} \lesssim \|f\|_{\mathscr{L}}^{q'-1} \|f\|_{X'},$$

and (6.33) would follow.

Since (cf. (6.34))

$$\int_0^1 \varrho(s)\,\mathrm{d}\,s = [f^*(1)]^{q'-1} \leqslant \|f\|_\infty^{q'-1} \leqslant \|f\|_{\mathscr{L}}^{q'-1},$$

and

$$\|t^{-\frac{1}{q}} \ell^{\alpha_\infty}(t) \ell\ell^{\beta_\infty}(t) \int_0^1 \varrho(s)\,\mathrm{d}\,s\|_{q,(1,\infty)} \approx \int_0^1 \varrho(s)\,\mathrm{d}\,s \leqslant \|f\|_{\mathscr{L}}^{q'-1},$$

we have from(6.31) that

$$(6.40) \qquad \begin{aligned} \|\overline{\varrho}\|_X &\approx \int_0^1 \varrho(s)\,\mathrm{d}\,s + \|t^{\frac{1}{q'}} \ell^{\alpha_\infty}(t) \ell\ell^{\beta_\infty}(t) \varrho^{**}(t)\|_{q,(1,\infty)} \\ &\lesssim \|f\|_{\mathscr{L}}^{q'-1} + \|t^{-\frac{1}{q}} \ell^{\alpha_\infty}(t) \ell\ell^{\beta_\infty}(t) \int_1^t \varrho(s)\,\mathrm{d}\,s\|_{q,(1,\infty)}. \end{aligned}$$

Thus, (6.39) will follow from (6.40) and (6.28) once we show that

$$\|t^{-\frac{1}{q}}\ell^{\alpha_\infty}(t)\ell\ell^{\beta_\infty}(t)\int_1^t \varrho(s)\,\mathrm{d}\,s\|_{q,(1,\infty)} \lesssim N(f)^{q'-1}.$$

Using (6.29) and (6.35), this inequality can be rewritten as

$$\|t^{-\frac{1}{q}}\ell^{\alpha_\infty}(t)\ell\ell^{\beta_\infty}(t)\int_1^t \varrho(s)\,\mathrm{d}\,s\|_{q,(1,\infty)} \lesssim \|t^{\frac{1}{q'}}\ell^{\alpha_\infty+1}(t)\ell\ell^{\beta_\infty}(t)\varrho(t)\|_{q,(1,\infty)} \tag{6.41}$$

if $\alpha_\infty + \frac{1}{q} < 0$, and

$$\|t^{-\frac{1}{q}}\ell^{\alpha_\infty}(t)\ell\ell^{\beta_\infty}(t)\int_1^t \varrho(s)\,\mathrm{d}\,s\|_{q,(1,\infty)} \lesssim \|t^{\frac{1}{q'}}\ell^{\frac{1}{q'}}(t)\ell\ell^{\beta_\infty+1}(t)\varrho(t)\|_{q,(1,\infty)} \tag{6.42}$$

if $\alpha_\infty + \frac{1}{q} = 0$ and $\beta_\infty + \frac{1}{q} < 0$. To verify (6.41) and (6.42) is a standard matter using the well-known criterion for the Hardy inequality (cf. [OK]). This proves (6.39), and in turn (6.33).

We shall now prove the converse inclusion to (6.32), i.e. $\mathscr{L} \hookrightarrow X'$. Thus, we need to verify that

$$\|f\|_{X'} \lesssim \|f\|_{\mathscr{L}} \quad \text{for all} \quad f \in \mathscr{L}. \tag{6.43}$$

Since for every $f \in \mathscr{M}(\mathscr{R},\mu)$,

$$\|f\|_{X'} = \sup_{\|g\|_X \leqslant 1} \int_{\mathscr{R}} f(x)g(x)\,\mathrm{d}\,\mu$$

and (cf. [BS, Chapter 2, Theorem 2.2])

$$\int_{\mathscr{R}} f(x)g(x)\,\mathrm{d}\,\mu \leqslant \int_0^\infty f^*(t)g^*(t)\,\mathrm{d}\,t,$$

we see that in order to prove (6.43) it suffices to show that

$$\int_0^\infty f^*(t)g^*(t)\,\mathrm{d}\,t \lesssim \|f\|_{\mathscr{L}}\;\|g\|_X \tag{6.44}$$

for every $f \in \mathscr{L}$ and $g \in X$. First, we have (cf. (6.28) and (6.31))

$$\int_0^1 f^*(t)g^*(t)\,\mathrm{d}\,t \leqslant \|f\|_\infty \int_0^1 g^*(t)\,\mathrm{d}\,t \lesssim \|f\|_{\mathscr{L}}\;\|g\|_X.$$

Therefore, it remains to prove that

$$\int_1^\infty f^*(t)g^*(t)\,\mathrm{d}\,t \lesssim \|f\|_{\mathscr{L}}\;\|g\|_X. \tag{6.45}$$

Assume that (6.45) is not satisfied. Then there are two sequences of functions $\{f_n\}, \{g_n\}$ such that $\|f_n\|_{\mathscr{L}} \leqslant 1$, $\|g_n\|_X \leqslant 1$, $n \in \mathbb{N}$, and

$$\int_1^\infty f_n^*(t)g_n^*(t)\,\mathrm{d}\,t \to \infty \quad \text{as} \quad n \to \infty. \tag{6.46}$$

Putting

$$h_n(x) = \min\{|g_n(x)|, g_n^*(1)\} \operatorname{sgn} g_n(x), \quad x \in \mathscr{R},\ n \in \mathbb{N}, \tag{6.47}$$

we have $|h_n| \leqslant |g_n|$ and since $X$ is a BFS (cf. Remark 3.6),

$$\|h_n\|_X \leqslant \|g_n\|_X \leqslant 1, \quad n \in \mathbb{N}.$$

Moreover, by (6.31),

$$g_n^*(1) \leqslant \int_0^1 g_n^*(t)\,\mathrm{d}\,t \lesssim \|g_n\|_X \leqslant 1, \quad n \in \mathbb{N}. \tag{6.48}$$

We now claim that $\{h_n\}$ is uniformly bounded in the space $Y$, where

$$Y = L_{(1,q;(1,\alpha_\infty),\mathbb{B})}. \tag{6.49}$$

To prove (6.49), note that (6.47) implies

$$h_n^*(s) = \min[g_n^*(s), g_n^*(1)], \quad s \geqslant 0.$$

Hence

$$h_n^*(s) = g_n^*(1),\ s \in (0,1] \quad \text{and} \quad h_n^*(s) = g_n^*(s),\ s \in (1,\infty).$$

This yields $h_n^{**}(s) = g_n^*(1)$ for $s \in (0,1]$ and $h_n^{**} \leqslant g_n^{**}$. Thus, using (6.48),

$$\begin{aligned} \|h_n\|_Y &\lesssim \|t^{1-\frac{1}{q}}\ell(t)\ell\ell^{\beta_0}(t)g_n^*(1)\|_{q,(0,1)} + \|t^{1-\frac{1}{q}}\ell^{\alpha_\infty}(t)\ell\ell^{\beta_\infty}(t)g_n^{**}\|_{q,(1,\infty)} \\ &\lesssim g_n^*(1)\|t^{1-\frac{1}{q}}\ell(t)\ell\ell^{\beta_0}(t)\|_{q,(0,1)} + \|g_n\|_X \lesssim 1, \end{aligned} \tag{6.50}$$

which proves the uniform boundedness of $\{h_n\}$ in $Y$.

Now, by part (i), we can determine the space $Y'$, namely

$$Y' = \begin{cases} L_{\infty,q';(-2,-\alpha_\infty-1),-\mathbb{B}} & \text{if } \alpha_\infty + \frac{1}{q} < 0 \\ L_{\infty,q';(-2,-\frac{1}{q'}),(-\beta_0,-\beta_\infty-1)} & \text{if } \alpha_\infty + \frac{1}{q} = 0,\ \beta_\infty + \frac{1}{q} < 0. \end{cases}$$

Hence, applying (6.28), (6.29), and the estimate

$$f_n^*(t) \leqslant \|f_n\|_\infty \leqslant \|f_n\|_{\mathscr{L}} \leqslant 1, \quad n \in \mathbb{N},$$

we obtain

$$\|f_n\|_{Y'} \lesssim \|t^{-\frac{1}{q'}}\ell^{-2}(t)\ell\ell^{-\beta_0}(t)f_n^*(t)\|_{q',(0,1)} + N(f_n) \lesssim \|f_n\|_{\mathscr{L}} \leqslant 1, \tag{6.51}$$

which means that $f_n$ are uniformly bounded in $Y'$. Now, (6.50) and (6.51) contradict (6.46) since $h_n^*(s) = g_n^*(s)$ for $s \in (1,\infty)$.

(iii) In the case when $q = \infty$, $\alpha_\infty = 0$, and $\beta_\infty = 0$, we have for all $g \in X$,

$$N_1 := \|t\ell^{\alpha_0}(t)\ell\ell^{\beta_0}(t)g^{**}(t)\|_{q,(0,1)} = \|\ell^{\alpha_0}(t)\ell\ell^{\beta_0}(t)\int_0^t g^*(s)\,\mathrm{d}\,s\|_{\infty,(0,1)},$$

$$N_2 := \|t\ell^{\alpha_\infty}(t)\ell\ell^{\beta_\infty}(t)g^{**}(t)\|_{q,(1,\infty)} = \|\int_0^t g^*(s)\,\mathrm{d}\,s\|_{\infty,(1,\infty)} = \int_0^\infty g^*(s)\,\mathrm{d}\,s,$$

and therefore

$$\|g\|_{(1,q;\mathbb{A},\mathbb{B})} = \max\{N_1, N_2\} \tag{6.52}$$
$$= \max\Big\{\|\ell^{\alpha_0}(t)\ell\ell^{\beta_0}(t)\int_0^t g^*(s)\,\mathrm{d}\,s\|_{\infty,(0,1)}, \int_0^\infty g^*(s)\,\mathrm{d}\,s\Big\}.$$

If either $\alpha_0 < 0$ or $\alpha_0 = 0$ and $\beta_0 \leqslant 0$, then $N_1 \approx \int_0^1 g^*(s)\,\mathrm{d}\,s$. This implies that $X = L^1$ and thus $X' = L^\infty = \mathscr{L}$.

Now let either $\alpha_0 > 0$ or $\alpha_0 = 0$ and $\beta_0 > 0$. We shall first show that (6.33) holds. Put (cf. (6.34) and (6.35)) for $t \in (0,\infty)$,

$$\varrho(t) = \begin{cases} t^{-1}\ell^{-\alpha_0-1}(t)\ell\ell^{-\beta_0}(t)\chi_{(0,1)}(t) & \text{if } \alpha_0 > 0 \\ t^{-1}\ell^{-1}(t)\ell\ell^{-\beta_0-1}(t)\chi_{(0,1)}(t) & \text{if } \alpha_0 = 0, \beta_0 > 0. \end{cases}$$

The function $\varrho$ is equivalent to a non-increasing function on $(0,\infty)$ and, by [BS, Chapter 2, Corollary 7.8], there is a $\overline{\varrho} \in \mathscr{M}(\mathscr{R},\mu)$ such that $\overline{\varrho}^* \approx \varrho$. This and our assumptions on $q, \mathbb{A}$, and $\mathbb{B}$ yield $\|\overline{\varrho}\|_X \approx 1$. Moreover, (6.30) and Hölder's inequality imply that for all step functions $f \in \mathscr{M}(\mathscr{R},\mu)$,

$$\|f\|_{\mathscr{L}} = \int_0^\infty \varrho(t) f^*(t)\,\mathrm{d}\,t \leqslant \|\overline{\varrho}\|_X\,\|f\|_{X'}$$

and (6.33) follows.

We have to prove the converse, that is, (6.43). To this end it suffices to verify (6.44). Since $f^*(1) \lesssim \|f\|_{\mathscr{L}}$, we have from (6.52) that for all $f \in \mathscr{L}$ and $g \in X$,

$$\int_1^\infty f^*(t)g^*(t)\,\mathrm{d}\,t \lesssim f^*(1)\int_1^\infty g^*(t)\,\mathrm{d}\,t \lesssim \|f\|_{\mathscr{L}}\,\|g\|_X.$$

Hence, it remains to prove that for all $f \in \mathscr{L}$ and $g \in X$,

$$\int_0^1 f^*(t)g^*(t)\,\mathrm{d}\,t \lesssim \|f\|_{\mathscr{L}}\,\|g\|_X. \tag{6.53}$$

Assume that (6.53) is not true. Then there are two sequences of functions $\{f_n\}$, $\{g_n\}$ such that $\|f_n\|_{\mathscr{L}} \leqslant 1$, $\|g_n\|_X \leqslant 1$, $n \in \mathbb{N}$, and

$$\int_0^1 f_n^*(t)g_n^*(t)\,\mathrm{d}\,t \to \infty \quad \text{as} \quad n \to \infty. \tag{6.54}$$

Define the functions $h_n$, $n \in \mathbb{N}$, by

$$h_n(x) = [|g_n(x)| - g_n^*(1)]^+ \operatorname{sgn} g_n(x), \quad x \in \mathscr{R},$$

and the space $Y$ by

$$Y = L_{(1,\infty;(\alpha_0,-1),\mathbb{B})}.$$

Then, by part (i),

$$Y' = \begin{cases} L_{\infty,1;(-\alpha_0-1,-2),-\mathbb{B}} & \text{if } \alpha_0 > 0 \\ L_{\infty,1;(-1,-2),(-\beta_0-1,-\beta_\infty)} & \text{if } \alpha_0 = 0, \beta_0 > 0. \end{cases} \tag{6.55}$$

Since $h_n^*(s) = [g_n^*(s) - g_n^*(1)]^+$, $s \geqslant 0$, we have supp $h_n^* \subset [0,1]$ and $h_n^*(s) = g_n^*(s) - g_n^*(1)$ for $s \in (0,1)$. Hence,

$$h_n^{**}(t) = g_n^{**}(t) - g_n^*(1),\ t \in (0,1), \quad \text{and} \quad h_n^{**}(t) = t^{-1}[g_n^{**}(1) - g_n^*(1)],\ t \in (1,\infty).$$

This yields

$$\begin{aligned}\|h_n\|_Y &\lesssim \|t\ell^{\alpha_0}(t)\ell\ell^{\beta_0}(t)g_n^{**}(t)\|_{\infty,(0,1)} + \|t\ell^{-1}(t)\ell\ell^{\beta_\infty}(t)t^{-1}g_n^{**}(1)\|_{\infty,(1,\infty)} \\ &\leqslant \|g_n\|_X + \int_0^1 g_n^*(s)\,\mathrm{d}\,s\,\|\ell^{-1}(t)\ell\ell^{\beta_\infty}(t)\|_{\infty,(1,\infty)} \\ &= \|g_n\|_X + \int_0^1 g_n^*(s)\,\mathrm{d}\,s,\end{aligned}$$

and, on using (6.52) with $g_n$ instead of $g$,

$$\|h_n\|_Y \lesssim \|g_n\|_X \leqslant 1 \quad \text{for all} \quad n \in \mathbb{N}. \tag{6.56}$$

Defining the functions $\psi_n$, $n \in \mathbb{N}$, by

$$\psi_n(x) = [|f_n(x)| - f^*(1)]^+ \operatorname{sgn} f_n(x), \quad x \in \mathscr{R},$$

we have $\psi_n^*(s) = [f_n^*(s) - f_n^*(1)]^+$, $s \geqslant 0$. Consequently, supp $\psi_n^* \subset [0,1]$ and $\psi_n^*(s) = f_n^*(s) - f_n^*(1)$ for $s \in (0,1)$. Together with (6.55) and (6.30), this yields

$$\|\psi_n\|_{Y'} \leqslant \|f_n\|_{\mathscr{L}} \leqslant 1 \quad \text{for all} \quad n \in \mathbb{N}. \tag{6.57}$$

By (6.52),

$$\int_0^1 g_n^*(s)\,\mathrm{d}\,s \lesssim \|g_n\|_X \lesssim 1,$$

which shows that, for all $n \in \mathbb{N}$,

$$g_n^*(1) \lesssim 1 \qquad \text{and} \qquad \int_0^1 h_n^*(s)\,\mathrm{d}\,s \lesssim 1. \tag{6.58}$$

Since, for every $t \in (0,1)$,

$$1 \lesssim \min\left\{t^{-1}\ell^{-\alpha_0-1}(t)\ell\ell^{-\beta_0}(t), t^{-1}\ell^{-1}(t)\ell\ell^{-\beta_0-1}(t)\right\},$$

we have from (6.30) for all $n \in \mathbb{N}$,

$$\int_0^1 f_n^*(s)\,\mathrm{d}\,s \lesssim \|f_n\|_{\mathscr{L}} \lesssim 1,$$

which in turn implies that

$$f^*(1) \lesssim 1 \qquad \text{and} \qquad \int_0^1 \psi_n^*(t)\,\mathrm{d}\,t \lesssim 1. \tag{6.59}$$

Finally, (6.56), (6.57), (6.58) and (6.59) contradict (6.54) as

$$\int_0^1 f_n^*(t)g_n^*(t)\,\mathrm{d}\,t = \int_0^1 (\psi_n^*(t)+f_n^*(1))(h_n^*(t)+g_n^*(1))\,\mathrm{d}\,t$$

$$= \int_0^1 \psi_n^*(t)h_n^*(t)\,\mathrm{d}\,t + f_n^*(1)\Big(\int_0^1 h_n^*(t)\,\mathrm{d}\,t + g_n^*(1)\Big) + g_n^*(1)\int_0^1 \psi_n^*(t)\,\mathrm{d}\,t.$$

The proof is complete. □

The following theorem is a complement of Theorem 6.7.

6.9. THEOREM. *Let* $0<q\leqslant 1$, $\mathbb{A}=(\alpha_0,\alpha_\infty)$, $\mathbb{B}=(\beta_0,\beta_\infty)\in\mathbb{R}^2$, *and assume that the space* $L_{(1,q;\mathbb{A},\mathbb{B})}$ *is not trivial* (cf. (3.3)). *Then* $(L_{(1,q;\mathbb{A},\mathbb{B})})'=\mathscr{L}$, *a space described below:*

(i) *Let either* $\alpha_0+\frac{1}{q}>0$ *or* $\alpha_0+\frac{1}{q}=0$ *and* $\beta_0+\frac{1}{q}\geqslant 0$. *Then* $\mathscr{L}$ *reads as*

$$\begin{array}{lll}
L_{\infty,\infty;-\mathbb{A}-\frac{1}{q},-\mathbb{B}} & \textit{if} & \alpha_0+\frac{1}{q}>0,\ \alpha_\infty+\frac{1}{q}<0;\\
L_{\infty,\infty;(-\alpha_0-\frac{1}{q},0),(-\beta_0,-\beta_\infty-\frac{1}{q})} & \textit{if} & \alpha_0+\frac{1}{q}>0,\ \alpha_\infty+\frac{1}{q}=0,\ \beta_\infty+\frac{1}{q}<0;\\
L_{\infty,\infty;(0,-\alpha_\infty-\frac{1}{q}),(-\beta_0-\frac{1}{q},-\beta_\infty)} & \textit{if} & \alpha_0+\frac{1}{q}=0,\ \beta_0+\frac{1}{q}>0,\ \alpha_\infty+\frac{1}{q}<0;\\
L_{\infty,\infty;(0,0),-\mathbb{B}-\frac{1}{q}} & \textit{if} & \alpha_0+\frac{1}{q}=0,\ \beta_0+\frac{1}{q}>0,\\
 & & \alpha_\infty+\frac{1}{q}=0,\ \beta_\infty+\frac{1}{q}<0;\\
L_{\infty,\infty;(0,-\alpha_\infty-\frac{1}{q}),(0,-\beta_\infty),(-\frac{1}{q},0)} & \textit{if} & \alpha_0+\frac{1}{q}=0,\ \beta_0+\frac{1}{q}=0,\ \alpha_\infty+\frac{1}{q}<0;\\
L_{\infty,\infty;(0,0),(0,-\beta_\infty-\frac{1}{q}),(-\frac{1}{q},0)} & \textit{if} & \alpha_0+\frac{1}{q}=0,\ \beta_0+\frac{1}{q}=0,\\
 & & \alpha_\infty+\frac{1}{q}=0,\ \beta_\infty+\frac{1}{q}<0.
\end{array}$$

(ii) *Let either* $\alpha_0+\frac{1}{q}<0$ *or* $\alpha_0+\frac{1}{q}=0$ *and* $\beta_0+\frac{1}{q}<0$. *Then*

$$\mathscr{L}=\begin{cases} L_{\infty,\infty;(0,-\alpha_\infty-\frac{1}{q}),(0,-\beta_\infty)} & \textit{if}\quad \alpha_\infty+\frac{1}{q}<0;\\ L_{\infty,\infty;(0,0),(0,-\beta_\infty-\frac{1}{q})} & \textit{if}\quad \alpha_\infty+\frac{1}{q}=0;\ \beta_\infty+\frac{1}{q}<0.\end{cases}$$

*Proof.* The assertion can be proved analogously to Theorem 6.6. The details are omitted. □

6.10. REMARK. If $q = 1$, then there is a simple proof of Theorem 6.9:

By Theorem 3.8 (i), any space in Theorem 6.9 coincides with its analogue in the norm of which $f^*$ is replaced by $f^{**}$. Thus, by Remark 3.6, any such space is a BFS, which implies that the space $\mathscr{L}$ is a BFS. Consequently, $\mathscr{L}'' = \mathscr{L}$. By Theorem 6.2 (iv) (and its analogue for spaces with three tiers of logarithms) and Theorem 3.8 (ii), $\mathscr{L}' = L_{(1,1;\mathbb{A},\mathbb{B})}$. Hence,

$$\mathscr{L} = \mathscr{L}'' = (L_{(1,1;\mathbb{A},\mathbb{B})})'.$$

To conclude this section, we list associate spaces of GLZ spaces of functions defined on a space of finite measure. With no loss of generality, we assume that $\mu(\mathscr{R}) = 1$. We omit the proofs.

6.11. THEOREM. *Let* $\mu(\mathscr{R}) = 1$. *Let* $0 < p, q \leqslant \infty$, $\alpha, \beta \in \mathbb{R}$, *and assume that the space* $L_{p,q;\alpha,\beta}$ *is not trivial* (*cf.* (3.10)). *Then* $(L_{p,q;\alpha,\beta})'$ *coincides with*

$\{0\}$ *if* *either* $0 < p < 1$
*or* $p = 1,\ 0 < q \leqslant 1,\ \alpha < 0$
*or* $p = 1,\ 0 < q \leqslant 1,\ \alpha = 0,\ \beta < 0$
*or* $p = 1,\ 1 < q \leqslant \infty,\ \alpha - \frac{1}{q'} < 0$
*or* $p = 1,\ 1 < q \leqslant \infty,\ \alpha - \frac{1}{q'} = 0,\ \beta - \frac{1}{q'} \leqslant 0$;

$L_{(p',q';-\alpha,-\beta)}$ *if* $1 < q \leqslant \infty$ *and either* $p = 1,\ \alpha > \frac{1}{q'}$
*or* $p = 1,\ \alpha = \frac{1}{q'},\ \beta \geqslant \frac{1}{q'}$
*or* $1 < p < \infty$;

$L_{(1,q';-\alpha-1,-\beta)}$ *if* $p = \infty,\ 1 < q < \infty,\ \alpha + \frac{1}{q} < 0$;

$L_{(1,q';-\frac{1}{q'},-\beta-1)}$ *if* $p = \infty,\ 1 < q < \infty,\ \alpha + \frac{1}{q} = 0,\ \beta + \frac{1}{q} < 0$;

$L_{1,1;-\alpha,-\beta}$ *if* $p = \infty,\ q = \infty$, *and either* $\alpha < 0$
*or* $\alpha = 0,\ \beta \leqslant 0$;

$L_{(p',\infty;-\alpha,-\beta)}$ *if* $0 < q \leqslant 1$, *and either* $1 < p < \infty$
*or* $p = 1,\ \alpha > 0$
*or* $p = 1,\ \alpha = 0,\ \beta \geqslant 0$;

$L_{(1,\infty;-\alpha-\frac{1}{q},-\beta)}$ *if* $p = \infty,\ 0 < q \leqslant 1,\ \alpha + \frac{1}{q} < 0$;

$L_{(1,\infty;0,-\beta-\frac{1}{q})}$ *if* $p = \infty,\ 0 < q \leqslant 1,\ \alpha + \frac{1}{q} = 0,\ \beta + \frac{1}{q} < 0$.

Let $0 < p, q \leqslant \infty$, and $\alpha, \beta \in \mathbb{R}$. Since $L_{(p,q;\alpha,\beta)} = L_{p,q;\alpha,\beta}$ if $1 < p \leqslant \infty$, $L_{(p,q;\alpha,\beta)} = L^1$ if $0 < p < 1$, and the associated spaces of $L_{p,q;\alpha,\beta}$ have already been described in the previous theorem, it remains to characterize the associated spaces of $L_{(1,q;\alpha,\beta)}$. This is done in the following theorem.

6.12. THEOREM. *Let* $\mu(\mathscr{R}) = 1$. *Let* $0 < q \leqslant \infty$, $\alpha, \beta \in \mathbb{R}$, *and assume that the space* $L_{(1,q;\alpha,\beta)}$ *is not trivial* (cf. (3.10)). *Then its associate space* $(L_{(1,q;\alpha,\beta)})'$ *coincides with*

$$
\begin{array}{lll}
L_{\infty,q';-\alpha-1,-\beta} & \textit{if} & 1 < q \leqslant \infty,\ \alpha + \dfrac{1}{q} > 0; \\
L_{\infty,q';-\frac{1}{q'},-\beta-1} & \textit{if} & 1 < q \leqslant \infty,\ \alpha + \dfrac{1}{q} = 0,\ \beta + \dfrac{1}{q} > 0; \\
L_{\infty,q';-\frac{1}{q'},-\frac{1}{q'},-1} & \textit{if} & 1 < q < \infty,\ \alpha + \dfrac{1}{q} = 0,\ \beta + \dfrac{1}{q} = 0; \\
L^{\infty} & \textit{if} & 0 < q \leqslant \infty,\ \textit{and either}\ \alpha + \dfrac{1}{q} < 0 \\
 & & \qquad \textit{or}\quad \alpha + \dfrac{1}{q} = 0,\ \beta + \dfrac{1}{q} < 0 \\
 & & \qquad \textit{or}\quad q = \infty,\ \alpha = 0,\ \beta = 0; \\
L_{\infty,\infty;-\alpha-\frac{1}{q},-\beta} & \textit{if} & 0 < q \leqslant 1,\ \alpha + \dfrac{1}{q} > 0; \\
L_{\infty,\infty;0,-\beta-\frac{1}{q}} & \textit{if} & 0 < q \leqslant 1,\ \alpha + \dfrac{1}{q} = 0,\ \beta + \dfrac{1}{q} > 0; \\
L_{\infty,\infty;0,0,-\frac{1}{q}} & \textit{if} & 0 < q \leqslant 1,\ \alpha + \dfrac{1}{q} = 0,\ \beta + \dfrac{1}{q} = 0.
\end{array}
$$

## 7. GLZ spaces as Banach function spaces

In this section we characterize GLZ spaces that are BFS. We begin with the spaces $L_{p,q;\mathbb{A},\mathbb{B}}$.

7.1. THEOREM. *Let* $0 < p, q \leqslant \infty$ *and* $\mathbb{A}, \mathbb{B} \in \mathbb{R}^2$. *Then the space* $X = L_{p,q;\mathbb{A},\mathbb{B}}$ *is a BFS if and only if one of the following conditions holds:*

$$
\begin{cases}
1 < p < \infty,\ 1 \leqslant q \leqslant \infty; \\
p = 1,\ q = 1,\ \alpha_0 > 0,\ \alpha_\infty < 0; \\
p = 1,\ q = 1,\ \alpha_0 > 0,\ \alpha_\infty = 0,\ \beta_\infty \leqslant 0; \\
p = 1,\ q = 1,\ \alpha_0 = 0,\ \beta_0 \geqslant 0,\ \alpha_\infty < 0; \\
p = 1,\ q = 1,\ \alpha_0 = 0,\ \beta_0 \geqslant 0,\ \alpha_\infty = 0,\ \beta_\infty \leqslant 0; \\
p = \infty,\ 1 \leqslant q \leqslant \infty,\ \alpha_0 + \frac{1}{q} < 0; \\
p = \infty,\ 1 \leqslant q \leqslant \infty,\ \alpha_0 + \frac{1}{q} = 0,\ \beta_0 + \frac{1}{q} < 0; \\
p = \infty,\ q = \infty,\ \alpha_0 = 0,\ \beta_0 = 0.
\end{cases} \tag{7.1}
$$

7.2. REMARKS. Some particular cases of Theorem 7.1 are worth noticing:

(i) A space $L_{1,q;\mathbb{A},\mathbb{B}}$, where $q \neq 1$, is not a BFS, for any $\mathbb{A}$ and $\mathbb{B}$.

(ii) A space $X = L_{1,1;\mathbb{A},\mathbb{B}}$ is a BFS if and only if $\varphi_X \in \mathcal{F}$.

(iii) A space $L_{\infty,q;\mathbb{A},\mathbb{B}}$ is a BFS if and only if $1 \leqslant q \leqslant \infty$ and $X$ is not trivial.

*Proof of Theorem 7.1.* (i) Let $1 < p \leqslant \infty$, $1 \leqslant q \leqslant \infty$, and $\mathbb{A}, \mathbb{B} \in \mathbb{R}^2$. Then, by Theorem 3.8 (i), $L_{p,q;\mathbb{A},\mathbb{B}} = L_{(p,q;\mathbb{A},\mathbb{B})}$ and by Remark 3.6 and Lemma 3.5 (ii), the latter space is a BFS if and only if either $1 < p < \infty$, or $p = \infty$ and either $\alpha_0 + \frac{1}{q} < 0$, or $\alpha_0 + \frac{1}{q} = 0$, $\beta_0 + \frac{1}{q} < 0$ or $q = \infty$, $\alpha_0 = 0$, $\beta_0 = 0$.

(ii) Let $p = 1$ and $1 < q \leqslant \infty$.

Assume first that

$$\text{either} \quad \alpha_0 + \frac{1}{q} < 1 \qquad \text{or} \qquad \alpha_0 + \frac{1}{q} = 1 \quad \text{and} \quad \beta_0 + \frac{1}{q} \leqslant 1.$$

Then, by Theorem 4.5 (vi), (vii), $L_{1,q;\mathbb{A},\mathbb{B}}$ is not locally embedded into $L^1$. Therefore, (P7) is not satisfied, and hence $L_{1,q;\mathbb{A},\mathbb{B}}$ is not a BFS.

Assume now that

$$\text{either} \quad \alpha_0 + \frac{1}{q} > 1 \qquad \text{or} \qquad \alpha_0 + \frac{1}{q} = 1 \quad \text{and} \quad \beta_0 + \frac{1}{q} > 1. \tag{7.2}$$

Then, by Theorem 6.2 (i),

$$(L_{1,q;\mathbb{A},\mathbb{B}})' = L_{\infty,q';-\mathbb{A},-\mathbb{B}} \tag{7.3}$$

and the conditions in (7.2) guarantee that the latter space is not trivial (cf. (3.2)). Recalling that $X = L_{1,q;\mathbb{A},\mathbb{B}}$, we get from Lemma 3.7 (i)

$$\varphi_X(t) \approx t\ell^{\alpha_0}(t)\ell\ell^{\beta_0}(t), \quad t \in (0,1).$$

By (7.3) and Lemma 3.7 (i), we have for $t \in (0,1)$,

$$\varphi_{X'}(t) \approx \begin{cases} \ell^{-\alpha_0+1/q'}(t)\ell\ell^{-\beta_0}(t) & \text{if} \quad \alpha_0 + \frac{1}{q} > 1 \\ \ell\ell^{-\beta_0+1/q'}(t) & \text{if} \quad \alpha_0 + \frac{1}{q} = 1,\ \beta_0 + \frac{1}{q} > 1. \end{cases}$$

Since $q > 1$, and thus $q' \neq +\infty$, the relation (2.2) is not satisfied, which implies that $X$ is not a BFS.

(iii) Let $p = 1$, $q = 1$ and let one of the conditions in (7.1) hold. We have

$$\|f\|_{1,1;\mathbb{A},\mathbb{B}} = \int_0^\infty f^*(t)\ell^{\mathbb{A}}(t)\ell\ell^{\mathbb{B}}(t)\ \mathrm{d}t.$$

Each of the conditions in (7.1) with $p = 1$ guarantees that $\varphi \in \mathcal{F}$, where

$$\varphi(t) = \int_0^t \ell^{\mathbb{A}}(s)\ell\ell^{\mathbb{B}}(s)\ \mathrm{d}s, \qquad t \in [0,\infty)$$

(since $\ell^{\mathbb{A}}(s)\ell\ell^{\mathbb{B}}(s)$, $s \in (0,\infty)$, is equivalent to a non-increasing function on $(0,\infty)$). Hence $L_{1,1;\mathbb{A},\mathbb{B}} = \Lambda_\varphi$ and therefore $L_{1,1;\mathbb{A},\mathbb{B}}$ is a BFS.

(iv) Let $p = 1$, $q = 1$, and assume that none of the conditions in (7.1) is satisfied. Then, by Lemma 6.5, $\varphi_X \notin \mathcal{F}$, and consequently $X$ is not a BFS.

(v) Let $0 < p < 1$. Then, by Lemma 3.7 (i), the fundamental function of $L_{p,q;\mathbb{A},\mathbb{B}}$ is not in $\mathcal{F}$. Consequently, $L_{p,q;\mathbb{A},\mathbb{B}}$ is not a BFS.

(vi) Finally, let $1 \leqslant p \leqslant \infty$ and $0 < q < 1$. There are three possibilities: either $X = \{0\}$, or $X \neq \{0\}$ and $\varphi_X \notin \mathcal{F}$, or $X \neq \{0\}$ and $\varphi_X \in \mathcal{F}$. It is clear (cf. (P1) and (P6)) that X is not a BFS when $X = \{0\}$, and also, as mentioned in Section 2 above, that X is not a BFS when $\varphi_X \notin \mathcal{F}$. Using Lemma 3.7 and embedding results of Section 4, we obtain that $\Lambda_{\varphi_X}$ is not embedded into $X$ when $X \neq \{0\}$ and $\varphi_X \in \mathcal{F}$. Thus, again, $X$ is not a BFS.

The proof is complete. □

We now turn our attention to the spaces $L_{(p,q;\mathbb{A},\mathbb{B})}$.

7.3. THEOREM. *Let $0 < p, q \leqslant \infty$ and $\mathbb{A}, \mathbb{B} \in \mathbb{R}^2$. Then the space $X = L_{(p,q;\mathbb{A},\mathbb{B})}$ is a BFS if and only if $1 \leqslant q \leqslant \infty$ and $X \neq \{0\}$.*

*Proof.* If $1 \leqslant q \leqslant \infty$, then, by Remark 3.6, $X$ is a BFS if and only if $X \neq \{0\}$.

Conversely, let either $X = \{0\}$ or $0 < q < 1$. If $X = \{0\}$, then it is evident from (P1) and (P6) that $X$ is not a BFS. If $0 < q < 1$, then $\Lambda_{\varphi_X}$ is not embedded into $X$. To see this, we use Theorem 3.8 (ii) to rewrite $\Lambda_{\varphi_X}$ as $L_{(1,1;\mathbb{L},\mathbb{E})}$ with appropriate $\mathbb{L}, \mathbb{E} \in \mathbb{R}^2$, and then apply embedding results of Section 5. Thus, $X$ is not a BFS. □

Finally, we present analogues of Theorems 7.1 and 7.3 for the case when $\mu(\mathscr{R}) < \infty$.

7.4. THEOREM. *Assume that $\mu(\mathscr{R}) < \infty$. Let $0 < p, q \leqslant \infty$ and $\alpha, \beta \in \mathbb{R}^2$. Then the space $L_{p,q;\alpha,\beta}$ is a BFS if and only if one of the following conditions holds:*

$$\begin{aligned}
&1 < p < \infty,\ 1 \leqslant q \leqslant \infty;\\
&p = 1,\ q = 1,\ \alpha > 0;\\
&p = 1,\ q = 1,\ \alpha = 0,\ \beta \geqslant 0;\\
&p = \infty,\ 1 \leqslant q \leqslant \infty,\ \alpha + \frac{1}{q} < 0;\\
&p = \infty,\ 1 \leqslant q \leqslant \infty,\ \alpha + \frac{1}{q} = 0,\ \beta + \frac{1}{q} < 0;\\
&p = \infty,\ q = \infty,\ \alpha = 0,\ \beta = 0.
\end{aligned}$$

7.5. THEOREM. *Assume that $\mu(\mathscr{R}) < \infty$. Let $0 < p, q \leqslant \infty$ and $\alpha, \beta \in \mathbb{R}^2$. Then the space $L_{(p,q;\alpha,\beta)}$ is a BFS if and only if $1 \leqslant q \leqslant \infty$ and one of the following conditions holds:*

$$\begin{aligned}
&0 < p < \infty;\\
&p = \infty,\ \alpha + \frac{1}{q} < 0;\\
&p = \infty,\ \alpha + \frac{1}{q} = 0,\ \beta + \frac{1}{q} < 0;\\
&p = \infty,\ q = \infty,\ \alpha = 0,\ \beta = 0.
\end{aligned}$$

## 8. GLZ spaces and Orlicz spaces

In this section we shall give a complete characterization of those GLZ spaces $L_{p,q;\mathbb{A},\mathbb{B}}$ and $L_{(p,q;\mathbb{A},\mathbb{B})}$ which coincide with Orlicz spaces. All such spaces are described by Theorems 8.8 and 8.11 below. To avoid trivial cases, we throughout this section use the following

CONVENTION. Since Orlicz spaces are BFS, we restrict ourselves to the values of $p,q;\mathbb{A},\mathbb{B}$ that satisfy (7.1) in the case of spaces $L_{p,q;\mathbb{A},\mathbb{B}}$, or (3.3) and $1 \leqslant q \leqslant \infty$ in the case of spaces $L_{(p,q;\mathbb{A},\mathbb{B})}$ (cf. Theorems 7.1 and 7.3).

Let us first consider the case when $p = q$. We shall first recall a result which follows from [EOP1, Lemmas 2.1 and 2.2], concerning spaces of functions defined on a finite-measure space.

8.1. LEMMA. *Let $\mu(\mathscr{R}) < \infty$. Let one of the following conditions hold:*

$$1 < p < \infty; \tag{8.1}$$

$$p = 1,\ \alpha > 0; \tag{8.2}$$

$$p = 1,\ \alpha = 0,\ \beta > 0; \tag{8.3}$$

$$p = \infty,\ \alpha < 0; \tag{8.4}$$

$$p = \infty,\ \alpha = 0,\ \beta < 0. \tag{8.5}$$

*Then*

$$L_{p,p;\alpha,\beta} = L_\Phi,$$

*where the Young function $\Phi$ satisfies for large $t$,*

$$\Phi(t) \approx \begin{cases} t^p \ell^{\alpha p}(t) \ell\ell^{\beta p}(t) & \text{if one of (8.1)–(8.3) holds,} \\ \exp\left(t^{-1/\alpha} \ell^{-\beta/\alpha}(t)\right) & \text{if (8.4) holds,} \\ \exp\exp(t^{-1/\beta}) & \text{if (8.5) holds.} \end{cases} \tag{8.6}$$

8.2. COROLLARY. *Let one of the conditions (8.1) – (8.5) hold, and let the Young function $\Phi$ be given for large $t$ by (8.6). Let $T \in (0,\infty)$ and $f \in \mathscr{M}(\mathscr{R},\mu)$. Then*

$$\int_0^T [f^*(t)\ell^\alpha(t)\ell\ell^\beta(t)]^p \, \mathrm{d}t < \infty$$

*if and only if there exists a $\lambda > 0$ such that*

$$\int_0^T \Phi(\lambda f^*(t)) \, \mathrm{d}t < \infty.$$

When $\mu(\mathscr{R}) = \infty$, the values of $\Phi(t)$ for $t$ small become important. To handle them properly, we shall need some auxiliary results. First, we formulate a modified version of the Young inequality.

8.3. LEMMA. (i) *Let* $\lambda > 0$ *and* $\varepsilon \in \mathbb{R}$. *Then for every* $a, b > 1$,

$$ab \lesssim \exp\left(a^{1/\lambda}\ell^{-\varepsilon/\lambda}(a)\right) + b\,\ell^{\lambda}(b)\ell\ell^{\varepsilon}(b). \tag{8.7}$$

(ii) *Let* $\varepsilon > 0$. *Then for every* $a, b > 1$,

$$ab \lesssim \exp\exp(a^{1/\varepsilon}) + b\,\ell\ell^{\varepsilon}(b).$$

*Proof.* This follows from the usual Young inequality by a straightforward calculation of complementary functions (cf. also the proof of [EOP1, Lemma 2.2 (vi)]). □

8.4. LEMMA. *Let* $\mu(\mathscr{R}) = \infty$, $1 \leqslant p < \infty$, *and let either* $\lambda > 0$ *or* $\lambda = 0$ *and* $\varepsilon > 0$. *Suppose that* $f \in \mathscr{M}(\mathscr{R}, \mu)$ *is such that* $f^*(1) < \infty$, *and put* $R = \mu(\operatorname{supp} f)$. *Then*

$$\int_1^R [f^*(t)]^p \ell^{\lambda}(t)\ell\ell^{\varepsilon}(t)\,\mathrm{d}t < \infty \tag{8.8}$$

*if and only if*

$$\int_1^R [f^*(t)]^p \ell^{\lambda}\left(f^*(t)\right)\ell\ell^{\varepsilon}\left(f^*(t)\right)\,\mathrm{d}t < \infty. \tag{8.9}$$

*Proof.* The assertion is trivial when $R < \infty$. Assume that $R = \infty$.

First, let us note that both (8.8) and (8.9) imply that

$$f^*(t) \to 0 \quad \text{as} \quad t \to \infty. \tag{8.10}$$

For $T \in [1, \infty)$, we denote

$$I_1(T) = \int_T^{\infty} [f^*(t)]^p \ell^{\lambda}(t)\ell\ell^{\varepsilon}(t)\,\mathrm{d}t, \qquad I_2(T) = \int_T^{\infty} [f^*(t)]^p \ell^{\lambda}\left(f^*(t)\right)\ell\ell^{\varepsilon}\left(f^*(t)\right)\,\mathrm{d}t.$$

We see that for any $T \in [1, \infty)$, $I_i(T) < \infty$ if and only if $I_i(1) < \infty$, $i = 1, 2$. Hence, using also (8.10), we may assume with no loss of generality that $f^*(t) \leqslant 1$ for every $t \in [1, \infty)$.

Suppose that (8.9) holds. Then we have

$$K_1 := \sup_{1<t<\infty} t^{1/p} f^*(t) < \infty. \tag{8.11}$$

Indeed, assuming that

$$t_k^{1/p} f^*(t_k) \to \infty \qquad \text{for some} \quad t_k \to \infty, \tag{8.12}$$

and using also the fact that the function $\ell^{\lambda}(x)\ell\ell^{\varepsilon}(x)$ is decreasing for $x \in (0,x_0)$ with $x_0$ small enough, we get

$$\int_1^{\infty}[f^*(t)]^p\ell^{\lambda}\big(f^*(t)\big)\ell\ell^{\varepsilon}\big(f^*(t)\big)\,\mathrm{d}\,t \geqslant \int_{t_k/2}^{t_k}[f^*(t)]^p\ell^{\lambda}\big(f^*(t)\big)\ell\ell^{\varepsilon}\big(f^*(t)\big)\,\mathrm{d}\,t$$
$$\geqslant \frac{t_k}{2}[f^*(t_k)]^p\ell^{\lambda}\Big(f^*\Big(\frac{t_{k_0}}{2}\Big)\Big)\ell\ell^{\varepsilon}\Big(f^*\Big(\frac{t_{k_0}}{2}\Big)\Big)$$

for all $k \geqslant k_0$, where $k_0 \in \mathbb{N}$ is chosen so that $f^*(t_{k_0}/2) < x_0$. Combined with (8.12), this estimate contradicts (8.9), and thus (8.11) holds.

By (8.11), $1 \leqslant t^{1/p}/K_1 \leqslant 1/f^*(t)$ if $t > \max\{1,K_1^p\}$. The function $\ell^{\lambda}(x)\ell\ell^{\varepsilon}(x)$ is increasing on $(x_1,\infty)$ for some $x_1$ large enough. Thus, taking $t > T := \max\{1,K_1^p, (x_1K_1)^p\}$, we get

$$\ell^{\lambda}(t^{1/p}/K_1)\ell\ell^{\varepsilon}(t^{1/p}/K_1) \leqslant \ell^{\lambda}(1/f^*(t))\ell\ell^{\varepsilon}(1/f^*(t)).$$

This yields immediately

$$\ell^{\lambda}(t)\ell\ell^{\varepsilon}(t) \lesssim \ell^{\lambda}(f^*(t))\ell\ell^{\varepsilon}(f^*(t)), \qquad t \in (T,\infty),$$

and (8.8) follows from (8.9).

Conversely, suppose that (8.8) holds. Let $n_0 \in \mathbb{N}\cup\{0\}$ be such that

$$e^{-(n_0+1)/p} < f^*(1) \leqslant e^{-n_0/p}.$$

We define

$$t_n = \inf\{t \geqslant 1; f^*(t) \leqslant e^{-n/p}\}, \quad n \geqslant n_0.$$

Then $\{t_n\}$ is a non-decreasing sequence such that $t_n \to \infty$ as $n \to \infty$. Since $f^*$ is right-continuous (cf. [BS, Chapter 2, Proposition 1.7]), we have

$$e^{-(n+1)/p} < f^*(t) \leqslant e^{-n/p}, \quad t \in [t_n,t_{n+1}).$$

This yields for all $t \in [t_n,t_{n+1})$,

$$[f^*(t)]^p \approx e^{-n}, \quad \ell^{\lambda}\big(f^*(t)\big) \approx n^{\lambda}, \quad \ell\ell^{\varepsilon}\big(f^*(t)\big) \approx \log^{\varepsilon}(n+2), \tag{8.13}$$

and hence

$$I_2(1) = \sum_{n\geqslant n_0}\int_{t_n}^{t_{n+1}}[f^*(t)]^p\ell^{\lambda}\big(f^*(t)\big)\ell\ell^{\varepsilon}\big(f^*(t)\big)\,\mathrm{d}\,t \tag{8.14}$$
$$\approx \sum_{n\geqslant n_0}e^{-n}n^{\lambda}(\log^{\varepsilon}(n+2))\,(t_{n+1}-t_n) = \sum_{n\geqslant n_0}e^{-n}a_nb_n,$$

where

$$a_n = (n/2)^{\lambda}\log^{\varepsilon}(n+2), \quad b_n = 2^{\lambda}(t_{n+1}-t_n).$$

We shall use (8.7) to estimate the right hand side of (8.14). Since

$$a_n^{1/\lambda}\ell^{-\varepsilon/\lambda}(a_n) \approx n/2$$

and

$$b_n\ell^{\lambda}(b_n)\ell\ell^{\varepsilon}(b_n) \approx (t_{n+1}-t_n)\ell^{\lambda}(t_{n+1}-t_n)\ell\ell^{\varepsilon}(t_{n+1}-t_n),$$

we get from (8.14) and (8.7),

$$I_2(1) \lesssim \sum_{n\geqslant n_0} e^{-n}e^{n/2} + \sum_{n\geqslant n_0} e^{-n}(t_{n+1}-t_n)\ell^{\lambda}(t_{n+1}-t_n)\ell\ell^{\varepsilon}(t_{n+1}-t_n) = S_1+S_2,$$

say. Obviously, $S_1 < \infty$. Now, we claim that, for large $n$, say $n \geqslant n_1$, we have

$$(t_{n+1}-t_n)\ell^{\lambda}(t_{n+1}-t_n)\ell\ell^{\varepsilon}(t_{n+1}-t_n) \lesssim t_{n+1}\ell^{\lambda}(t_{n+1})\ell\ell^{\varepsilon}(t_{n+1}) - t_n\ell^{\lambda}(t_n)\ell\ell^{\varepsilon}(t_n). \tag{8.15}$$

Once this is shown, the assertion will follow easily as, by (8.15),

$$S_2 \lesssim \sum_{n=n_0}^{n_1-1} e^{-n}(t_{n+1}-t_n)\ell^{\lambda}(t_{n+1}-t_n)\ell\ell^{\varepsilon}(t_{n+1}-t_n) + S_3,$$

where

$$S_3 = \sum_{n\geqslant n_1} e^{-n}[t_{n+1}\ell^{\lambda}(t_{n+1})\ell\ell^{\varepsilon}(t_{n+1}) - t_n\ell^{\lambda}(t_n)\ell\ell^{\varepsilon}(t_n)].$$

But, by (8.13) and (8.8),

$$S_3 \lesssim \sum_{n\geqslant n_1} e^{-n}\int_{t_n}^{t_{n+1}} \ell^{\lambda}(t)\ell\ell^{\varepsilon}(t)\,\mathrm{d}t \lesssim I_1(1) < \infty.$$

It remains to prove (8.15). Assume first that $t_{n+1} \geqslant 2t_n$, i.e., $t_{n+1} \leqslant 2(t_{n+1}-t_n)$. Then

$$t_{n+1}\ell^{\lambda}(t_{n+1})\ell\ell^{\varepsilon}(t_{n+1}) \leqslant 2(t_{n+1}-t_n)\ell^{\lambda}(t_{n+1})\ell\ell^{\varepsilon}(t_{n+1}). \tag{8.16}$$

Since the function $\ell^{\lambda}(t)\ell\ell^{\varepsilon}(t)$ is increasing near $\infty$, we obtain from (8.16) that, for large $n$,

$$t_{n+1}\ell^{\lambda}(t_{n+1})\ell\ell^{\varepsilon}(t_{n+1}) \leqslant 2[t_{n+1}\ell^{\lambda}(t_{n+1})\ell\ell^{\varepsilon}(t_{n+1}) - t_n\ell^{\lambda}(t_n)\ell\ell^{\varepsilon}(t_n)],$$

and (8.15) follows.

Now assume that $t_{n+1} \leqslant 2t_n$, i.e., $t_{n+1}-t_n \leqslant t_n$. Since the function $F(t) = t\ell^{\lambda}(t)\ell\ell^{\varepsilon}(t)$ is convex near $\infty$, it is equivalent to an increasing convex function on entire $[1,\infty)$. With no loss of generality, we shall assume that $F$ itself is increasing and convex on $[1,\infty)$. Thus, for every $n$,

$$F'(t_n)(t_{n+1}-t_n) \lesssim F(t_{n+1}) - F(t_n), \tag{8.17}$$

and

$$\ell^{\lambda}(t_n)\ell\ell^{\varepsilon}(t_n) \approx F'(t_n). \tag{8.18}$$

Similarly, the function $\ell^{\lambda}(t)\ell\ell^{\varepsilon}(t)$ is equivalent to an increasing function on $[1,\infty)$. Therefore, using also $t_{n+1}-t_n \leqslant t_n$, (8.18) and (8.17), we obtain for every $n$,

$$\begin{aligned}(t_{n+1}-t_n)&\ell^{\lambda}(t_{n+1}-t_n)\ell\ell^{\varepsilon}(t_{n+1}-t_n)\\ &\leqslant (t_{n+1}-t_n)\ell^{\lambda}(t_n)\ell\ell^{\varepsilon}(t_n) \approx (t_{n+1}-t_n)F'(t_n)\\ &\lesssim t_{n+1}\ell^{\lambda}(t_{n+1})\ell\ell^{\varepsilon}(t_{n+1}) - t_n\ell^{\lambda}(t_n)\ell\ell^{\varepsilon}(t_n)\end{aligned}$$

and (8.15) follows. The proof is complete. □

The following simple lemma is of independent interest as it indicates a relation between Orlicz spaces and Marcinkiewicz spaces (compare [Lo2, Theorem 2]).

8.5. LEMMA. *Let $f \in \mathscr{M}(\mathscr{R},\mu)$ and let $\Phi$ be a Young function on $(0,\infty)$ such that*

$$\int_0^{\infty} \Phi\Big(\gamma\Phi^{-1}\Big(\frac{1}{t}\Big)\Big)\,\mathrm{d}t < \infty \qquad \textit{for some} \quad \gamma > 0. \tag{8.19}$$

*Then $f \in L_\Phi$ if and only if there exists a constant $K = K(f)$ such that*

$$\sup_{0<t<\infty} \frac{f^*(t)}{\Phi^{-1}(1/t)} = K < \infty. \tag{8.20}$$

8.6. REMARK. Since (cf. (2.9)) $\varphi_{L_\Phi}(t) = \frac{1}{\Phi^{-1}(1/t)}$, $t \in (0,\infty)$, the condition (8.20) can be rewritten as

$$\sup_{0<t<\infty} f^*(t)\varphi_{L_\Phi}(t) = K < \infty.$$

*Proof of Lemma 8.5.* If (8.20) holds, then we have for $\lambda < \gamma K^{-1}$

$$\int_0^{\infty} \Phi\big(\lambda f^*(t)\big)\,\mathrm{d}t \leqslant \int_0^{\infty} \Phi\Big(\gamma\Phi^{-1}\Big(\frac{1}{t}\Big)\Big)\,\mathrm{d}t < \infty,$$

in other words, $f \in L_\Phi$.

Conversely, assume that $f \in L_\Phi$. Then there is a $\lambda_0 > 0$ such that

$$\int_0^{\infty} \Phi\big(\lambda_0 f^*(t)\big)\,\mathrm{d}t \leqslant \frac{1}{2}. \tag{8.21}$$

Let $a > 0$. Then, by (8.21),

$$\frac{1}{2} \geqslant \int_{a/2}^{a} \Phi\big(\lambda_0 f^*(t)\big)\,\mathrm{d}t \geqslant \Phi\big(\lambda_0 f^*(a)\big)\frac{a}{2},$$

which is equivalent to

$$f^*(a) \leqslant \lambda_0^{-1}\Phi^{-1}\Big(\frac{1}{a}\Big), \qquad a > 0,$$

and (8.20) follows with $K = \lambda_0^{-1}$. (Let us note that $f \in L_\Phi$ implies (8.20) for *any* Young function $\Phi$, regardless of the validity of (8.19).) $\square$

The following example follows by an easy calculation.

8.7. EXAMPLE. Let $\Phi$ be a Young function such that, for $0 < t \leqslant 1$, either

$$\Phi(t) \approx \exp\big(-t^\alpha \log^\beta(1/t)\big), \qquad \alpha < 0,\ \beta \in \mathbb{R},$$

or

$$\Phi(t) \approx \exp(-\exp t^\beta), \qquad \beta < 0,$$

and, for $1 < t < \infty$, either

$$\Phi(t) \approx \exp(t^\gamma \log^\delta t), \qquad \gamma > 0,\ \delta \in \mathbb{R},$$

or

$$\Phi(t) \approx \exp\exp t^\delta, \qquad \delta > 0.$$

Then $\Phi$ satisfies (8.19).

Now we are in a position to prove the main result of this section.

8.8. THEOREM. *Assume that* $\mu(\mathscr{R}) = \infty$ *and* $\mathbb{A} = (\alpha_0, \alpha_\infty)$, $\mathbb{B} = (\beta_0, \beta_\infty) \in \mathbb{R}^2$.

(i) *Let* $1 < p < \infty$. *Then* $L_{p,p;\mathbb{A},\mathbb{B}} = L_\Phi$, *where*

$$\Phi(t) \approx t^p \ell^{\tilde{\mathbb{A}}p}(t)\ell\ell^{\tilde{\mathbb{B}}p}(t), \qquad t \in (0,\infty) \tag{8.22}$$

(*recall that* $\tilde{\mathbb{A}} = (\alpha_\infty, \alpha_0)$ *and* $\tilde{\mathbb{B}} = (\beta_\infty, \beta_0)$).

(ii) *Let* $\alpha_\infty > 0$ *and either* $\alpha_0 < 0$ *or* $\alpha_0 = 0$ *and* $\beta_0 < 0$. *Then* $L_{\infty,\infty;\mathbb{A},\mathbb{B}} = L_\Phi$, *where*

$$\Phi(t) \approx \exp\big(-t^{-1/\alpha_\infty}\ell^{-\beta_\infty/\alpha_\infty}(t)\big) \qquad \textit{for} \quad t \in (0,1), \tag{8.23}$$

*and, for* $t \in (1,\infty)$,

$$\Phi(t) \approx \begin{cases} \exp\big(t^{-1/\alpha_0}\ell^{-\beta_0/\alpha_0}(t)\big) & \textit{if} \quad \alpha_0 < 0, \\ \exp\exp(t^{-1/\beta_0}) & \textit{if} \quad \alpha_0 = 0,\ \beta_0 < 0. \end{cases} \tag{8.24}$$

(iii) *Let* $\alpha_\infty = 0$, $\beta_\infty > 0$, *and either* $\alpha_0 < 0$ *or* $\alpha_0 = 0$ *and* $\beta_0 < 0$. *Then* $L_{\infty,\infty;\mathbb{A},\mathbb{B}} = L_\Phi$, *where*

$$\Phi(t) \approx \exp(-\exp t^{-1/\beta_\infty}) \qquad \textit{for} \quad t \in (0,1),$$

*and* $\Phi$ *satisfies* (8.24) *for* $t \in (1,\infty)$.

(iv) *Let either* $\alpha_0 > 0$ *or* $\alpha_0 = 0$ *and* $\beta_0 > 0$, *and let either* $\alpha_\infty < 0$ *or* $\alpha_\infty = 0$ *and* $\beta_\infty < 0$. *Then* $L_{1,1;\mathbb{A},\mathbb{B}} = L_\Phi$, *where*

$$\Phi(t) \approx t\ell^{\tilde{\mathbb{A}}}(t)\ell\ell^{\tilde{\mathbb{B}}}(t), \qquad t \in (0,\infty). \tag{8.25}$$

*Proof.* (i) Let $\Phi$ satisfy (8.22). Then $\Phi \in \Delta_2$. Hence, $f \in L_\Phi$ if and only if

$$\int_0^\infty \Phi\big(f^*(t)\big)\,\mathrm{d}t < \infty. \tag{8.26}$$

Put $R = \mu(\operatorname{supp} f)$. Define $T = \inf\{t > 0; f^*(t) \leqslant 1\}$. Using the fact that $1 < f^*(t) < \infty$ for $t \in (0,T)$ and $f^*(t) \leqslant 1$ for $t \in [T,\infty)$, and (8.22), we get from the estimate (8.26),

$$\int_0^T \big(f^*(t)\big)^p \ell^{\alpha_0 p}\big(f^*(t)\big)\ell\ell^{\beta_0 p}\big(f^*(t)\big)\,\mathrm{d}t + \int_T^R \big(f^*(t)\big)^p \ell^{\alpha_\infty p}\big(f^*(t)\big)\ell\ell^{\beta_\infty p}\big(f^*(t)\big)\,\mathrm{d}t < \infty. \tag{8.27}$$

By Corollary 8.2 and Lemma 8.4, (8.27) holds if and only if

$$\int_0^R \big(f^*(t)\big)^p \ell^{\mathbb{A}p}(t)\ell\ell^{\mathbb{B}p}(t)\,\mathrm{d}t < \infty$$

in the case $\alpha_\infty > 0$ or $\alpha_\infty = 0$, $\beta_\infty > 0$. This proves (i) in such case. If $\alpha_\infty < 0$ or $\alpha_\infty = 0$, $\beta_\infty < 0$, the assertion follows via duality (Theorem 6.2). In the remaining case $\alpha_\infty = 0$, $\beta_\infty = 0$, the assertion is obvious.

(ii) Let $\Phi$ satisfy (8.23) and (8.24).Then, by Example 8.7, $\Phi$ satisfies (8.19). By Lemma 8.5, $f \in L_\Phi$ if and only if (8.20) holds. However, it is easy to see that (8.20) with our $\Phi$ is equivalent to $f \in L_{\infty,\infty;\mathbb{A},\mathbb{B}}$.

(iii) The proof is analogous to that of (ii).

(iv) By Theorem 6.6 (i) we have

$$\Big(L_{1,1;\mathbb{A},\mathbb{B}}\Big)' = L_{\infty,\infty;-\mathbb{A},-\mathbb{B}}. \tag{8.28}$$

If $\alpha_\infty < 0$, then, by (ii) with $\mathbb{A},\mathbb{B}$ replaced by $-\mathbb{A},-\mathbb{B}$, we get

$$L_{\infty,\infty;-\mathbb{A},-\mathbb{B}} = L_\Psi, \tag{8.29}$$

where

$$\Psi(t) \approx \begin{cases} \exp\big(-t^{1/\alpha_\infty}\ell^{-\beta_\infty/\alpha_\infty}(t)\big), & t \in (0,1), \\ \exp\big(t^{1/\alpha_0}\ell^{-\beta_0/\alpha_0}(t)\big), & t \in (1,\infty),\ \alpha_0 > 0, \\ \exp\exp(t^{1/\beta_0}), & t \in (1,\infty),\ \alpha_0 = 0,\ \beta_0 > 0. \end{cases}$$

A direct calculation shows that the complementary function of $\Psi$ is given by (8.25). Together with (8.28) and (8.29) this yields

$$L_{1,1;\mathbb{A},\mathbb{B}} = \Big(L_{1,1;\mathbb{A},\mathbb{B}}\Big)'' = \Big(L_{\infty,\infty;-\mathbb{A},-\mathbb{B}}\Big)' = \big(L_\Psi\big)' = L_\Phi,$$

which is the desired result.

If $\alpha_\infty = 0$ and $\beta_\infty < 0$, we adopt an analogous argument using (iii) rather than (ii). $\square$

Lemma 8.10 below completes the results of Theorem 8.8. To prove it, we shall need the following assertion, which follows easily from (2.8) and (2.9).

8.9. LEMMA. *Let $X$ be an r.i. space. If its fundamental function $\varphi_X$ is equivalent near 0 or near $\infty$ either to $t$ or to $1$, then $X$ is not an Orlicz space.*

8.10. LEMMA. *Suppose that $\mu(\mathscr{R}) = \infty$.*

(i) *Let either $\alpha_0 < 0$ or $\alpha_0 = 0$ and $\beta_0 \leqslant 0$, and let either $\alpha_\infty < 0$ or $\alpha_\infty = 0$ and $\beta_\infty \leqslant 0$. Then $L_{\infty,\infty;\mathbb{A},\mathbb{B}}$ is not an Orlicz space.*

(ii) *Let one of the following conditions be satisfied:*

$$\begin{aligned} &\alpha_0 = 0, \quad \beta_0 = 0, \quad \alpha_\infty < 0;\\ &\alpha_0 = 0, \quad \beta_0 = 0, \quad \alpha_\infty = 0, \quad \beta_\infty \leqslant 0;\\ &\alpha_0 > 0, \quad \alpha_\infty = 0, \quad \beta_\infty = 0;\\ &\alpha_0 = 0, \quad \beta_0 \geqslant 0, \quad \alpha_\infty = 0, \quad \beta_\infty = 0. \end{aligned}$$

*Then $L_{1,1;\mathbb{A},\mathbb{B}}$ is not an Orlicz space.*

*Proof.* (i) Put $X = L_{\infty,\infty;\mathbb{A},\mathbb{B}}$. By Corollary 3.10, $X = L_{\infty,\infty;(\alpha_0,0),(\beta_0,0)}$. Thus, by Lemma 3.7 (i), $\varphi_X(t) \approx 1$ for all $t \in (1,\infty)$, and the result follows from Lemma 8.9.

(ii) The proof is analogous. $\square$

Now we turn our attention to spaces $L_{(p,q;\mathbb{A},\mathbb{B})}$. In view of Lemma 3.5 (ii), Theorem 3.8 (i) and Theorem 8.8 (i)–(ii), it suffices to consider the case $p = 1$.

8.11. THEOREM. *Assume that $\mu(\mathscr{R}) = \infty$ and $\mathbb{A} = (\alpha_0, \alpha_\infty)$, $\mathbb{B} = (\beta_0, \beta_\infty) \in \mathbb{R}^2$. Let*

$$\text{either} \qquad \alpha_0 + 1 > 0 \qquad \text{or} \qquad \alpha_0 + 1 = 0,\ \beta_0 + 1 \geqslant 0,$$

*and*

$$\text{either} \qquad \alpha_\infty + 1 < 0 \qquad \text{or} \qquad \alpha_\infty + 1 = 0,\ \beta_\infty + 1 < 0.$$

*Then $L_{(1,1;\mathbb{A},\mathbb{B})} = L_\Phi$, where, for $t \in (0,1)$,*

$$\Phi(t) \approx \begin{cases} t\ell^{\alpha_\infty+1}(t)\ell\ell^{\beta_\infty}(t) & \text{if} \quad \alpha_\infty + 1 < 0,\\ t\ell\ell^{\beta_\infty+1}(t) & \text{if} \quad \alpha_\infty + 1 = 0,\ \beta_\infty + 1 < 0, \end{cases}$$

*and, for $t \in [1,\infty)$,*

$$\Phi(t) \approx \begin{cases} t\ell^{\alpha_0+1}(t)\ell\ell^{\beta_0}(t) & \text{if} \quad \alpha_0 + 1 > 0,\\ t\ell\ell^{\beta_0+1}(t) & \text{if} \quad \alpha_0 + 1 = 0,\ \beta_0 + 1 > 0,\\ t\ell\ell\ell(t) & \text{if} \quad \alpha_0 + 1 = 0,\ \beta_0 + 1 = 0. \end{cases}$$

*Proof.* For example, assume that either $\alpha_0 + 1 > 0$ and $\alpha_\infty + 1 < 0$. Then, by Theorem 3.8 (ii) and Theorem 8.8 (iv), $L_{(1,1;\mathbb{A},\mathbb{B})} = L_{1,1;\mathbb{A}+1,\mathbb{B}} = L_\Phi$, where

$\Phi(t) \approx t\ell^{\tilde{\mathbb{A}}+1}(t)\ell\ell^{\tilde{\mathbb{B}}}(t)$, $t \in (0,\infty)$, and the result follows. The proof in other cases is similar. $\square$

To complete our analysis, we shall show that if $p \neq q$, then neither the space $L_{p,q;\mathbb{A},\mathbb{B}}$ nor the space $L_{(p,q;\mathbb{A},\mathbb{B})}$ coincides with an Orlicz space. (Consequently, Theorems 8.8, and 8.11 cover all possible cases when a GLZ space is an Orlicz space.) It is thus enough to consider those spaces which are BFS. Further reduction is enabled by Lemma 8.9.

Let us first deal with the spaces $L_{p,q;\mathbb{A},\mathbb{B}}$. It follows from Theorem 7.1, Lemma 8.9 and Lemma 3.7 (i) that we have only to consider one of the cases

$$(8.30)\qquad \begin{cases} 1 < p < \infty,\ 1 \leqslant q \leqslant \infty,\ p \neq q; \\ p = \infty,\ 1 \leqslant q < \infty,\ \alpha_0 + \frac{1}{q} < 0,\ \alpha_\infty + \frac{1}{q} > 0; \\ p = \infty,\ 1 \leqslant q < \infty,\ \alpha_0 + \frac{1}{q} < 0,\ \alpha_\infty + \frac{1}{q} = 0,\ \beta_\infty + \frac{1}{q} \geqslant 0; \\ p = \infty,\ 1 \leqslant q < \infty,\ \alpha_0 + \frac{1}{q} = 0,\ \beta_0 + \frac{1}{q} < 0,\ \alpha_\infty + \frac{1}{q} > 0; \\ p = \infty,\ 1 \leqslant q < \infty,\ \alpha_0 + \frac{1}{q} = 0,\ \beta_0 + \frac{1}{q} < 0, \\ \qquad\qquad \alpha_\infty + \frac{1}{q} = 0,\ \beta_\infty + \frac{1}{q} \geqslant 0. \end{cases}$$

We shall use the following auxiliary result.

8.12. LEMMA. *Let $X$ and $Y$ be two r.i. spaces such that $X \neq Y$ and $\varphi_X \approx \varphi_Y$. Suppose that there is a Young function $\Phi$ such that $Y = L_\Phi$. Then $X \neq L_\Psi$ for any Young function $\Psi$.*

*Proof.* Assume the contrary, that is, $X = L_\Psi$ for some Young function $\Psi$. Then, by (2.9),

$$\frac{1}{\Phi^{-1}(1/t)} \approx \varphi_X(t) \approx \varphi_Y(t) \approx \frac{1}{\Psi^{-1}(1/t)}, \qquad t \in (0,\infty).$$

Thus $\Phi \approx \Psi$ on $(0,\infty)$, and therefore $X = L_\Phi = L_\Psi = Y$, a contradiction. $\square$

8.13. COROLLARY. *Let $0 < p < \infty$, $0 < q \leqslant \infty$, $p \neq q$, and $\mathbb{A} = (\alpha_0, \alpha_\infty)$, $\mathbb{B} = (\beta_0, \beta_\infty) \in \mathbb{R}^2$. Put $X = L_{p,p;\mathbb{A},\mathbb{B}}$ and $Y = L_{p,q;\mathbb{A},\mathbb{B}}$. Suppose that there is a Young function $\Phi$ such that $X = L_\Phi$. Then $Y$ is not an Orlicz space.*

*Proof.* Embedding results of Section 4 show that $X \neq Y$. Moreover, by Lemma 3.7 (i), $\varphi_X \approx \varphi_Y$ on $(0,\infty)$. The result now follows from Lemma 8.12. $\square$

Now we are in a position to finish the analysis concerning the spaces $L_{p,q;\mathbb{A},\mathbb{B}}$. The following theorem completes the picture.

8.14. THEOREM. *Let $0 < p, q \leqslant \infty$ and $\mathbb{A} = (\alpha_0, \alpha_\infty)$, $\mathbb{B} = (\beta_0, \beta_\infty) \in \mathbb{R}^2$. Put $X = L_{p,q;\mathbb{A},\mathbb{B}}$. Let one of the conditions in* (8.30) *be satisfied. Then $X$ is not an Orlicz space.*

*Proof.* If the first condition in (8.30) holds, then the result follows immediately from Corollary 8.13. In all other cases we use a method which will be illustrated on the case

$$(8.31)\qquad p = \infty, \quad 1 \leqslant q < \infty, \quad \alpha_0 + \frac{1}{q} < 0, \quad \alpha_\infty + \frac{1}{q} > 0.$$

Other cases are left to the reader.

Assume (8.31). Then, by Lemma 3.7 (i), $\varphi_X(t) \approx \ell^{\mathbb{A}+1/q}(t)\ell\ell^{\mathbb{B}}(t)$ for $t \in (0,\infty)$. Consequently, $\varphi_X$ is on $(0,\infty)$ equivalent to an increasing concave function $\varphi$ such that $\varphi(0_+) = 0$ and $\varphi(\infty_-) = \infty$. Therefore, putting $\Phi(t) = 1/\varphi^{-1}(1/t)$, $t \in (0,\infty)$, we obtain (cf. (2.9)) that $\Phi$ is equivalent on $(0,\infty)$ to a Young function. Observe that

$$\Phi(t) \approx \begin{cases} \exp\big(-t^{-1/(\alpha_\infty+1/q)}\ell^{-\beta_\infty/(\alpha_\infty+1/q)}(t)\big) & \text{if} \quad t \in (0,1) \\ \exp\big(t^{-1/(\alpha_0+1/q)}\ell^{-\beta_0/(\alpha_0+1/q)}(t)\big) & \text{if} \quad t \in [1,\infty). \end{cases}$$

By Example 8.7, $\Phi$ satisfies (8.19). Hence, $f \in L_\Phi$ if and only if (8.20) holds.

We have to show that $X \neq L_\Phi$ (then the result will follow from Lemma 8.12). Note that the function $f(t) = 1/\varphi(t) = \Phi^{-1}(1/t)$, $t \in (0,\infty)$, satisfies (8.20), but

$$\|f\|_X^q \geqslant \int_0^1 t^{-1}\ell^{-1}(t)\,\mathrm{d}t = \infty.$$

The proof (in the case (8.31)) is thus complete. □

Now let us deal with the spaces $L_{(p,q;\mathbb{A},\mathbb{B})}$. Since $L_{(p,q;\mathbb{A},\mathbb{B})} = \{0\}$ for $0 < p < 1$, and $L_{(p,q;\mathbb{A},\mathbb{B})} = L_{p,q;\mathbb{A},\mathbb{B}}$ when $1 < p < \infty$, it is enough to consider the case $p = 1$. Moreover, using Theorem 7.3, Lemma 8.9 and Lemma 3.7 (ii), we see that we may restrict ourselves to the situation when $1 < q \leqslant \infty$ and one of the following conditions is satisfied:

$$(8.32) \qquad \begin{cases} \alpha_0 + \frac{1}{q} > 0,\ \alpha_\infty + \frac{1}{q} < 0; \\ \alpha_0 + \frac{1}{q} > 0,\ \alpha_\infty + \frac{1}{q} = 0,\ \beta_\infty + \frac{1}{q} < 0; \\ \alpha_0 + \frac{1}{q} = 0,\ \beta_0 + \frac{1}{q} > 0,\ \beta_\infty + \frac{1}{q} < 0, \\ \alpha_0 + \frac{1}{q} = 0,\ \beta_0 + \frac{1}{q} > 0,\ \alpha_\infty + \frac{1}{q} = 0,\ \beta_\infty + \frac{1}{q} < 0; \\ 1 < q < \infty,\ \alpha_0 + \frac{1}{q} = 0,\ \beta_0 + \frac{1}{q} = 0,\ \alpha_\infty + \frac{1}{q} < 0; \\ 1 < q < \infty,\ \alpha_0 + \frac{1}{q} = 0,\ \beta_0 + \frac{1}{q} = 0,\ \alpha_\infty + \frac{1}{q} = 0,\ \beta_\infty + \frac{1}{q} < 0. \end{cases}$$

8.15. THEOREM. *Let* $1 < q \leqslant \infty$ *and* $\mathbb{A} = (\alpha_0, \alpha_\infty)$, $\mathbb{B} = (\beta_0, \beta_\infty) \in \mathbb{R}^2$. *Put* $X = L_{(1,q;\mathbb{A},\mathbb{B})}$. *Let one of the conditions in* (8.32) *be satisfied. Then* $X$ *is not an Orlicz space.*

*Proof.* It follows from Theorem 6.7 (i), complemented with its analogue involving three tiers of logarithms, and Theorem 8.14, that $X'$ is not an Orlicz space. Therefore, neither is $X$. □

## 9. Absolute continuity of the norm

The aim of this section is to characterize those GLZ spaces $L_{p,q;\mathbb{A},\mathbb{B}}$ and $L_{(p,q;\mathbb{A},\mathbb{B})}$ that have absolutely continuous (quasi-)norm.

If $\mu(\mathscr{R}) < \infty$, then it is easy to see that the space $L_{p,q;\mathbb{A},\mathbb{B}}$ (or $L_{(p,q;\mathbb{A},\mathbb{B})}$) with $0 < q < \infty$ has absolutely continuous (quasi-)norm. Indeed, taking $f \in X = L_{p,q;\mathbb{A},\mathbb{B}}$ and $\{E_n\}_{n=1}^{\infty} \subset \mathscr{R}$ satisfying $E_n \searrow \emptyset$ $\mu$-a.e., we have $\mu(E_n) \leqslant \mu(\mathscr{R}) < \infty$ for all $n \in \mathbb{N}$, which implies

$$\lim_{n\to\infty} \mu(E_n) = \mu\Big(\bigcap_{n=1}^{\infty} E_n\Big) = \mu(\emptyset) = 0.$$

Thus, as $n \to \infty$,

$$(f\chi_{E_n})^* \leqslant f^*\chi_{(0,\mu(E_n))} \to 0,$$

and, by Lebesgue's Dominated Convergence Theorem,

$$\|f\chi_{E_n}\|_X \leqslant \Big(\int_0^{\infty} \Big[t^{\frac{1}{p}-\frac{1}{q}}\ell^{\mathbb{A}}(t)\ell\ell^{\mathbb{B}}(t)f^*(t)\chi_{(0,\mu(E_n))}(t)\Big]^q \,\mathrm{d}t\Big)^{1/q} \to 0, \qquad n \to \infty,$$

and the result follows.

However, when $\mu(\mathscr{R}) = \infty$, it may happen that $E_n \searrow \emptyset$ as $n \to \infty$ but $\mu(E_n) = \infty$ for every $n \in \mathbb{N}$. Then $\chi_{(0,\mu(E_n))} = \chi_{(0,\infty)}$, and we do not obtain $f^*\chi_{(0,\mu(E_n))} \to 0$ from the trivial estimate $(f\chi_{E_n})^* \leqslant f^*\chi_{(0,\mu(E_n))}$ as above.

In the case of infinite measure a deeper analysis is needed and we find the following lemmas useful in this.

9.1. LEMMA. *Let $f_n, f \in \mathscr{M}^+(\mathscr{R},\mu)$, $n \in \mathbb{N}$, be such that $\limsup_{n\to\infty} f_n \leqslant f$ $\mu$-a.e. Assume that there are $g \in \mathscr{M}^+(\mathscr{R},\mu)$ and $n_0 \in \mathbb{N}$ satisfying*

(9.1) $\quad g \geqslant f_n \quad \mu$*-a.e. for all* $\quad n \geqslant n_0$;

(9.2) $\quad \mu_g(\lambda) < \infty \quad$ *for all* $\quad \lambda \in [0,\infty)$.

*Then for all $\lambda \in [0,\infty)$,*

$$\limsup_{n\to\infty} \mu_{f_n}(\lambda) \leqslant \mu_f(\lambda). \tag{9.3}$$

*Proof.* For $\lambda \in [0,\infty)$ and $F \in \mathscr{M}(\mathscr{R},\mu)$ put

$$E_\lambda(F) = \big\{x \in \mathscr{R}; |F(x)| > \lambda\big\}.$$

Since ($\mu$-a.e.)

$$f(x) \geqslant \limsup_{n\to\infty} f_n(x) = \lim_{n\to\infty} h_n(x),$$

where $h_n(x) = \sup_{m\geqslant n} f_m(x)$, the function $h(x) := \lim_{n\to\infty} h_n(x)$ satisfies $h_n \searrow h$ and $h \leqslant f$. Consequently,

$$\mu_h(\lambda) \leqslant \mu_f(\lambda), \qquad \lambda \in [0,\infty). \tag{9.4}$$

Moreover, for all $\lambda \in [0,\infty)$,

$$E_\lambda(h) = \bigcap_{n=1}^{\infty} E_\lambda(h_n) = \bigcap_{n=1}^{\infty} \bigcup_{m\geqslant n} E_\lambda(f_m). \tag{9.5}$$

The assumptions (9.1) and (9.2) imply

$$\mu\Big(\bigcup_{m\geqslant n_0} E_\lambda(f_m)\Big) \leqslant \mu(E_\lambda(g)) = \mu_g(\lambda) < \infty.$$

Together with (9.5), this yields

$$\mu_h(\lambda) = \lim_{n\to\infty} \mu\Big(\bigcup_{m\geqslant n} E_\lambda(f_m)\Big).$$

Further, observe that

$$\mu\Big(\bigcup_{m\geqslant n} E_\lambda(f_m)\Big) \geqslant \sup_{k\geqslant n} \mu\Big(E_\lambda(f_k)\Big) = \sup_{k\geqslant n} \mu_{f_k}(\lambda).$$

Thus,

$$\mu_h(\lambda) \geqslant \lim_{n\to\infty} \sup_{k\geqslant n} \mu_{f_k}(\lambda) = \limsup_{n\to\infty} \mu_{f_n}(\lambda),$$

which, together with (9.4), implies (9.3). □

Combining Lemma 9.1 with a symmetric assertion, cf. [BS, Chapter 2, (1.5)], one can prove the following result.

9.2. LEMMA. *Let* $f_n, f \in \mathscr{M}^+(\mathscr{R},\mu)$, $n \in \mathbb{N}$, *be such that* $\lim_{n\to\infty} f_n = f$ $\mu$*-a.e. Assume that there are* $g \in \mathscr{M}^+(\mathscr{R},\mu)$ *and* $n_0 \in \mathbb{N}$ *such that* (9.1) *and* (9.2) *hold. Then for all* $\lambda \in [0,\infty)$,

$$\mu_f(\lambda) = \lim_{n\to\infty} \mu_{f_n}(\lambda).$$

9.3. LEMMA. *Let* $f_n, f \in \mathscr{M}^+(\mathscr{R},\mu)$, $n \in \mathbb{N}$, *be such that* $\limsup_{n\to\infty} f_n \leqslant f$ $\mu$*-a.e. Assume that there are* $g \in \mathscr{M}^+(\mathscr{R},\mu)$ *and* $n_0 \in \mathbb{N}$ *satisfying*

(9.6) $\quad g \geqslant f_n \quad \mu$*-a.e.* *for all* $\quad n \geqslant n_0$;

(9.7) $\quad g^*(t) < \infty \quad$ *for all* $\quad t \in (0,\infty)$;

(9.8) $\quad \lim_{t\to\infty} g^*(t) = 0.$

*Then*

$$\limsup_{n\to\infty} f_n^* \leqslant f^*. \tag{9.9}$$

*Proof.* Assuming that $\mu_g(\lambda_0) = \infty$ for some $\lambda_0 \in [0,\infty)$, we get for any $t > 0$,

$$g^*(t) = \inf\{\lambda; \mu_g(\lambda) \leqslant t\} \geqslant \lambda_0,$$

which contradicts (9.8). Consequently, $\mu_g(\lambda) < \infty$ for all $\lambda \in [0,\infty)$. Thus, by Lemma 9.1,

$$\limsup_{n\to\infty} \mu_{f_n} \leqslant \mu_f.$$

Let $m$ denote the Lebesgue measure on $(0,\infty)$. Applying Lemma 9.1 again, this time to $\{\mu_{f_n}\}$, $\mu_g$, $\mu_f$, and $m$ instead of $\{f_n\}$, $g$, $f$, and $\mu$, we get

$$\limsup_{n\to\infty} m_{\mu_{f_n}} \leqslant m_{\mu_f},$$

which is (9.9), since for any $h \in \mathscr{M}(\mathscr{R},\mu)$ we have $m_{\mu_h} = h^*$ (cf. [BS, Chapter 2, (1.10)]). □

Combining Lemma 9.3 with a symmetric assertion (cf. [BS, Chapter 2, (1.17)]), one can prove the following result.

9.4. LEMMA. *Let* $f_n, f \in \mathscr{M}^+(\mathscr{R},\mu)$, $n \in \mathbb{N}$, *be such that* $\lim_{n\to\infty} f_n = f$ $\mu$*-a.e. Assume that there are* $g \in \mathscr{M}^+(\mathscr{R},\mu)$ *and* $n_0 \in \mathbb{N}$ *such that* (9.6) – (9.8) *hold. Then* $\lim_{n\to\infty} f_n^* = f^*$.

Now we are in a position to prove the main result of this section.

9.5. THEOREM. *Let* $0 < p,q \leqslant \infty$, $\mathbb{A},\mathbb{B} \in \mathbb{R}^2$ *and let* $X$ *be one of the spaces* $L_{p,q;\mathbb{A},\mathbb{B}}$ *or* $L_{(p,q;\mathbb{A},\mathbb{B})}$. *Assume that* $X \neq \{0\}$. *Then* $X$ *has absolutely continuous (quasi-)norm if and only if* $0 < q < \infty$.

*Proof.* (i) Assume first that $X = L_{p,q;\mathbb{A},\mathbb{B}}$. Let $0 < q < \infty$. It is a consequence of (P3) (cf. Section 2) that $X$ has absolutely continuous (quasi-)norm if (ACN) holds for every $f \in X \cap \mathscr{M}^+(\mathscr{R},\mu)$. For such $f$ we clearly have $f^*(t) < \infty$, $t \in (0,\infty)$, and $\lim_{t\to\infty} f^*(t) = 0$. Let $\{E_n\} \subset \mathscr{R}$ satisfy $E_n \searrow \emptyset$ $\mu$-a.e. Putting $f_n = f\chi_{E_n}$, we have $0 \leqslant f_n \leqslant f$, $n \in \mathbb{N}$, and, by Lemma 9.4, $f_n^*(t) \to 0$ as $n \to \infty$ for all $t \in (0,\infty)$. Since $f_n^* \leqslant f^*$, $n \in \mathbb{N}$, the Lebesgue Dominated Convergence Theorem shows that

$$\|f\chi_{E_n}\|_{p,q;\mathbb{A},\mathbb{B}} = \Big(\int_0^\infty [t^{\frac{1}{p}-\frac{1}{q}}\ell^{\mathbb{A}}(t)\ell\ell^{\mathbb{B}}(t)f_n^*(t)]^q \,\mathrm{d}t\Big)^{1/q} \to 0.$$

Hence $X$ has absolutely continuous (quasi-)norm.

Let $q = \infty$. Since $X \neq \{0\}$, either $p < \infty$, or $p = \infty$, $\alpha_0 < 0$, or $p = \infty$, $\alpha_0 = 0$, $\beta_0 \leqslant 0$. Thus, there exists $n_0 \in \mathbb{N}$ such that the function $h(t) = t^{-1/p}\ell^{-\alpha_0}(t)\ell\ell^{-\beta_0}(t)\chi_{(0,1/n_0)}(t)$, $t \in (0,\infty)$, is non-increasing on $(0,\infty)$. By [BS, Chapter 2, Corollary 7.8], there is a $g \in \mathscr{M}(\mathscr{R},\mu)$ satisfying $g^* = h$. Furthermore, by [BS, Chapter 2, Corollary 7.6], there is a measure-preserving transformation $\sigma$ from the set $G = \operatorname{supp} g$ onto $[0,1/n_0] = \operatorname{supp} h$ such that $g = h\circ\sigma$ $\mu$–a.e. on $G$. Put $h_n = h\chi_{(0,1/n)}$ and $g_n = h_n \circ \sigma$ for $n \in \mathbb{N}$, $n \geqslant n_0$. Then (cf. [BS, Chapter 2, Proposition 7.2]) $g_n^* = h_n$. It is clear from the definition of $g_n$ that the sequence $\{E_n\}$ of $\mu$–measurable subsets of $\mathscr{R}$ given by $E_n := \operatorname{supp} g_n$, $n \geqslant n_0$, satisfies $E_{n+1} \subset E_n \subset G$ and $g_n = g\chi_{E_n}$ $\mu$–a.e. on $G$. Since $\mu(E_n) = |(0,1/n)| = 1/n$ (where $|E|$ denotes the usual Lebesgue measure of a set $E$), we see that $E_n \searrow \emptyset$ as $n \to \infty$. Moreover,

$$\|g\chi_{E_n}\|_X = \|g_n\|_{p,\infty;\mathbb{A},\mathbb{B}} = \sup_{0<t<1/n} t^{1/p}\ell^{\mathbb{A}}(t)\ell\ell^{\mathbb{B}}(t)h(t) = \sup_{0<t<1/n} 1 = 1$$

for every $n \geqslant n_0$. Since $g \in X$, the space $X$ does not have absolutely continuous (quasi-)norm.

(ii) Now let $X = L_{(p,q;\mathbb{A},\mathbb{B})}$. Since $X \neq \{0\}$, one of the conditions in (3.3) is satisfied. In particular, $1 \leqslant p \leqslant \infty$. By Theorem 3.8 (i), $X = L_{p,q;\mathbb{A},\mathbb{B}}$ if $1 < p \leqslant \infty$. This case is covered by part (i). It remains to consider the case when $p = 1$.

Let $0 < q < \infty$ and $f \in X = L_{(1,q;\mathbb{A},\mathbb{B})} \cap \mathscr{M}^+(\mathscr{R},\mu)$. For such $f$ we clearly have $f^{**}(t) < \infty$, $t \in (0,\infty)$, and $\lim_{t\to\infty} f^{**}(t) = 0$, which in turn implies $f^*(t) < \infty$ for $t \in (0,\infty)$, and $\lim_{t\to\infty} f^*(t) = 0$. Let $\{E_n\} \subset \mathscr{R}$ satisfy $E_n \searrow \emptyset$ $\mu$–a.e. Define $f_n$ as in part (i). Since $f_n^* \leqslant f^*$, $n \in \mathbb{N}$, and $f_n^*(t) \to 0$ as $n \to \infty$ for all $t \in (0,\infty)$, the Lebesgue Dominated Convergence Theorem implies that for every $s \in (0,\infty)$,

$$f_n^{**}(s) = s^{-1} \int_0^s f_n^*(t)\,\mathrm{d}\,t \to 0.$$

Since moreover $f_n^{**} \leqslant f^{**}$, $n \in \mathbb{N}$, one more application of the Lebesgue Dominated Convergence Theorem shows that

$$\|f\chi_{E_n}\|_{(1,q;\mathbb{A},\mathbb{B})} = \Big( \int_0^\infty [s^{1-\frac{1}{q}} \ell^{\mathbb{A}}(s) \ell\ell^{\mathbb{B}}(s) f_n^{**}(s)]^q \,\mathrm{d}\,s \Big)^{1/q} \to 0 \qquad \text{as} \quad n \to \infty.$$

Finally, let $p = 1$ and $q = \infty$. Assume first that

(9.10) $$\text{either} \qquad \alpha_0 > 0$$

(9.11) $$\text{or} \qquad \alpha_0 = 0 \quad \text{and} \quad \beta_0 > 0.$$

If (9.10) holds, then there is a $n_0 \in \mathbb{N}$ such that the function

(9.12) $$h(t) = t^{-1} \ell^{-\alpha_0 - 1}(t) \ell\ell^{-\beta_0}(t) \chi_{(0,1/n_0)}(t), \qquad t \in (0,\infty),$$

is non-increasing on $(0,\infty)$. The same argument as the one used in part (i) above implies that there is a $g \in \mathscr{M}(\mathscr{R},\mu)$ and a sequence $\{E_n\} \subset \mathscr{R}$ with $E_n \searrow \emptyset$ as $n \to \infty$ such that the functions $g_n := g\chi_{E_n}$ satisfy $g_n^* = h_n := h\chi_{(0,1/n)}$, $n \geqslant n_0$. Hence, for all $n \geqslant n_0$ and every $t \in (0, 1/n)$,

$$g_n^{**}(t) = \frac{1}{t} \int_0^t h(s)\,\mathrm{d}\,s \approx \frac{1}{t} \ell^{-\alpha_0}(t) \ell\ell^{-\beta_0}(t).$$

Consequently,

$$\|g\chi_{E_n}\|_X = \|g_n\|_{(1,\infty;\mathbb{A},\mathbb{B})} \geqslant \sup_{0<t<1/n} t \ell^{\mathbb{A}}(t) \ell\ell^{\mathbb{B}}(t) g_n^{**}(t) \approx 1$$

for every $n \geqslant n_0$. Since $g \in X$, the space $X$ does not have absolutely continuous (quasi-)norm.

When (9.11) is satisfied, we use the same argument as above with (9.12) replaced by

$$h(t) = t^{-1} \ell^{-1}(t) \ell\ell^{-\beta_0 - 1}(t) \chi_{(0,1/n_0)}(t), \qquad t \in (0,\infty),$$

and the result follows again.

Assume now that either $\alpha_0 < 0$ or $\alpha_0 = 0$ and $\beta_0 \leqslant 0$. Then, by Corollary 3.12, $X = L_{(1,\infty;\mathbb{A},\mathbb{B})} = L_{(1,\infty;(0,\alpha_\infty),(0,\beta_\infty))}$. The same argument as above with (9.12) replaced by $h = \chi_{(0,1)}$ shows that $L_{(1,\infty;(0,\alpha_\infty),(0,\beta_\infty))}(= X)$ does not have absolutely continuous (quasi-)norm. □

## 10. Appendix

In this section we prove Theorems 5.1, 5.2 and 5.5. To derive conditions for the embedding

$$L_{(P_1,Q;\mathbb{L},\mathbb{E})} \hookrightarrow L_{(P_2,R;\mathbb{S},\mathbb{W})}, \tag{10.1}$$

we need the following lemma.

10.1. LEMMA. *Let* $0 < q \leqslant \infty$, $\mu(\mathscr{R}) = \infty$, $\mathbb{A} = (\alpha_0, \alpha_\infty)$, $\mathbb{B} = (\beta_0, \beta_\infty) \in \mathbb{R}^2$, *and* $L_{(1,q;\mathbb{A},\mathbb{B})} \neq \{0\}$. *Then, for any* $\varrho \in (0,\infty)$, *there exists a function* $g_\varrho \in \mathscr{M}(\mathscr{R},\mu)$ *(independent of* $q, \mathbb{A}$, *and* $\mathbb{B}$ *) such that for all* $\varrho \in (0,1)$,

$$\|g_\varrho\|_{(1,q;\mathbb{A},\mathbb{B})} \approx \begin{cases} \varrho \ell^{\alpha_0+\frac{1}{q}}(\varrho)\ell\ell^{\beta_0}(\varrho) & \text{if } \alpha_0+\frac{1}{q} > 0 \\ \varrho \ell\ell^{\beta_0+\frac{1}{q}}(\varrho) & \text{if } \alpha_0+\frac{1}{q} = 0,\ \beta_0+\frac{1}{q} > 0 \\ \varrho \ell\ell\ell^{\frac{1}{q}}(\varrho) & \text{if } \alpha_0+\frac{1}{q} = 0,\ \beta_0+\frac{1}{q} = 0 \\ \varrho & \text{if } \alpha_0+\frac{1}{q} = 0,\ \beta_0+\frac{1}{q} < 0 \\ & \quad \text{or } \alpha_0+\frac{1}{q} < 0 \end{cases}$$

*and for all* $\varrho \in (1,\infty)$,

$$\|g_\varrho\|_{(1,q;\mathbb{A},\mathbb{B})} \approx \begin{cases} \varrho \ell^{\alpha_\infty+\frac{1}{q}}(\varrho)\ell\ell^{\beta_\infty}(\varrho) & \text{if } \alpha_\infty+\frac{1}{q} < 0 \\ \varrho \ell\ell^{\beta_\infty+\frac{1}{q}}(\varrho) & \text{if } \alpha_\infty+\frac{1}{q} = 0,\ \beta_\infty+\frac{1}{q} < 0 \\ \varrho & \text{if } q = \infty,\ \alpha_\infty = \beta_\infty = 0. \end{cases}$$

*Proof.* Put $Y = L_{(1,q;\mathbb{A},\mathbb{B})}$. Since the underlying measure space $(\mathscr{R},\mu)$ is non-atomic, for any $\varrho \in (0,\mu(\mathscr{R}))$ there exists a function $g_\varrho \in \mathscr{M}(\mathscr{R},\mu)$ such that $g_\varrho^* = \chi_{(0,\varrho)}$, and hence $\|g_\varrho\|_Y = \varphi_Y(\varrho)$, where $\varphi_y$ stands for the fundamental function of $Y$. Thus, Lemma 10.1 is a consequence of Theorem 3.7 (ii). □

The next assertion provides us with conditions on the second components of vector exponents of logarithmic functions which are necessary for the embedding (10.1).

10.2. LEMMA. *Let* $0 < Q, R \leqslant \infty$, $\mu(\mathscr{R}) = \infty$, $L_{(1,Q;\mathbb{L},\mathbb{E})} \neq \{0\}$, *and*

$$L_{(1,Q;\mathbb{L},\mathbb{E})} \hookrightarrow L_{(1,R;\mathbb{S},\mathbb{W})}. \tag{10.2}$$

*Then one of the following conditions is satisfied:*

$$\lambda_\infty + \frac{1}{Q} > \sigma_\infty + \frac{1}{R};$$
$$\lambda_\infty + \frac{1}{Q} = \sigma_\infty + \frac{1}{R} < 0, \quad \varepsilon_\infty \geqslant \omega_\infty;$$
$$\lambda_\infty + \frac{1}{Q} = \sigma_\infty + \frac{1}{R} = 0, \quad \varepsilon_\infty + \frac{1}{Q} \geqslant \omega_\infty + \frac{1}{R}.$$

*Proof.* By our assumption $L_{(1,Q;\mathbb{L},\mathbb{E})} \neq \{0\}$, and hence by (10.2), $L_{(1,R;\mathbb{S},\mathbb{W})} \neq \{0\}$. Thus, Lemma 3.5 (ii) implies that one of the following conditions is satisfied:

$$\text{(10.3)} \qquad \begin{cases} \lambda_\infty + \frac{1}{Q} < 0; \\ \lambda_\infty + \frac{1}{Q} = 0,\ \varepsilon_\infty + \frac{1}{Q} < 0; \\ Q = \infty, \quad \lambda_\infty = 0, \quad \varepsilon_\infty = 0 \end{cases}$$

and also that one of the following conditions is satisfied:

$$\text{(10.4)} \qquad \begin{cases} \sigma_\infty + \frac{1}{R} < 0; \\ \sigma_\infty + \frac{1}{R} = 0, \quad \omega_\infty + \frac{1}{R} < 0; \\ R = \infty, \quad \sigma_\infty = 0, \quad \omega_\infty = 0. \end{cases}$$

By Lemma 10.1, there is a function $g_\varrho \in \mathscr{M}(\mathscr{R},\mu)$ such that for all $\varrho \in (1,\infty)$,

$$\text{(10.5)} \qquad \|g_\varrho\|_{(1,Q;\mathbb{L},\mathbb{E})} \approx \begin{cases} \varrho \ell^{\lambda_\infty + \frac{1}{Q}}(\varrho)\ell\ell^{\varepsilon_\infty}(\varrho) & \text{if } \lambda_\infty + \frac{1}{Q} < 0 \\ \varrho \ell\ell^{\varepsilon_\infty + \frac{1}{Q}}(\varrho) & \text{if } \lambda_\infty + \frac{1}{Q} = 0,\ \varepsilon_\infty + \frac{1}{Q} < 0 \\ \varrho & \text{if } Q = \infty,\ \lambda_\infty = \varepsilon_\infty = 0, \end{cases}$$

and similarly,

$$\text{(10.6)} \qquad \|g_\varrho\|_{(1,R;\mathbb{S},\mathbb{W})} \approx \begin{cases} \varrho \ell^{\sigma_\infty + \frac{1}{R}}(\varrho)\ell\ell^{\omega_\infty}(\varrho) & \text{if } \sigma_\infty + \frac{1}{R} < 0 \\ \varrho \ell\ell^{\omega_\infty + \frac{1}{R}}(\varrho) & \text{if } \sigma_\infty + \frac{1}{R} = 0,\ \omega_\infty + \frac{1}{R} < 0 \\ \varrho & \text{if } R = \infty,\ \sigma_\infty = \omega_\infty = 0. \end{cases}$$

Assume that

$$\text{(10.7)} \qquad \lambda_\infty + \frac{1}{Q} < \sigma_\infty + \frac{1}{R}.$$

Then

$$\text{(10.8)} \qquad \lambda_\infty + \frac{1}{Q} < 0$$

and we get from (10.2), (10.5), and (10.6) that for all $\varrho \in (1,\infty)$,

$$\varrho \ell^{\lambda_\infty + \frac{1}{Q}}(\varrho)\ell\ell^{\varepsilon_\infty}(\varrho) \gtrsim \begin{cases} \varrho \ell^{\sigma_\infty + \frac{1}{R}}(\varrho)\ell\ell^{\omega_\infty}(\varrho) & \text{if } \sigma_\infty + \frac{1}{R} < 0 \\ \varrho \ell\ell^{\omega_\infty + \frac{1}{R}}(\varrho) & \text{if } \sigma_\infty + \frac{1}{R} = 0,\ \omega_\infty + \frac{1}{R} < 0 \\ \varrho & \text{if } R = \infty,\ \sigma_\infty = \omega_\infty = 0, \end{cases}$$

which contradicts (10.7) and (10.8). Consequently $\lambda_\infty + \frac{1}{Q} \geqslant \sigma_\infty + \frac{1}{R}$.

Let $\lambda_\infty + \frac{1}{Q} = \sigma_\infty + \frac{1}{R} < 0$. Then (10.2), (10.5), and (10.6) imply that for all $\varrho \in (1,\infty)$,

$$\varrho \ell^{\lambda_\infty + \frac{1}{Q}}(\varrho) \ell\ell^{\varepsilon_\infty}(\varrho) \gtrsim \varrho \ell^{\sigma_\infty + \frac{1}{R}}(\varrho) \ell\ell^{\omega_\infty}(\varrho),$$

which yields $\varepsilon_\infty \geqslant \omega_\infty$.

Finally, let $\lambda_\infty + \frac{1}{Q} = \sigma_\infty + \frac{1}{R} = 0$. Assume that

$$\varepsilon_\infty + \frac{1}{Q} < \omega_\infty + \frac{1}{R}. \tag{10.9}$$

Then

$$\varepsilon_\infty + \frac{1}{Q} < 0 \tag{10.10}$$

and we get from (10.2), (10.5), and (10.6) that for all $\varrho \in (1,\infty)$,

$$\varrho \ell\ell^{\varepsilon_\infty + \frac{1}{Q}}(\varrho) \gtrsim \begin{cases} \varrho \ell\ell^{\omega_\infty + \frac{1}{R}}(\varrho) & \text{if } \omega_\infty + \frac{1}{R} < 0 \\ \varrho & \text{if } R = \infty,\ \omega_\infty = 0, \end{cases}$$

which contradicts (10.9) and (10.10). Consequently, $\varepsilon_\infty + \frac{1}{Q} \geqslant \omega_\infty + \frac{1}{R}$, and the proof is complete. □

Now, we are going to derive conditions on the first components of vector exponents of logarithmic functions which are necessary for the embedding (10.2).

10.3. LEMMA. *Let* $0 < Q, R \leqslant \infty$, $\mu(\mathscr{R}) = \infty$, $L_{(1,Q;\mathbb{L},\mathbb{E})} \neq \{0\}$, *and*

$$L_{(1,Q;\mathbb{L},\mathbb{E})} \hookrightarrow L_{(1,R;\mathbb{S},\mathbb{W})}. \tag{10.11}$$

*Then the following implications hold:*

$$0 < \max\{\lambda_0 + \frac{1}{Q},\ \sigma_0 + \frac{1}{R}\} \implies \lambda_0 + \frac{1}{Q} \geqslant \sigma_0 + \frac{1}{R}; \tag{10.12}$$

$$0 = \lambda_0 + \frac{1}{Q} \implies \lambda_0 + \frac{1}{Q} \geqslant \sigma_0 + \frac{1}{R}; \tag{10.13}$$

$$0 < \lambda_0 + \frac{1}{Q} = \sigma_0 + \frac{1}{R} \implies \varepsilon_0 \geqslant \omega_0; \tag{10.14}$$

$$\left.\begin{aligned} 0 = \lambda_0 + \frac{1}{Q} = \sigma_0 + \frac{1}{R} \\ 0 \leqslant \max\{\varepsilon_0 + \frac{1}{Q},\ \omega_0 + \frac{1}{Q}\} \end{aligned}\right\} \implies \varepsilon_0 + \frac{1}{Q} \geqslant \omega_0 + \frac{1}{R}. \tag{10.15}$$

*Proof.* Let $g_\varrho \in \mathscr{M}(\mathscr{R},\mu)$ be the function from Lemma 10.1.

a) Let $0 < \lambda_0 + \frac{1}{Q}$. Suppose that

$$\lambda_0 + \frac{1}{Q} < \sigma_0 + \frac{1}{R}. \tag{10.16}$$

Then for all $\varrho \in (0,1)$,

$$\|g_\varrho\|_{(1,Q;\mathbb{L},\mathbb{E})} \approx \varrho \ell^{\lambda_0+\frac{1}{Q}}(\varrho)\ell\ell^{\varepsilon_0}(\varrho), \tag{10.17}$$

$$\|g_\varrho\|_{(1,R;\mathbb{S},\mathbb{W})} \approx \varrho \ell^{\sigma_0+\frac{1}{R}}(\varrho)\ell\ell^{\omega_0}(\varrho), \tag{10.18}$$

and thus the embedding (10.11) shows that the inequality (10.16) cannot hold.

b) Let $\sigma_0 + \frac{1}{R} > 0$. Suppose that

$$\lambda_0 + \frac{1}{Q} \leqslant 0. \tag{10.19}$$

Then for all $\varrho \in (0,1)$,

$$\|g_\varrho\|_{(1,Q;\mathbb{L},\mathbb{E})} \approx \begin{cases} \varrho \ell\ell^{\varepsilon_0+\frac{1}{Q}}(\varrho) & \text{if } \lambda_0 + \frac{1}{Q} = 0,\ \varepsilon_0 + \frac{1}{Q} > 0 \\ \varrho \ell\ell\ell^{\frac{1}{Q}}(\varrho) & \text{if } \lambda_0 + \frac{1}{Q} = 0,\ \varepsilon_0 + \frac{1}{Q} = 0 \\ \varrho & \text{if } \lambda_0 + \frac{1}{Q} = 0,\ \varepsilon_0 + \frac{1}{Q} < 0 \\ & \quad \text{or } \lambda_0 + \frac{1}{Q} < 0 \end{cases} \tag{10.20}$$

and

$$\|g_\varrho\|_{(1,R;\mathbb{S},\mathbb{W})} \approx \varrho \ell^{\sigma_0+\frac{1}{R}}(\varrho)\ell\ell^{\omega_0}(\varrho).$$

These two estimates and the embedding (10.11) show that (10.19) cannot hold. Thus $\lambda_0 + \frac{1}{Q} > 0$ and, by part a), $\lambda_0 + \frac{1}{Q} \geqslant \sigma_0 + \frac{1}{R}$.

Now, (10.12) follows from a) and b).

c) Let $0 = \lambda_0 + \frac{1}{Q}$. Assume that (10.16) holds. Then $\sigma_0 + \frac{1}{R} > 0$, and hence the estimates (10.18), (10.20), and the embedding (10.11) show that (10.16) cannot hold. This proves (10.13).

d) Let $0 < \lambda_0 + \frac{1}{Q} = \sigma_0 + \frac{1}{R}$. Then the estimates (10.17), (10.18) and the embedding (10.11) imply that $\varepsilon_0 \geqslant \omega_0$. Consequently, the implication (10.14) holds.

e) Let $0 = \lambda_0 + \frac{1}{Q} = \sigma_0 + \frac{1}{R}$ and $\varepsilon_0 + \frac{1}{Q} \geqslant 0$. Assume that

$$\varepsilon_0 + \frac{1}{Q} < \omega_0 + \frac{1}{R}. \tag{10.21}$$

Then $\omega_0 + \frac{1}{R} > 0$. We have for all $\varrho \in (0,1)$,

$$\|g_\varrho\|_{(1,Q;\mathbb{L},\mathbb{E})} \approx \begin{cases} \varrho \ell\ell^{\varepsilon_0+\frac{1}{Q}}(\varrho) & \text{if } \varepsilon_0 + \frac{1}{Q} > 0 \\ \varrho \ell\ell\ell^{\frac{1}{Q}}(\varrho) & \text{if } \varepsilon_0 + \frac{1}{Q} = 0 \end{cases}$$

and

$$\|g_\varrho\|_{(1,R;\mathbb{S},\mathbb{W})} \approx \varrho \ell\ell^{\omega_0+\frac{1}{R}}(\varrho).$$

The last two estimates and the embedding (10.11) show that (10.21) cannot hold.

f) Let $0 = \lambda_0 + \frac{1}{Q} = \sigma_0 + \frac{1}{R}$ and $\omega_0 + \frac{1}{R} \geqslant 0$. Assume that

$$\varepsilon_0 + \frac{1}{Q} < 0. \tag{10.22}$$

Then, for all $\varrho \in (0,1)$,

$$\|g_\varrho\|_{(1,Q;\mathbb{L},\mathbb{E})} \approx \varrho,$$
$$\|g_\varrho\|_{(1,R;\mathbb{S},\mathbb{W})} \approx \begin{cases} \varrho \ell\ell^{\omega_0+\frac{1}{R}}(\varrho) & \text{if } \omega_0 + \frac{1}{R} > 0 \\ \varrho \ell\ell\ell^{\frac{1}{R}}(\varrho) & \text{if } \omega_0 + \frac{1}{R} = 0 \end{cases}$$

and the embedding (10.11) shows that (10.22) cannot hold. Thus $\varepsilon_0 + \frac{1}{Q} \geqslant 0$ and, by part e), $\varepsilon_0 + \frac{1}{Q} \geqslant \omega_0 + \frac{1}{R}$.

We have from e) and f) that the implication (10.15) is satisfied. □

In the next lemma we consider the case when $0 < R < Q < \infty$.

10.4. LEMMA. *Let* $0 < R < Q \leqslant \infty$, $\mu(\mathscr{R}) = \infty$,

$$L_{(1,Q;\mathbb{L},\mathbb{E})} \neq \{0\}, \tag{10.23}$$

*and*

$$L_{(1,Q;\mathbb{L},\mathbb{E})} \hookrightarrow L_{(1,R;\mathbb{S},\mathbb{W})}. \tag{10.24}$$

*Then either*

$$\lambda_\infty + \frac{1}{Q} > \sigma_\infty + \frac{1}{R} \tag{10.25}$$

*or*

$$\lambda_\infty + \frac{1}{Q} = \sigma_\infty + \frac{1}{R} \quad \text{and} \quad \varepsilon_\infty + \frac{1}{Q} > \omega_\infty + \frac{1}{R}. \tag{10.26}$$

*Moreover, the following conditions hold:*

$$0 < \max\{\lambda_0 + \frac{1}{Q}, \sigma_0 + \frac{1}{R}\} \implies \lambda_0 + \frac{1}{Q} \geqslant \sigma_0 + \frac{1}{R}; \tag{10.27}$$

$$0 = \lambda_0 + \frac{1}{Q} \implies \lambda_0 + \frac{1}{Q} \geqslant \sigma_0 + \frac{1}{R}; \tag{10.28}$$

$$0 < \lambda_0 + \frac{1}{Q} = \sigma_0 + \frac{1}{R} \implies \varepsilon_0 + \frac{1}{Q} > \omega_0 + \frac{1}{R}; \tag{10.29}$$

$$\left.\begin{array}{r} 0 = \lambda_0 + \frac{1}{Q} = \sigma_0 + \frac{1}{R} \\ 0 < \max\{\varepsilon_0 + \frac{1}{Q},\ \omega_0 + \frac{1}{Q}\} \end{array}\right\} \implies \varepsilon_0 + \frac{1}{Q} > \omega_0 + \frac{1}{R}. \tag{10.30}$$

*Proof.* The implications (10.27) and (10.28) follow from Lemma 10.3.

For $\varrho \in (0,1)$ and $\nu, \theta, \eta \in \mathbb{R}$ put

$$f_\varrho(t) = t^{-1}\ell^{\nu}(t)\ell\ell^{\theta}(t)\ell\ell\ell^{\eta}(t)\chi_{(0,\varrho)}(t), \quad t \in (0,\infty). \tag{10.31}$$

Then $f_\varrho$ is equivalent to a non-increasing function on $(0,\infty)$ and, as the measure space $(\mathscr{R},\mu)$ is non-atomic, there is a function $g_\varrho \in \mathscr{M}(\mathscr{R},\mu)$ such that $g_\varrho^* \approx f_\varrho$ (cf. [BS, Chapter 2, Corollary 7.8]). Consequently,

$$g_\varrho^{**}(t) \approx \frac{1}{t}\int_0^t f_\varrho(s)ds, \quad t \in (0,\infty). \tag{10.32}$$

If $\nu + 1 < 0$ and $\varrho \in (0,1)$ are *fixed*, then

$$g_\varrho^{**}(t) \approx \begin{cases} t^{-1}\ell^{\nu+1}(t)\ell\ell^{\theta}(t)\ell\ell\ell^{\eta}(t), & t \in (0,\varrho] \\ t^{-1}\ell^{\nu+1}(\varrho)\ell\ell^{\theta}(\varrho)\ell\ell\ell^{\eta}(\varrho), & t \in (\varrho,\infty). \end{cases}$$

This implies that for all $t \in (0,\infty)$,

$$g_\varrho^{**}(t) \approx t^{-1}\ell^{\nu+1}(t)\ell\ell^{\theta}(t)\ell\ell\ell^{\eta}(t)\chi_{(0,\varrho]}(t) + t^{-1}\chi_{(\varrho,\infty)}(t). \tag{10.33}$$

a) Let $0 < \lambda_0 + \frac{1}{Q} = \sigma + \frac{1}{R}$. Suppose that

$$\varepsilon_0 + \frac{1}{Q} \leqslant \omega_0 + \frac{1}{R}. \tag{10.34}$$

Taking

$$\nu{+}1 = -\Big(\lambda_0{+}\frac{1}{Q}\Big) = -\Big(\sigma_0{+}\frac{1}{R}\Big),\ \theta \in \Big[-\Big(\omega_0{+}\frac{1}{R}\Big), -\Big(\varepsilon_0{+}\frac{1}{Q}\Big)\Big],\ \eta = -\frac{1}{2}\Big(\frac{1}{Q}{+}\frac{1}{R}\Big),$$

we have $\nu + 1 < 0$ and, on using (10.33), we get

$$\|g_\varrho\|_{(1,Q;\mathbb{L},\mathbb{E})} \lesssim N_1(\varrho) + N_2(\varrho),$$

where

$$N_1(\varrho) = \|t^{-\frac{1}{Q}}\ell^{\lambda_0+\nu+1}(t)\ell\ell^{\varepsilon_0+\theta}(t)\ell\ell\ell^{\eta}(t)\|_{Q,(0,\varrho)},$$
$$N_2(\varrho) = \|t^{-\frac{1}{Q}}\ell^{\mathbb{L}}(t)\ell\ell^{\mathbb{E}}(t)\|_{Q,(\varrho,\infty)}.$$

Since

$$\lambda_0 + \frac{1}{Q} + \nu + 1 = 0, \quad \varepsilon_0 + \frac{1}{Q} + \theta \leqslant 0, \quad \eta + \frac{1}{Q} = \frac{1}{2}\Big(\frac{1}{Q} - \frac{1}{R}\Big) < 0,$$

we have $N_1(\varrho) < \infty$. Moreover, the assumption (10.23) implies that one of conditions (10.3) is satisfied, which in turn yields that $N_2(\varrho) < \infty$. Consequently,

$$\|g_\varrho\|_{(1,Q;\mathbb{L},\mathbb{E})} < \infty. \tag{10.35}$$

On the other hand, we have from (10.33) that

$$\|g_\varrho\|_{(1,R;\mathbb{S},\mathbb{W})} \gtrsim \|t^{-\frac{1}{R}}\ell^{\sigma_0+\nu+1}(t)\ell\ell^{\omega_0+\theta}(t)\ell\ell\ell^{\eta}(t)\|_{R,(0,\varrho)} = \infty \tag{10.36}$$

since

$$\sigma_0 + \frac{1}{R} + \nu + 1 = 0, \quad \omega_0 + \frac{1}{R} + \theta \geqslant 0, \quad \eta + \frac{1}{R} = \frac{1}{2}\Big(\frac{1}{R} - \frac{1}{Q}\Big) > 0.$$

The estimates (10.35) and (10.36) contradict (10.24). Thus, (10.34) cannot hold and the implication (10.29) is verified.

b) Let $0 = \lambda_0 + \frac{1}{Q} = \sigma_0 + \frac{1}{R}$ and $\varepsilon_0 + \frac{1}{Q} \geqslant 0$. Suppose that (10.34) holds. In (10.31) we take

$$\nu + 1 = 0, \quad \theta + 1 \in \Big[ -\Big(\omega_0 + \frac{1}{R}\Big), -\Big(\varepsilon_0 + \frac{1}{Q}\Big)\Big],$$

$$\eta = \begin{cases} -\frac{1}{2}\Big(\frac{1}{Q} + \frac{1}{R}\Big) & \text{if } \theta + 1 < 0 \\ -\frac{1}{2}\Big(\frac{1}{Q} + \frac{1}{R}\Big) - 1 & \text{if } \theta + 1 = 0. \end{cases}$$

Then $\theta + 1 \leqslant 0$ and it may be $\theta + 1 = 0$ only if $\varepsilon_0 + \frac{1}{Q} = 0$. Using (10.32), we get

$$g_\varrho^{**}(t) \approx t^{-1}\ell\ell^{\theta+1}(t)\ell\ell\ell^{\eta}(t)\chi_{(0,\varrho]}(t) + t^{-1}\ell\ell^{\theta+1}(\varrho)\ell\ell\ell^{\eta}(\varrho)\chi_{(\varrho,\infty)}(t) \ \text{ if } \theta + 1 < 0,$$
$$g_\varrho^{**}(t) \approx t^{-1}\ell\ell\ell^{\eta+1}(t)\chi_{(0,\varrho]}(t) + t^{-1}\ell\ell\ell^{\eta+1}(\varrho)\chi_{(\varrho,\infty)}(t) \quad \text{if } \theta + 1 = 0.$$

One can prove analogously as in part a) that

$$\|g_\varrho\|_{(1,Q;\mathbb{L},\mathbb{E})} < \infty \quad \text{and} \quad \|g_\varrho\|_{(1,R;\mathbb{S},\mathbb{W})} = \infty \tag{10.37}$$

since $\lambda_0 + \frac{1}{Q} = 0, \quad \sigma_0 + \frac{1}{R} = 0$, and

$$\theta + 1 < 0 \Longrightarrow \begin{cases} \varepsilon_0 + \frac{1}{Q} + \theta + 1 \leqslant 0, \quad \eta + \frac{1}{Q} = \frac{1}{2}\Big(\frac{1}{Q} - \frac{1}{R}\Big) < 0, \\ \omega_0 + \frac{1}{R} + \theta + 1 \geqslant 0, \quad \eta + \frac{1}{R} = \frac{1}{2}\Big(\frac{1}{R} - \frac{1}{Q}\Big) > 0, \end{cases}$$

$$\theta + 1 = 0 \Longrightarrow \begin{cases} \varepsilon_0 + \frac{1}{Q} = 0, \quad \eta + 1 + \frac{1}{Q} = \frac{1}{2}\Big(\frac{1}{Q} - \frac{1}{R}\Big) < 0, \\ \omega_0 + \frac{1}{R} \geqslant 0, \quad \eta + 1 + \frac{1}{R} = \frac{1}{2}\Big(\frac{1}{R} - \frac{1}{Q}\Big) > 0. \end{cases}$$

However, (10.37) contradicts (10.24). Thus (10.34) cannot hold and consequently $\varepsilon_0 + \frac{1}{Q} > \omega_0 + \frac{1}{R}$.

c) Let $0 = \lambda_0 + \frac{1}{Q} = \sigma_0 + \frac{1}{R}$ and $\omega_0 + \frac{1}{R} \geqslant 0$. Then we have from (10.15) of Lemma 10.3 that $\varepsilon_0 + \frac{1}{Q} \geqslant 0$. Consequently, by part b), $\varepsilon_0 + \frac{1}{Q} > \omega_0 + \frac{1}{R}$.

We have from b) and c) that the implication (10.30) is satisfied.

It remains to prove that either (10.25) or (10.26) holds. However, it follows from Lemma 10.2 that we only need to show

$$\lambda_\infty + \frac{1}{Q} = \sigma_\infty + \frac{1}{R} \quad \Longrightarrow \quad \varepsilon_\infty + \frac{1}{Q} > \omega_\infty + \frac{1}{R}. \tag{10.38}$$

For $\nu_0, \nu_\infty, \theta_0, \theta_\infty, \eta \in \mathbb{R}$ put

$$f(t) = \begin{cases} t^{-1}\ell^{\nu_0}(t)\ell\ell^{\theta_0}(t), & t \in (0,1] \\ t^{-1}\ell^{\nu_\infty}(t)\ell\ell^{\theta_\infty}(t)\ell\ell\ell^{\eta}(t), & t \in (1,\infty). \end{cases} \tag{10.39}$$

Then there exists $g \in \mathscr{M}(\mathscr{R},\mu)$ such that $g^* \approx f$. Consequently,

$$g^{**}(t) \approx \frac{1}{t}\int_0^t f(s)ds. \tag{10.40}$$

Taking $\nu_0 < \min\{-1, -1-\lambda_0-\frac{1}{Q}\}$, we have

$$g^{**}(t) \approx t^{-1}\ell^{\nu_0+1}(t)\ell\ell^{\theta_0}(t), \quad t \in (0,1], \tag{10.41}$$

and hence

$$\|g\|_{(1,Q;\mathbb{L},\mathbb{E})(0,1)} \approx \|t^{-\frac{1}{Q}}\ell^{\lambda_0+\nu_0+1}(t)\ell\ell^{\varepsilon_0+\theta_0}(t)\|_{Q,(0,1)} < \infty. \tag{10.42}$$

The assumption (10.23) implies that one of conditions in (10.3) holds. Moreover, assume that the premise in (10.38) is satisfied. We shall distinguish two cases:

1) Let $\lambda_\infty + \frac{1}{Q} = \sigma_\infty + \frac{1}{R} < 0$. Suppose that

$$\varepsilon_\infty + \frac{1}{Q} \leqslant \omega_\infty + \frac{1}{R}. \tag{10.43}$$

Take $1+\nu_\infty = -(\lambda_\infty + \frac{1}{Q}) = -(\sigma_\infty + \frac{1}{R})$. Then $1+\nu_\infty > 0$ and (10.39)–(10.41) yield

$$\begin{aligned} g^{**}(t) &\approx t^{-1}\Big[\int_0^1 f(s)ds + \int_1^t f(s)ds\Big] \\ &\approx t^{-1}\Big[1 + \ell^{\nu_\infty+1}(t)\ell\ell^{\theta_\infty}\ell\ell\ell^{\eta}(t)\Big] \\ &\approx t^{-1}\ell^{\nu_\infty+1}(t)\ell\ell^{\theta_\infty}(t)\ell\ell\ell^{\eta}(t), \qquad t \in (1,\infty). \end{aligned} \tag{10.44}$$

Furthermore, let

$$\theta_\infty \in \Big[-\Big(\omega_\infty + \frac{1}{R}\Big), -\Big(\varepsilon_\infty + \frac{1}{Q}\Big)\Big] \quad \text{and} \quad \eta = -\frac{1}{2}\Big(\frac{1}{Q}+\frac{1}{R}\Big).$$

We have from (10.44) that

$$\|g\|_{(1,Q;\mathbb{L},\mathbb{E})(1,\infty)} \approx \|t^{-\frac{1}{Q}}\ell^{\lambda_\infty+\nu_\infty+1}(t)\ell\ell^{\varepsilon_\infty+\theta_\infty}(t)\ell\ell\ell^{\eta}(t)\|_{Q,(1,\infty)} \approx 1 \tag{10.45}$$

since

$$\lambda_\infty + \frac{1}{Q} + \nu_\infty + 1 = 0, \quad \varepsilon_\infty + \frac{1}{Q} + \theta_\infty \leqslant 0, \quad \eta + \frac{1}{Q} = \frac{1}{2}\Big(\frac{1}{Q} - \frac{1}{R}\Big) < 0.$$

We see from (10.42) and (10.45) that

$$\|g\|_{(1,Q;\mathbb{L},\mathbb{E})} < \infty. \tag{10.46}$$

On the other hand, we have from (10.44) that

$$\|g\|_{(1,R;\mathbb{S},\mathbb{W})} \geqslant \|t^{-\frac{1}{R}}\ell^{\sigma_\infty+\nu_\infty+1}(t)\ell\ell^{\omega_\infty+\theta_\infty}(t)\ell\ell\ell^{\eta}(t)\|_{R,(1,\infty)} = \infty \tag{10.47}$$

since

$$\sigma_\infty + \frac{1}{R} + \nu_\infty + 1 = 0, \quad \omega_\infty + \frac{1}{R} + \theta_\infty \geqslant 0, \quad \eta + \frac{1}{R} = \frac{1}{2}\Big(\frac{1}{R} - \frac{1}{Q}\Big) > 0.$$

However, (10.46) and (10.47) contradict (10.24). Thus (10.43) cannot hold. Consequently, $\varepsilon_\infty + \frac{1}{Q} > \omega_\infty + \frac{1}{R}$.

2) Let $\lambda_\infty + \frac{1}{Q} = \sigma_\infty + \frac{1}{R} = 0$. Suppose that (10.43) holds. Take

$$\nu_\infty = -1, \quad \text{i.e.,} \quad 1 + \nu_\infty = -\Big(\lambda_\infty + \frac{1}{Q}\Big) = -\Big(\sigma_\infty + \frac{1}{R}\Big),$$
$$1 + \theta_\infty \in \Big[-\Big(\omega_\infty + \frac{1}{R}\Big), -\Big(\varepsilon_\infty + \frac{1}{Q}\Big)\Big], \quad \eta = -\frac{1}{2}\Big(\frac{1}{Q} + \frac{1}{R}\Big).$$

Our assumptions (10.23) and (10.24) imply that one of conditions in (10.4) holds. Together with the facts that $\sigma_\infty + \frac{1}{R} = 0$ and $R < \infty$ this yields $\omega_\infty + \frac{1}{R} < 0$, which in turn shows that $1 + \theta_\infty > 0$. Consequently, we have from (10.39)–(10.41) that

$$g^{**}(t) \approx t^{-1}\Big[\int_0^1 f(s)ds + \int_1^t f(s)ds\Big] \approx t^{-1}\Big[1 + \ell\ell^{\theta_\infty+1}(t)\ell\ell\ell^{\eta}(t)\Big]$$
$$\approx t^{-1}\ell\ell^{\theta_\infty+1}(t)\ell\ell\ell^{\eta}(t), \quad t \in (1,\infty).$$

Similar argument to that of part 1) shows that (10.43) cannot hold. Thus $\varepsilon_\infty + \frac{1}{Q} > \omega_\infty + \frac{1}{R}$.

It follows from 1) and 2) that the implication (10.38) is true. The proof is complete. □

Now we shall look for sufficient conditions for the embedding (10.1). We need the following lemma.

10.5. LEMMA. *Assume that* $0 < Q < R < \infty$, $\mu(\mathscr{R}) = \infty$ *and* $\mathbb{L} = (\lambda_0, \lambda_\infty)$, $\mathbb{E} = (\varepsilon_0, \varepsilon_\infty)$, $\mathbb{S} = (\sigma_0, \sigma_\infty)$, $\mathbb{W} = (\omega_0, \omega_\infty) \in \mathbb{R}^2$.

*Put*

$$\alpha_0 = \frac{\sigma_0 R - \lambda_0 Q}{R - Q} \qquad \text{and} \qquad \beta_0 = \frac{\omega_0 R - \varepsilon_0 Q}{R - Q}.$$

*Then for all* $f \in L_{(1,Q;\mathbb{L},\mathbb{E})}$,

$$\|f\|_{(1,R;\sigma_0,\omega_0)(0,1)} \leqslant \|f\|_{(1,Q;\lambda_0,\varepsilon_0)(0,1)}^{\frac{Q}{R}} \, \|f\|_{(1,\infty;\alpha_0,\beta_0)(0,1)}^{1-\frac{Q}{R}}.$$

*Put*

$$\alpha_\infty = \frac{\sigma_\infty R - \lambda_\infty Q}{R - Q} \qquad \text{and} \qquad \beta_\infty = \frac{\omega_\infty R - \varepsilon_\infty Q}{R - Q}.$$

*Then for all* $f \in L_{(1,Q;\mathbb{L},\mathbb{E})}$,

$$\|f\|_{(1,R;\sigma_\infty,\omega_\infty)(1,\infty)} \leqslant \|f\|_{(1,Q;\lambda_\infty,\varepsilon_\infty)(1,\infty)}^{\frac{Q}{R}} \, \|f\|_{(1,\infty;\alpha_\infty,\beta_\infty)(1,\infty)}^{1-\frac{Q}{R}}. \tag{10.48}$$

*In particular, if we define* $\mathbb{A} = (\alpha_0, \alpha_\infty)$ *and* $\mathbb{B} = (\beta_0, \beta_\infty)$ *by*

$$\mathbb{A} = \frac{\mathbb{S}R - \mathbb{L}Q}{R - Q} \qquad \text{and} \qquad \mathbb{B} = \frac{\mathbb{W}R - \mathbb{E}Q}{R - Q}, \tag{10.49}$$

*then for all* $f \in L_{(1,Q;\mathbb{L},\mathbb{E})}$,

$$\|f\|_{(1,R;\mathbb{S},\mathbb{W})} \leqslant \|f\|_{(1,Q;\mathbb{L},\mathbb{E})}^{\frac{Q}{R}} \|f\|_{(1,\infty;\mathbb{A},\mathbb{B})}^{1-\frac{Q}{R}}. \tag{10.50}$$

*Proof.* Put $I_0 = (0,1)$ and $I_\infty = (1,\infty)$. If $f \in L_{(1,Q;\mathbb{L},\mathbb{E})}$ and $i \in \{0,\infty\}$, then

$$\begin{aligned}
\|f\|_{(1,R;\sigma_i,\omega_i)\,I_i}^R &= \|t^{1-\frac{1}{R}}\ell^{\sigma_i}(t)\ell\ell^{\omega_i}(t)f^{**}(t)\|_{R,I_i}^R \\
&= \int_{I_i} \Big[t\ell^{\lambda_i}(t)\ell\ell^{\varepsilon_i}(t)f^{**}(t)\Big]^Q \Big[t\ell^{\alpha_i}(t)\ell\ell^{\beta_i}(t)f^{**}(t)\Big]^{R-Q}\frac{\mathrm{d}\,t}{t} \\
&\leqslant \|t\ell^{\alpha_i}(t)\ell\ell^{\beta_i}(t)f^{**}(t)\|_{\infty,I_i}^{R-Q}\, \|t^{1-\frac{1}{Q}}\ell^{\lambda_i}(t)\ell\ell^{\varepsilon_i}(t)f^{**}(t)\|_{Q,I_i}^Q \\
&= \|f\|_{(1,Q;\lambda_i,\varepsilon_i)\,I_i}^Q\, \|f\|_{(1,\infty;\alpha_i,\beta_i)\,I_i}^{R-Q}. \qquad \square
\end{aligned}$$

10.6. LEMMA. *Let* $0 < Q \leqslant R \leqslant \infty$, $\mu(\mathscr{R}) = \infty$, $L_{(1,Q;\mathbb{L},\mathbb{E})} \neq \{0\}$, *and*

$$\lambda_\infty + \frac{1}{Q} > \sigma_\infty + \frac{1}{R}, \tag{10.51}$$

*or*

$$0 > \lambda_\infty + \frac{1}{Q} = \sigma_\infty + \frac{1}{R} \qquad \textit{and} \qquad \varepsilon_\infty \geqslant \omega_\infty, \tag{10.52}$$

*or*

$$0 = \lambda_\infty + \frac{1}{Q} = \sigma_\infty + \frac{1}{R} \qquad \textit{and} \qquad \varepsilon_\infty + \frac{1}{Q} \geqslant \omega_\infty + \frac{1}{R}. \tag{10.53}$$

*Moreover, let one of the following conditions hold:*

$$0 \leqslant \lambda_0 + \frac{1}{Q} \qquad \textit{and} \quad \lambda_0 + \frac{1}{Q} > \sigma_0 + \frac{1}{R}, \tag{10.54}$$

$$0 < \lambda_0 + \frac{1}{Q} = \sigma_0 + \frac{1}{R} \quad \textit{and} \quad \varepsilon_0 \geqslant \omega_0, \tag{10.55}$$

$$0 = \lambda_0 + \frac{1}{Q} = \sigma_0 + \frac{1}{R},\ \varepsilon_0 + \frac{1}{Q} \geqslant 0,\ \varepsilon_0 + \frac{1}{Q} \geqslant \omega_0 + \frac{1}{R}. \tag{10.56}$$

*Then*

$$L_{(1,Q;\mathbb{L},\mathbb{E})} \hookrightarrow L_{(1,R;\mathbb{S},\mathbb{W})}. \tag{10.57}$$

*Proof.* (i) If $0 < Q = R \leqslant \infty$, then it is clear from (10.51)–(10.53) and (10.54)–(10.56) that (10.57) holds for all $f \in L_{(1,Q;\mathbb{L},\mathbb{E})}$.

(ii) Let $0 < Q < R = \infty$. We have for all $t \in (0,1)$,

$$\|s^{-\frac{1}{Q}}\ell^{\lambda_0}(s)\ell\ell^{\varepsilon_0}(s)\|_{Q,(t,1)} \approx \begin{cases} \ell^{\lambda_0+\frac{1}{Q}}(t)\ell\ell^{\varepsilon_0}(t) & \text{if } \lambda_0+\frac{1}{Q}>0 \\ \ell\ell^{\varepsilon_0+\frac{1}{Q}}(t) & \text{if } \lambda_0+\frac{1}{Q}=0,\ \varepsilon_0+\frac{1}{Q}>0 \\ \ell\ell\ell^{\frac{1}{Q}}(t) & \text{if } \lambda_0+\frac{1}{Q}=0,\ \varepsilon_0+\frac{1}{Q}=0 \\ 1 & \text{if } \lambda_0+\frac{1}{Q}<0 \\ & \text{or } \lambda_0+\frac{1}{Q}=0,\ \varepsilon_0+\frac{1}{Q}<0. \end{cases} \tag{10.58}$$

If $\lambda_0 + \frac{1}{Q} > 0$, then (10.54), (10.55), and (10.58) yield for all $t \in (0,1)$,

$$\ell^{\sigma_0}(t)\ell\ell^{\omega_0}(t) \lesssim \ell^{\lambda_0+\frac{1}{Q}}(t)\ell\ell^{\varepsilon_0}(t) \approx \|s^{-\frac{1}{Q}}\ell^{\lambda_0}(s)\ell\ell^{\varepsilon_0}(s)\|_{Q(t,1)}. \tag{10.59}$$

If $\lambda_0 + \frac{1}{Q} = 0$, then (10.54), (10.56), and (10.58) yield for all $t \in (0,1)$,

$$\ell^{\sigma_0}(t)\ell\ell^{\omega_0}(t) \lesssim \ell^{\lambda_0+\frac{1}{Q}}(t)\ell\ell^{\varepsilon_0+\frac{1}{Q}}(t) \lesssim \|s^{-\frac{1}{Q}}\ell^{\lambda_0}(s)\ell\ell^{\varepsilon_0}(s)\|_{Q,(t,1)}. \tag{10.60}$$

Using (10.59) and (10.60), we get

$$\begin{aligned} \sup_{0<t<1} t^{1-\frac{1}{R}}\ell^{\sigma_0}(t)\ell\ell^{\omega_0}(t)f^{**}(t) &= \sup_{0<t<1} \ell^{\sigma_0}(t)\ell\ell^{\omega_0}(t)\int_0^t f^*(\tau)\,\mathrm{d}\,\tau \\ &\lesssim \sup_{0<t<1} \|s^{-\frac{1}{Q}}\ell^{\lambda_0}(s)\ell\ell^{\varepsilon_0}(s)\|_{Q,(t,1)} \int_0^t f^*(\tau)\,\mathrm{d}\,\tau \\ &\leqslant \sup_{0<t<1} \|s^{1-\frac{1}{Q}}\ell^{\lambda_0}(s)\ell\ell^{\varepsilon_0}(s)f^{**}(s)\|_{Q,(t,1)} = \|f\|_{(1,Q;\mathbb{L},\mathbb{E})(0,1)} \end{aligned} \tag{10.61}$$

Since $L_{(1,Q;\mathbb{L},\mathbb{E})} \neq \{0\}$ and $Q < \infty = R$, either $\lambda_\infty + \frac{1}{Q} < 0$ or $\lambda_\infty + \frac{1}{Q} = 0$ and $\varepsilon_\infty + \frac{1}{Q} < 0$ (cf. (10.3)). We have for all $t \in [1,\infty)$,

$$\|s^{-\frac{1}{Q}}\ell^{\lambda_\infty}(s)\ell\ell^{\varepsilon_\infty}(s)\|_{Q,(t,\infty)} \approx \begin{cases} \ell^{\lambda_\infty+\frac{1}{Q}}(t)\ell\ell^{\varepsilon_\infty}(t) & \text{if } \lambda_\infty+\frac{1}{Q}<0 \\ \ell\ell^{\varepsilon_\infty+\frac{1}{Q}}(t) & \text{if } \lambda_\infty+\frac{1}{Q}=0, \\ & \quad \varepsilon_\infty+\frac{1}{Q}<0. \end{cases} \tag{10.62}$$

If $\lambda_\infty + \frac{1}{Q} < 0$, then (10.51), (10.52), and (10.62) imply for all $t \in [1,\infty)$,

$$\ell^{\sigma_\infty}(t)\ell\ell^{\omega_\infty}(t) \lesssim \ell^{\lambda_\infty+\frac{1}{Q}}(t)\ell\ell^{\varepsilon_\infty}(t) \approx \|s^{-\frac{1}{Q}}\ell^{\lambda_\infty}(s)\ell\ell^{\varepsilon_\infty}(s)\|_{Q,(t,\infty)}. \tag{10.63}$$

If $\lambda_0 + \frac{1}{Q} = 0$ and $\varepsilon_\infty + \frac{1}{Q} < 0$, then (10.51), (10.53), and (10.62) yield for all $t \in [1,\infty)$,

$$\ell^{\sigma_\infty}(t)\ell\ell^{\omega_\infty}(t) \lesssim \ell^{\lambda_\infty+\frac{1}{Q}}(t)\ell\ell^{\varepsilon_\infty+\frac{1}{Q}}(t) \approx \|s^{-\frac{1}{Q}}\ell^{\lambda_\infty}(s)\ell\ell^{\varepsilon_\infty}(t)\|_{Q,(t,\infty)}. \tag{10.64}$$

Using (10.63) and (10.64), we obtain

$$
\begin{aligned}
(10.65)\qquad \sup_{1\leqslant t<\infty} t^{1-\frac{1}{R}}\ell^{\sigma_\infty}(t)\ell\ell^{\omega_\infty}(t)f^{**}(t) &= \sup_{1\leqslant t<\infty} \ell^{\sigma_\infty}(t)\ell\ell^{\omega_\infty}(t)\int_0^t f^*(\tau)d\tau \\
&\lesssim \sup_{1\leqslant t<\infty} \|s^{-\frac{1}{Q}}\ell^{\lambda_\infty}(s)\ell\ell^{\varepsilon_\infty}(s)\|_{Q,(t,\infty)}\int_0^t f^*(\tau)d\tau \\
&\leqslant \|s^{1-\frac{1}{Q}}\ell^{\lambda_\infty}(s)\ell\ell^{\varepsilon_\infty}(s)f^{**}(s)\|_{Q,(1,\infty)} = \|f\|_{(1,Q;\mathbb{L},\mathbb{E})(1,\infty)}.
\end{aligned}
$$

Moreover, by (10.61) and (10.65),

$$
\begin{aligned}
\|f\|_{(1,\infty;\mathbb{S},\mathbb{W})} &= \sup_{0<t<\infty} t\ell^{\mathbb{S}}(t)\ell\ell^{\mathbb{W}}(t)f^{**}(t) \\
&= \max\{\sup_{0<t<1} t\ell^{\sigma_0}(t)\ell\ell^{\omega_0}(t)f^{**}(t),\ \sup_{1\leqslant t<\infty} t\ell^{\sigma_\infty}(t)\ell\ell^{\omega_\infty}(t)f^{**}(t)\} \\
&\leqslant \|f\|_{(1,Q;\mathbb{L},\mathbb{E})}
\end{aligned}
$$

and (10.57) follows.

(iii) Let $0<Q<R<\infty$. Then, by Lemma 10.5,

$$
\|f\|_{(1,R;\mathbb{S},\mathbb{W})} \leqslant \|f\|^{\frac{Q}{R}}_{(1,Q;\mathbb{L},\mathbb{E})}\|f\|^{1-\frac{Q}{R}}_{(1,\infty;\mathbb{A},\mathbb{B})},
$$

where $\mathbb{A}$ and $\mathbb{B}$ are given by (10.49). Thus, (10.57) will be proved if we show that for all $f\in L_{(1,Q;\mathbb{L},\mathbb{E})}$,

$$
(10.66)\qquad \|f\|_{(1,\infty;\mathbb{A},\mathbb{B})} \lesssim \|f\|_{(1,Q;\mathbb{L},\mathbb{E})}.
$$

However, (10.66) will follow on using part (ii) above with $\mathbb{S},\mathbb{W}$ replaced by $\mathbb{A},\mathbb{B}$, respectively, once we show that each of conditions (10.51)–(10.56) implies the same one with $R,\sigma_i,\omega_i$ replaced by $\infty,\alpha_i,\beta_i$ $(i=0,\infty)$, respectively. We are going to verify it in the case when one of the conditions (10.51)–(10.53) holds; the proof is similar when one of the conditions (10.54)–(10.56) is satisfied.

Suppose that (10.51) holds. Then, by (10.49),

$$
\begin{aligned}
(10.67)\qquad \sigma_\infty+\frac{1}{R} &= \frac{\sigma_\infty R-\lambda_\infty Q}{R-Q}+\frac{Q}{R-Q}\Big(\lambda_\infty+\frac{1}{Q}-\Big(\sigma_\infty+\frac{1}{R}\Big)\Big) \\
&> \alpha_\infty = \alpha_\infty+\frac{1}{\infty},
\end{aligned}
$$

which, together with (10.51), gives $\lambda_\infty+\frac{1}{Q}>\alpha_\infty+\frac{1}{\infty}$.

If $\lambda_\infty+\frac{1}{Q}=\sigma_\infty+\frac{1}{R}$, then the first equality in (10.67) implies that $\lambda_\infty+\frac{1}{Q}=\alpha_\infty+\frac{1}{\infty}$.

If (10.52) is satisfied, then, using (10.49), we have

$$
\varepsilon_\infty = \frac{\varepsilon_\infty R-\varepsilon_\infty Q}{R-Q} \geqslant \frac{\omega_\infty R-\varepsilon_\infty Q}{R-Q} = \beta_\infty.
$$

Finally, assume that (10.53) holds. Then, by (10.49),

$$
\omega_\infty+\frac{1}{R} = \frac{\omega_\infty R-\omega_\infty Q}{R-Q}+\frac{Q}{R-Q}\Big(\varepsilon_\infty+\frac{1}{Q}-\Big(\omega_\infty+\frac{1}{R}\Big)\Big) \geqslant \beta_\infty = \beta_\infty+\frac{1}{\infty}.
$$

Together with (10.53), this gives $\varepsilon_\infty + \frac{1}{Q} \geqslant \beta_\infty + \frac{1}{\infty}$. □

Using the method of the proof of Lemma 10.6 and employing estimate (10.48) instead of (10.50), one can prove the following result.

10.7. LEMMA. *Let* $0 < Q \leqslant R \leqslant \infty$, $\mu(\mathscr{R}) = \infty$, $L_{(1,Q;\mathbb{L},\mathbb{E})} \neq \{0\}$, *and let one of conditions* (10.51)–(10.53) *hold. Then for all* $f \in L_{(1,Q;\mathbb{L},\mathbb{E})}$,

$$\|f\|_{(1,R;\mathbb{S},\mathbb{W})(1,\infty)} \lesssim \|f\|_{(1,Q;\mathbb{L},\mathbb{E})(1,\infty)}. \tag{10.68}$$

The next lemma provides the "complementary inequality" to (10.68) in the case when the assumptions of Lemma 10.6 on $\lambda_0, \varepsilon_0, \sigma_0$, and $\omega_0$ may not be satisfied.

10.8. LEMMA. *Let* $0 < Q, R \leqslant \infty$, $\mu(\mathscr{R}) = \infty$, $L_{(1,Q;\mathbb{L},\mathbb{E})} \neq \{0\}$, *and let one of the following conditions be satisfied:*

$$\sigma_0 + \frac{1}{R} < 0; \tag{10.69}$$

$$\sigma_0 + \frac{1}{R} = 0; \quad \omega_0 + \frac{1}{R} < 0; \tag{10.70}$$

$$R = \infty, \quad \sigma_0 = 0, \quad \omega_0 = 0. \tag{10.71}$$

*Then for all* $f \in L_{(1,Q;\mathbb{L},\mathbb{E})}$,

$$\|f\|_{(1,R;\mathbb{S},\mathbb{W})(0,1)} \lesssim \|f\|_{(1,Q;\mathbb{L},\mathbb{E})(1,\infty)}.$$

*Proof.* Let $f \in L_{(1,Q;\mathbb{L},\mathbb{E})}$. Then, using (10.69)–(10.71) and (10.3), we obtain

$$\begin{aligned}
&\|f\|_{(1,R;\mathbb{S},\mathbb{W})(0,1)} = \|t^{-\frac{1}{R}}\ell^{\sigma_0}(t)\ell\ell^{\omega_0}(t)\int_0^t f^*(\tau)d\tau\|_{R,(0,1)}\\
&\leqslant \|t^{-\frac{1}{R}}\ell^{\sigma_0}(t)\ell\ell^{\omega_0}(t)\|_{R,(0,1)}\int_0^1 f^*(\tau)d\tau \approx \int_0^1 f^*(\tau)d\tau\\
&\approx \|t^{-\frac{1}{Q}}\ell^{\lambda_\infty}(t)\ell\ell^{\varepsilon_\infty}(t)\|_{Q,(1,\infty)}\int_0^1 f^*(\tau)d\tau\\
&\leqslant \|t^{-\frac{1}{Q}}\ell^{\lambda_\infty}(t)\ell\ell^{\varepsilon_\infty}(t)\int_0^t f^*(\tau)d\tau\|_{Q,(1,\infty)} = \|f\|_{(1,Q;\mathbb{L},\mathbb{E})(1,\infty)}. \qquad \square
\end{aligned}$$

10.9. REMARK. Suppose that all the assumptions of Lemma 10.8 are satisfied. Then for all $f \in L_{(1,Q;\mathbb{L},\mathbb{E})}$,

$$\|f\|_{(1,R;\mathbb{S},\mathbb{W})(0,1)} \lesssim \|f\|_{(1,Q;\mathbb{L},\mathbb{E})(0,1)}.$$

Indeed, if $f \in L_{(1,Q;\mathbb{L},\mathbb{E})}$, then

$$\begin{aligned}
\|f\|_{(1,R;\mathbb{S},\mathbb{W})(0,1)} &\lesssim \int_0^1 f^*(\tau)d\tau = f^{**}(1) \approx \|t^{1-\frac{1}{Q}}\ell^{\lambda_0}(t)\ell\ell^{\varepsilon_0}(t)\|_{Q,(0,1)}f^{**}(1)\\
&\leqslant \|t^{1-\frac{1}{Q}}\ell^{\lambda_0}(t)\ell\ell^{\varepsilon_0}(t)f^{**}(t)\|_{Q,(0,1)} = \|f\|_{(1,Q;\mathbb{L},\mathbb{E})(0,1)}.
\end{aligned}$$

Now, we turn our attention to the case when $0 < R < Q \leqslant \infty$.

10.10. LEMMA. *Suppose that* $0 < R < Q \leqslant \infty$, $\mu(\mathscr{R}) = \infty$, *and* $L_{(1,Q;\mathbb{L},\mathbb{E})} \neq \{0\}$.

(i) *Let either*

$$\lambda_\infty + \frac{1}{Q} > \sigma_\infty + \frac{1}{R} \tag{10.72}$$

*or*

$$\lambda_\infty + \frac{1}{Q} = \sigma_\infty + \frac{1}{R} \quad \textit{and} \quad \varepsilon_\infty + \frac{1}{Q} > \omega_\infty + \frac{1}{R}. \tag{10.73}$$

*Then for all* $f \in L_{(1,Q;\mathbb{L},\mathbb{E})}$,

$$\|f\|_{(1,R;\mathbb{S},\mathbb{W})(1,\infty)} \lesssim \|f\|_{(1,Q;\mathbb{L},\mathbb{E})(1,\infty)}.$$

(ii) *Let either*

$$\lambda_0 + \frac{1}{Q} > \sigma_0 + \frac{1}{R} \tag{10.74}$$

*or*

$$\lambda_0 + \frac{1}{Q} = \sigma_0 + \frac{1}{R} \quad \textit{and} \quad \varepsilon_0 + \frac{1}{Q} > \omega_0 + \frac{1}{R}. \tag{10.75}$$

*Then for all* $f \in L_{(1,Q;\mathbb{L},\mathbb{E})}$,

$$\|f\|_{(1,R;\mathbb{S},\mathbb{W})(0,1)} \lesssim \|f\|_{(1,Q;\mathbb{L},\mathbb{E})(0,1)}.$$

*Proof.* Put $I_0 = (0,1)$ and $I_\infty = (1,\infty)$. If $f \in L_{(1,Q;\mathbb{L},\mathbb{E})}$, $i \in \{0,\infty\}$, and let one of conditions (10.72), (10.73), or (10.74), (10.75), respectively, hold, if $i = \infty$ or $i = 0$. Since $R < \infty$, we get by the Hölder inequality with respect to the measure $\frac{\mathrm{d}t}{t}$ and with exponents $Q/R$ and $Q/(Q-R)$ if $Q < \infty$ and immediately if $Q = \infty$ that

$$\begin{aligned} \|f\|^R_{(1,R;\sigma_i,\omega_i)\,I_i} &= \int\limits_{I_i} \left[ t\ell^{\lambda_i}(t)\ell\ell^{\varepsilon_i}(t) f^{**}(t)\ell^{\sigma_i-\lambda_i}(t)\ell\ell^{\omega_i-\varepsilon_i}(t)\right]^R \frac{\mathrm{d}t}{t} \\ &\leqslant C_i \,\|f\|^R_{(1,Q;\lambda_i,\varepsilon_i)\,I_i}, \end{aligned}$$

where

$$C_i = \left( \int\limits_{I_i} \left[\ell^{\sigma_i-\lambda_i}(t)\ell\ell^{\omega_i-\varepsilon_i}(t)\right]^{1/(\frac{1}{R}-\frac{1}{Q})} \frac{\mathrm{d}t}{t}\right)^{1-\frac{R}{Q}}.$$

Since our assumptions imply that $C_i < \infty$, the result follows. □

Now, we are able to prove Theorems 5.1 and 5.2.

*Proof of Theorem 5.1. Necessity* follows from Lemmas 10.2 and 10.3. *Sufficiency* is a consequence of Lemmas 10.6–10.8. □

*Proof of Theorem 5.2. Necessity* follows from Lemma 10.4. *Sufficiency* is a consequence of Lemmas 10.8 and 10.10. □

We conclude this section with the proof of Theorem 5.5.

*Proof of Theorem 5.5.* We shall use the following notation:

$$
\begin{aligned}
\mathscr{S}^* &= \{(P_1,P_2,\mathbb{L},\mathbb{E},\mathbb{S},\mathbb{W});\ 0<P_1,P_2\leqslant\infty,\ P_1\neq P_2,\ \mathbb{L},\mathbb{E},\mathbb{S},\mathbb{W}\in\mathbb{R}^2\},\\
\mathscr{S}_1 &= \{(P_1,P_2,\mathbb{L},\mathbb{E},\mathbb{S},\mathbb{W})\in\mathscr{S}^*;\ 1\leqslant P_2<P_1\leqslant\infty\},\\
\widetilde{\mathscr{S}}_1 &= \{(P_1,P_2,\mathbb{L},\mathbb{E},\mathbb{S},\mathbb{W})\in\mathscr{S}^*;\ 1\leqslant P_1<P_2\leqslant\infty\},\\
\mathscr{S}_2 &= \{(P_1,P_2,\mathbb{L},\mathbb{E},\mathbb{S},\mathbb{W})\in\mathscr{S}^*;\ 0<P_2<1\},\\
\mathscr{S}_3^* &= \{(P_1,P_2,\mathbb{L},\mathbb{E},\mathbb{S},\mathbb{W})\in\mathscr{S}^*;\ 0<P_1<1,\ P_2=1\},\\
\mathscr{S}_{3,1} &= \{(P_1,P_2,\mathbb{L},\mathbb{E},\mathbb{S},\mathbb{W})\in\mathscr{S}_3^*;\ \sigma_0+\frac{1}{R}<0\},\\
\mathscr{S}_{3,2} &= \{(P_1,P_2,\mathbb{L},\mathbb{E},\mathbb{S},\mathbb{W})\in\mathscr{S}_3^*;\ \sigma_0+\frac{1}{R}=0,\ \omega_0+\frac{1}{R}<0\},\\
\mathscr{S}_{3,3} &= \{(P_1,P_2,\mathbb{L},\mathbb{E},\mathbb{S},\mathbb{W})\in\mathscr{S}_3^*;\ R=\infty,\ \sigma_0=0,\ \omega_0=0\},\\
\mathscr{S}_3 &= \mathscr{S}_{3,1}\cup\mathscr{S}_{3,2}\cup\mathscr{S}_{3,3},\qquad \widetilde{\mathscr{S}}_3=\mathscr{S}_3^*\setminus\mathscr{S}_3,\\
\widetilde{\mathscr{S}}_4 &= \{(P_1,P_2,\mathbb{L},\mathbb{E},\mathbb{S},\mathbb{W})\in\mathscr{S}^*;\ 0<P_1<1,\ 1<P_2\leqslant\infty\},\\
\mathscr{S} &= \mathscr{S}_1\cup\mathscr{S}_2\cup\mathscr{S}_3,\qquad\qquad \widetilde{\mathscr{S}}=\widetilde{\mathscr{S}}_1\cup\widetilde{\mathscr{S}}_3\cup\widetilde{\mathscr{S}}_4.
\end{aligned}
$$

Obviously, $\mathscr{S}^*=\mathscr{S}\cup\widetilde{\mathscr{S}}$.

*Necessity.* We have to prove that (5.5) does not hold if either $\mu(\mathscr{R})=\infty$ or $\mu(\mathscr{R})<\infty$ and $(P_1,P_2,\mathbb{L},\mathbb{E},\mathbb{S},\mathbb{W})\in\widetilde{\mathscr{S}}$.

Assume first that $\mu(\mathscr{R})=\infty$. Then (5.4), (5.5), and Lemma 3.5 (ii) implies that $1\leqslant P_1,P_2\leqslant\infty$. Putting

$$X=L_{(P_1,Q;\mathbb{L},\mathbb{E})}\quad\text{and}\quad Y=L_{(P_2,R;\mathbb{S},\mathbb{W})},$$

we obtain from (5.5) that

$$\varphi_Y\lesssim\varphi_X, \tag{10.76}$$

where $\varphi_X$ and $\varphi_Y$ are the fundamental functions of $X$ and $Y$, respectively. Moreover, one can see from (10.76) and Lemma 3.7 (ii) that (5.5) cannot hold.

Assume now that $\mu(\mathscr{R})<\infty$. Then (10.76) and Lemma 3.7 (ii) implies, again, that (5.5) cannot hold if $(P_1,P_2,\mathbb{L},\mathbb{E},\mathbb{S},\mathbb{W})\in\widetilde{\mathscr{S}}$.

*Sufficiency.* let $M:=\mu(\mathscr{R})<\infty$. We have to prove that (5.5) holds provided that $(P_1,P_2,\mathbb{L},\mathbb{E},\mathbb{S},\mathbb{W})\in\mathscr{S}=\mathscr{S}_1\cup\mathscr{S}_2\cup\mathscr{S}_3$.

Assume that $(P_1,P_2,\mathbb{L},\mathbb{E},\mathbb{S},\mathbb{W})\in\mathscr{S}_1$ (i.e. (5.6) holds). We shall distinguish two cases:

a) Let $0 < R < Q \leqslant \infty$. Then for all $f \in \mathscr{M}(\mathscr{R},\mu)$,

$$\|f\|_{(P_2,R;\mathbb{S},\mathbb{W})} = \Big(\int_0^M \Big[t^{\frac{1}{P_2}}\ell^{\mathbb{S}}(t)\ell\ell^{\mathbb{W}}(t)f^{**}(t)\Big]^R \frac{\mathrm{d}\,t}{t}\Big)^{\frac{1}{R}} \tag{10.77}$$

$$\leqslant \|f\|_{(P_1,Q;\mathbb{L},\mathbb{E})}\Big(\int_0^M \Big[t^{\frac{1}{P_2}-\frac{1}{P_1}}\ell^{\mathbb{S}-\mathbb{L}}(t)\ell\ell^{\mathbb{W}-\mathbb{E}}(t)\Big]^{1/(\frac{1}{R}-\frac{1}{Q})}\frac{dt}{t}\Big)^{\frac{1}{R}-\frac{1}{Q}}$$

$$\lesssim \|f\|_{(P_1,Q;\mathbb{L},\mathbb{E})}$$

and (5.5) follows.

b) Let $0 < Q \leqslant R \leqslant \infty$. Put

$$\sigma = \sigma_0, \quad \omega = \omega_0, \quad \lambda = \lambda_0, \quad \varepsilon = \varepsilon_0. \tag{10.78}$$

If $\widetilde{\lambda}, \widetilde{\varepsilon} \in \mathbb{R}$, we have for all $f \in \mathscr{M}(\mathscr{R},\mu)$ that

$$\|f\|_{(P_2,R;\sigma,\omega)} = \|t^{\frac{1}{P_1}}\ell^{\widetilde{\lambda}}(t)\ell\ell^{\widetilde{\varepsilon}}(t)f^{**}(t)t^{\frac{1}{P_2}-\frac{1}{P_1}-\frac{1}{R}}\ell^{\sigma-\widetilde{\lambda}}(t)\ell\ell^{\omega-\widetilde{\varepsilon}}(t)\|_{R,(0,M)} \tag{10.79}$$

$$\leqslant C\|f\|_{(P_1,\infty;\widetilde{\lambda},\widetilde{\varepsilon})},$$

where $C = \|t^{\frac{1}{P_2}-\frac{1}{P_1}-\frac{1}{R}}\ell^{\sigma-\widetilde{\lambda}}(t)\ell\ell^{\omega-\widetilde{\varepsilon}}(t)\|_{R,(0,M)} < \infty$ since $P_1 > P_2$. Consequently, for any $\widetilde{\lambda}, \widetilde{\varepsilon} \in \mathbb{R}$,

$$L_{(P_1,\infty;\widetilde{\lambda},\widetilde{\varepsilon})} \hookrightarrow L_{(P_2,R;\sigma,\omega)}. \tag{10.80}$$

Since $P_1 > P_2 \geqslant 1$,

$$L_{(P_1,\infty;\widetilde{\lambda},\widetilde{\varepsilon})} = L_{P_1,\infty;\widetilde{\lambda},\widetilde{\varepsilon}} \quad \text{and} \quad L_{(P_1,Q;\lambda,\varepsilon)} = L_{P_1,Q;\lambda,\varepsilon}. \tag{10.81}$$

Taking $\widetilde{\lambda} < \lambda$, we obtain from Theorem 4.5 that

$$L_{P_1,Q;\lambda,\varepsilon} \hookrightarrow L_{P_1,\infty;\widetilde{\lambda},\widetilde{\varepsilon}}$$

and hence, on using (10.81),

$$L_{(P_1,Q;\lambda,\varepsilon)} \hookrightarrow L_{(P_1,\infty;\widetilde{\lambda},\widetilde{\varepsilon})}.$$

Together with (10.80) this yields

$$L_{(P_1,Q;\lambda,\varepsilon)} \hookrightarrow L_{(P_2,R;\sigma,\omega)} \tag{10.82}$$

and (5.5) follows.

Assume that $(P_1,P_2,\mathbb{L},\mathbb{E},\mathbb{S},\mathbb{W}) \in \mathscr{S}_2$. Putting

$$\mathscr{S}_{2,0} = \{(P_1,P_2,\mathbb{L},\mathbb{E},\mathbb{S},\mathbb{W}) \in \mathscr{S}_2;\ 0 < P_1 < 1\},$$
$$\mathscr{S}_{2,1} = \{(P_1,P_2,\mathbb{L},\mathbb{E},\mathbb{S},\mathbb{W}) \in \mathscr{S}_2;\ 1 \leqslant P_1 \leqslant \infty\},$$

we have $\mathscr{S}_2 = \mathscr{S}_{2,0} \cup \mathscr{S}_{2,1}$.

Let $(P_1, P_2, \mathbb{L}, \mathbb{E}, \mathbb{S}, \mathbb{W}) \in \mathscr{S}_{2,1}$ (i.e. (5.7) holds). If $0 < R < Q \leqslant \infty$, we obtain (cf. (10.77)) that (5.5) is satisfied. If $0 < Q \leqslant R \leqslant \infty$, we have (cf. (10.79)) that (10.80) holds. Moreover, if $P_1 > 1$, we have (10.81), and (5.5) follows using the same argument as that of part b) above. If $P_1 = 1$, we choose $\widetilde{\lambda} < 0$ and $\widetilde{\varepsilon} \in \mathbb{R}$. Then, by Theorem 5.3,

$$L_{(1,Q;\lambda,\varepsilon)} \hookrightarrow L_{(1,\infty;\widetilde{\lambda},\widetilde{\varepsilon})}$$

with $\lambda$ and $\varepsilon$ from (10.78). Together with (10.80) this yields (10.82) and (5.5) follows.

Let $(P_1, P_2, \mathbb{L}, \mathbb{E}, \mathbb{S}, \mathbb{W}) \in \mathscr{S}_{2,0}$ (i.e. (5.8) holds). Then (5.5) is satisfied since, by Lemma 3.15,

$$L_{(P_1,Q;\mathbb{L},\mathbb{E})} = L^1, \qquad L_{(P_2,R;\mathbb{S},\mathbb{W})} = L^1. \tag{10.83}$$

Finally, let $(P_1, P_2, \mathbb{L}, \mathbb{E}, \mathbb{S}, \mathbb{W}) \in \mathscr{S}_3$. Then, Lemma 3.15 again implies (10.83) and (5.5) follows. □

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