Mathematical
Inequalities
& Applications
Volume 2, Number 3 (1999), 473–477

SIMPLIFIED PROOF OF TANAHASHI'S RESULT ON THE BEST POSSIBILITY OF GENERALIZED FURUTA INEQUALITY

TAKEAKI YAMAZAKI

(*communicated by T. Furuta*)

Abstract. We give a simplified proof of Tanahashi's result on the best possibility of generalized Furuta inequality by using Tanahashi's result on the best possibility of Furuta inequality.

1. Introduction

In what follows, a capital letter means a bounded linear operator on a complex Hilbert space H. An operator T is said to be positive (denoted by $T \geqslant 0$) if $(Tx, x) \geqslant 0$ for all $x \in H$. Also, an operator T is strictly positive (denoted by $T > 0$) if T is positive and invertible. The celebrated Löwner-Heinz theorem asserts "$A \geqslant B \geqslant 0$ *ensures* $A^{\alpha} \geqslant B^{\alpha}$ *for any* $\alpha \in [0,1]$." It is well known that $A \geqslant B \geqslant 0$ does not always assure $A^{\alpha} \geqslant B^{\alpha}$ for any $\alpha > 1$ in general. For the sake of convenience on application, the following Theorem F was established in 1987.

THEOREM F. (Furuta inequality) [8]
If $A \geqslant B \geqslant 0$, *then for each* $r \geqslant 0$,

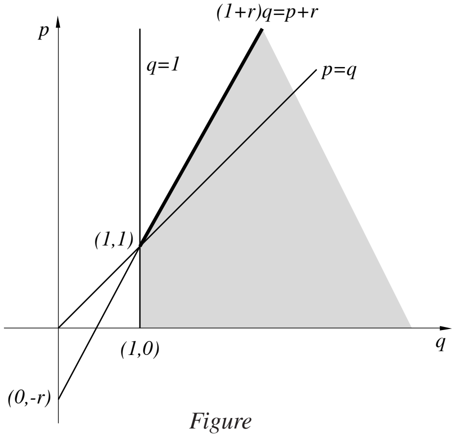
(i) $$(B^{\frac{r}{2}} A^p B^{\frac{r}{2}})^{\frac{1}{q}} \geqslant (B^{\frac{r}{2}} B^p B^{\frac{r}{2}})^{\frac{1}{q}}$$

and

(ii) $$(A^{\frac{r}{2}} A^p A^{\frac{r}{2}})^{\frac{1}{q}} \geqslant (A^{\frac{r}{2}} B^p A^{\frac{r}{2}})^{\frac{1}{q}}$$

hold for $p \geqslant 0$ *and* $q \geqslant 1$ *with*

$$(1+r)q \geqslant p+q.$$



Figure

We remark that Theorem F yields the Löwner-Heinz theorem when we put $r = 0$. Alternative proofs of Theorem F are given in [3] and [18] and also an elementary one page proof in [9]. Tanahashi [19] shows that the domain drawn for p, q and r in the Figure is the best possible one for Theorem F as follows.

Mathematics subject classification (1991): 47A63.
Key words and phrases: positive operator, Furuta inequality, generalized Furuta inequality.

Paper MIA-02-37

THEOREM A. ([19]) *Let* $p > 0$ *and* $r \geqslant 0$. *If* $1 > q > 0$ *or* $(1+r)q < p+r$, *then there exist positive operators* A *and* B *on* $\mathbb{R}^2$ *such that* $A \geqslant B > 0$ *and*

$$A^{\frac{p+r}{q}} \ngeqslant (A^{\frac{r}{2}} B^p A^{\frac{r}{2}})^{\frac{1}{q}}.$$

Especially in case $p \geqslant 1$, Theorem F can be rewritten as follows.

THEOREM F'. *If* $A \geqslant B \geqslant 0$, *then*

$$A^{1+r} \geqslant (A^{\frac{r}{2}} B^p A^{\frac{r}{2}})^{\frac{1+r}{p+r}} \quad \text{holds for } p \geqslant 1 \text{ and } r \geqslant 0.$$

And Theorem A in case $p \geqslant 1$ can be rewritten as follows.

THEOREM A'. *Let* $p \geqslant 1$ *and* $r \geqslant 0$. *If* $\alpha > 1$, *then there exist positive operators* A *and* B *on* $\mathbb{R}^2$ *such that* $A \geqslant B > 0$ *and*

$$A^{(1+r)\alpha} \ngeqslant (A^{\frac{r}{2}} B^p A^{\frac{r}{2}})^{\frac{1+r}{p+r}\alpha}.$$

Ando [1] shows that $\log A \geqslant \log B$ is equivalent to that $A^r \geqslant (A^{\frac{r}{2}} B^r A^{\frac{r}{2}})^{\frac{1}{2}}$ holds for any $r \geqslant 0$ and for positive invertible operators A and B. The following Theorem B is an extension of this characterization.

THEOREM B. ([4, 5, 10]) *Let* A *and* B *be positive invertible operators. Then the following assertions are mutually equivalent:*

(i) $\log A \geqslant \log B$.
(ii) $A^r \geqslant (A^{\frac{r}{2}} B^p A^{\frac{r}{2}})^{\frac{r}{p+r}}$ *for any* $p \geqslant 0$ *and* $r \geqslant 0$.

Very recently, the following Theorem C which asserts the best possibility of (ii) of Theorem B was obtained as a parallel result to Theorem A' which asserts the best possibility of Theorem F'.

THEOREM C. ([21]) *Let* $p > 0$ *and* $r > 0$. *If* $\alpha > 1$, *then there exist positive invertible operators* A *and* B *on* $\mathbb{R}^2$ *such that* $\log A \geqslant \log B$ *and*

$$A^{r\alpha} \ngeqslant (A^{\frac{r}{2}} B^p A^{\frac{r}{2}})^{\frac{r}{p+r}\alpha}.$$

On the other hand, the following Theorem G is an extension of Theorem F.

THEOREM G. (Generalized Furuta inequality [11]) *If* $A \geqslant B \geqslant 0$ *with* $A > 0$, *then for each* $t \in [0,1]$,

$$A^{1-t+r} \geqslant \{A^{\frac{r}{2}} (A^{\frac{-t}{2}} B^p A^{\frac{-t}{2}})^s A^{\frac{r}{2}}\}^{\frac{1-t+r}{(p-t)s+r}} \tag{1.1}$$

holds for $p \geqslant 1$, $s \geqslant 1$ *and* $r \geqslant t$.

Alternative proof of Theorem G is shown in [6] and also very recently an elementary one page proof of Theorem G was shown in [12]. Related results on Theorem G are discussed in [13], [14], [15], [16] and [17]. Ando-Hiai [2] established excellent log majorization results and proved the useful inequality equivalent to the main log majorization theorem as follows: *If* $A \geqslant B \geqslant 0$ *with* $A > 0$, *then*

$$A^r \geqslant \{A^{\frac{r}{2}}(A^{\frac{-1}{2}}B^pA^{\frac{-1}{2}})^rA^{\frac{r}{2}}\}^{\frac{1}{p}}$$

holds for any $p \geqslant 1$ *and* $r \geqslant 1$. Theorem G interpolates the inequality stated above by Ando-Hiai and Theorem F itself. Tanahashi [20] shows that the outside exponents of both sides of (1.1) in Theorem G are the best possible as follows.

THEOREM D. ([20]) *Let* $p \geqslant 1$, $t \in [0,1]$, $r \geqslant t$ *and* $s \geqslant 1$. *If* $\alpha > 1$, *then there exist positive operators* A *and* B *on* $\mathbb{R}^2$ *such that* $A \geqslant B > 0$ *and*

$$A^{(1-t+r)\alpha} \ngeqslant \{A^{\frac{r}{2}}(A^{\frac{-t}{2}}B^pA^{\frac{-t}{2}})^sA^{\frac{r}{2}}\}^{\frac{1-t+r}{(p-t)s+r}\alpha}.$$

We remark that Tanahashi [19, 20] proved Theorem A and Theorem D independently by using ingenious technique. Especially the proof of Theorem D required a lot of elaborate calculations. In this paper, we obtain a simplified proof of Theorem D by using Theorem F', Theorem A, Theorem B and Theorem C.

2. Simplified proof of Theorem D

Proof of Theorem D. (a) In case $t \in [0,1)$. Assume that

$$S \geqslant T > 0 \quad \text{ensures} \quad S^{(1-t+r)\alpha} \geqslant \{S^{\frac{r}{2}}(S^{\frac{-t}{2}}T^pS^{\frac{-t}{2}})^sS^{\frac{r}{2}}\}^{\frac{1-t+r}{(p-t)s+r}\alpha} \tag{2.1}$$

for $p \geqslant 1$, $t \in [0,1)$, $r \geqslant t$, $s \geqslant 1$ and $\alpha > 1$.

On the other hand, $A \geqslant B > 0$ assures the following (2.2) by Theorem F':

$$A^{1+r_1} \geqslant (A^{\frac{r_1}{2}}B^{p_1}A^{\frac{r_1}{2}})^{\frac{1+r_1}{p_1+r_1}} \quad \text{for } p_1 \geqslant 1 \text{ and } r_1 \geqslant 0. \tag{2.2}$$

Put $p_1 = \frac{p-t}{1-t} \geqslant 1$ and $r_1 = \frac{t}{1-t} \geqslant 0$. Then (2.2) is equivalent to (2.3).

$$A^{\frac{1}{1-t}} \geqslant (A^{\frac{t}{2(1-t)}}B^{\frac{p-t}{1-t}}A^{\frac{t}{2(1-t)}})^{\frac{1}{p}}. \tag{2.3}$$

Put $S = A^{\frac{1}{1-t}}$ and $T = (A^{\frac{t}{2(1-t)}}B^{\frac{p-t}{1-t}}A^{\frac{t}{2(1-t)}})^{\frac{1}{p}}$. Then $S \geqslant T > 0$ by (2.3) and applying (2.1), we have

$$S^{(1-t+r)\alpha} \geqslant \{S^{\frac{r}{2}}(S^{\frac{-t}{2}}T^pS^{\frac{-t}{2}})^sS^{\frac{r}{2}}\}^{\frac{1-t+r}{(p-t)s+r}\alpha}. \tag{2.4}$$

(2.4) is equivalent to the following:

$$\begin{aligned} A^{(1+\frac{r}{1-t})\alpha} &\geqslant [A^{\frac{r}{2(1-t)}}\{A^{\frac{-t}{2(1-t)}}(A^{\frac{t}{2(1-t)}}B^{\frac{p-t}{1-t}}A^{\frac{t}{2(1-t)}})^{\frac{p}{p}}A^{\frac{-t}{2(1-t)}}\}^sA^{\frac{r}{2(1-t)}}]^{\frac{1-t+r}{(p-1)s+r}\alpha} \\ &= (A^{\frac{r}{2(1-t)}}B^{\frac{p-t}{1-t}s}A^{\frac{r}{2(1-t)}})^{\frac{1+\frac{r}{1-t}}{\frac{p-t}{1-t}s+\frac{r}{1-t}}\alpha}. \end{aligned} \tag{2.5}$$

Put $r_2 = \frac{r}{1-t} \geqslant 0$ and $p_2 = \frac{p-t}{1-t}s \geqslant 1$ in (2.5). Then (2.5) is equivalent to

$$A^{(1+r_2)\alpha} \geqslant (A^{\frac{r_2}{2}}B^{p_2}A^{\frac{r_2}{2}})^{\frac{1+r_2}{p_2+r_2}\alpha} \quad \text{for } p_2 \geqslant 1,\ r_2 \geqslant 0 \text{ and } \alpha > 1.$$

This contradiction proves the result in case $t \in [0,1)$ by Theorem A'.

(b) In case $t=1$. Assume that

$$S \geqslant T > 0 \quad \text{ensures} \quad S^{r\alpha} \geqslant \{S^{\frac{r}{2}}(S^{\frac{-1}{2}}T^pS^{\frac{-1}{2}})^sS^{\frac{r}{2}}\}^{\frac{r}{(p-1)s+r}\alpha} \tag{2.6}$$

for $p \geqslant 1$, $r \geqslant 1$, $s \geqslant 1$ and $\alpha > 1$.

For positive invertible operators A and B, $\log A \geqslant \log B$ assures the following (2.7) by Theorem B:

$$A \geqslant (A^{\frac{1}{2}}B^{p-1}A^{\frac{1}{2}})^{\frac{1}{p}}. \tag{2.7}$$

Put $S=A$ and $T=(A^{\frac{1}{2}}B^{p-1}A^{\frac{1}{2}})^{\frac{1}{p}}$. Then $S \geqslant T > 0$ by (2.7) and applying (2.6), we have

$$S^{r\alpha} \geqslant \{S^{\frac{r}{2}}(S^{\frac{-1}{2}}T^pS^{\frac{-1}{2}})^sS^{\frac{r}{2}}\}^{\frac{r}{(p-1)s+r}\alpha}. \tag{2.8}$$

(2.8) is equivalent to the following:

$$\begin{aligned} A^{r\alpha} &\geqslant [A^{\frac{r}{2}}\{A^{\frac{-1}{2}}(A^{\frac{1}{2}}B^{p-1}A^{\frac{1}{2}})^{\frac{p}{p}}A^{\frac{-1}{2}}\}^sA^{\frac{r}{2}}]^{\frac{r}{(p-1)s+r}\alpha} \\ &= (A^{\frac{r}{2}}B^{(p-1)s}A^{\frac{r}{2}})^{\frac{r}{(p-1)s+r}\alpha}. \end{aligned} \tag{2.9}$$

Put $p_1=(p-1)s \geqslant 0$ in (2.9). Then we have

$$A^{r\alpha} \geqslant (A^{\frac{r}{2}}B^{p_3}A^{\frac{r}{2}})^{\frac{r}{p_3+r}\alpha} \quad \text{for } p_3 \geqslant 0,\ r \geqslant 1 \text{ and } \alpha > 1.$$

This contradiction proves the result in case $t=1$ by Theorem C.

Hence the proof of Theorem D is complete.

Addendum. Recently M. Fujii, A. Matsumoto and R. Nakamoto [7] show a short proof of Theorem D without use of Theorem C by scrutinizing our paper. The proof in case $p=t=1$ can be reduced to Löwner-Heinz theorem, so we omit to describe it (see [7]).

Acknowledgement. I would like to express my cordial thanks to Professor Takayuki Furuta for his guidance and encouragement.

REFERENCES

[1] T. ANDO, *On some operator inequalities*, Math. Ann., **279** (1987), 157–159.
[2] T. ANDO AND F. HIAI, *Log majorization and complementary Golden-Thompson type inequalities*, Linear Algebra Appl., **197, 198** (1994), 113–131.
[3] M. FUJII, *Furuta's inequality and its mean theoretic approach*, J. Operator Theory, **23** (1990), 67–72.
[4] M. FUJII, T. FURUTA AND E. KAMEI, *Furuta's inequality and its application to Ando's theorem*, Linear Algebra Appl., **179** (1993), 161–169.
[5] M. FUJII, J. F. JIANG AND E. KAMEI, *Characterization of chaotic order and its application to Furuta inequality*, Proc. Amer. Math. Soc., **125** (1997), 3655–3658.
[6] M. FUJII AND E. KAMEI, *Mean theoretic approach to the grand Furuta inequality*, Proc. Amer. Math. Soc., **124** (1996), 2751–2756.
[7] M. FUJII, A. MATSUMOTO AND R. NAKAMOTO, *A short proof of the best possibility for the grand Furuta inequality*, preprint.

[8] T. FURUTA, *$A \geqslant B \geqslant 0$ assures $(B^rA^pB^r)^{1/q} \geqslant B^{(p+2r)/q}$ for $r \geqslant 0$, $p \geqslant 0$, $q \geqslant 1$ with $(1+2r)q \geqslant p+2r$*, Proc. Amer. Math. Soc., **101** (1987), 85–88.
[9] T. FURUTA, *An elementary proof of an order preserving inequality*, Proc. Japan Acad., **65** (1989), 126.
[10] T. FURUTA, *Applications of order preserving operator inequalities*, Oper. Theory Adv. Appl., **59** (1992), 180–190.
[11] T. FURUTA, *Extension of the Furuta inequality and Ando-Hiai log-majorization*, Linear Algebra Appl., **19** (1995), 139–155.
[12] T. FURUTA, *Simplified proof of an order preserving operator inequality*, Proc. Japan Acad., **74** (1998), 114.
[13] T. FURUTA, M. HASHIMOTO AND M. ITO, *Equivalence relation between generalized Furuta inequality and related operator functions*, Scientiae Mathematicae, **1** (1998), 257–259.
[14] T. FURUTA AND D. WANG, *A decreasing operator function associated with Furuta inequality*, Proc. Amer. Math. Soc., **126** (1998), 2427–2432.
[15] T. FURUTA, T. YAMAZAKI AND M. YANAGIDA, *Operator functions implying generalized Furuta inequality*, Math. Ineq. Appl., **1** (1998), 123–130.
[16] T. FURUTA, T. YAMAZAKI AND M. YANAGIDA, *Order preserving operator inequalities via Furuta inequality*, Math. Japon., **48** (1998), 471–476.
[17] T. FURUTA, T. YAMAZAKI AND M. YANAGIDA, *Order preserving operator function via Furuta inequality "$A \geqslant B \geqslant 0$ ensures $(A^{\frac{r}{2}}A^pA^{\frac{r}{2}})^{\frac{1+r}{p+r}} \geqslant (A^{\frac{r}{2}}B^pA^{\frac{r}{2}})^{\frac{1+r}{p+r}}$ for $p \geqslant 1$ and $r \geqslant 0$"*, to appear in Proc. 96-IWOTA.
[18] E. KAMEI, *A satellite to Furuta's inequality*, Math. Japon., **33** (1988), 883–886.
[19] K. TANAHASHI, *Best possibility of the Furuta inequality*, Proc. Amer. Math. Soc., **124** (1996), 141–146.
[20] K. TANAHASHI, *The best possibility of the grand Furuta inequality*, to appear in Proc. Amer. Math. Soc.
[21] M. YANAGIDA, *Some applications of Tanahashi's result on the best possibility of Furuta inequality*, to appear in Math. Ineq. Appl.

(Received February 10, 1999)
(Revised March 10, 1999)

Takeaki Yamazaki
Department of Applied Mathematics
Faculty of Science
Science University of Tokyo
1-3 Kagurazaka, Shinjuku
Tokyo 162-8601, Japan
e-mail: tyamaz@am.kagu.sut.ac.jp