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FUNCTIONAL STOLARSKY MEANS

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Abstract. A functional generalization is given for Stolarsky mean and its basic properties investigated. Functional means in n variables are also considered.

1. Introduction

As a response to the needs of diverse applications, a considerable variety of particular means of sets of numbers have been proposed and studied in the literature. See, for example, the compendious treatment of Bullen, Mitrinović and Vasić [1]. Valuable work has been done in systematizing and unifying this area *via* the judicious introduction of parameters.

A helpful paradigm is due to Stolarsky [18]. See also Tobey [19]. The Stolarsky mean $E_{r,s}(x,y)$ of two positive numbers x and y is given by $E_{r,s}(x,x) = x$ when the numbers coincide and otherwise by

$$E_{r,s}(x,y) = \begin{cases} \left[\dfrac{r}{s}\dfrac{y^s - x^s}{y^r - x^r}\right]^{1/(s-r)} & , \ r \neq s, \ r,s \neq 0 \\ \left[\dfrac{1}{r}\dfrac{y^r - x^r}{\ln y - \ln x}\right]^{1/r} & , \ r \neq 0, \ s = 0 \\ e^{-1/r}\left(\dfrac{x^{x^r}}{y^{y^r}}\right)^{1/(x^r - y^r)} & , \ s = r \neq 0 \\ \sqrt{xy} & , r = s = 0. \end{cases} \tag{1.1}$$

For various choices of the parameters r, s, this subsumes a number of commonly employed means as special cases. Apart from direct application, it has theoretical interest. Thus there is a comparison theorem prescribing for which pairs (r,s), (u,v) the inequality $E_{r,s}(x,y) < E_{u,v}(x,y)$ holds for all $x \neq y$ (see Leach and Sholander [7] and Páles [10, 11]). A trivial special case is the familiar inequality between the geometric and arithmetic means of a pair of distinct positive numbers.

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An interesting representation has been found [13] linking the Stolarsky mean with power means and integral means. The power mean $m_r(x,y;t)$ of order r and weights t and $1-t$ (for $t \in [0,1]$) of positive numbers x,y is defined by

$$m_r(x,y;t) = \begin{cases} (tx^r + (1-t)y^r)^{1/r}, & \text{if } r \neq 0, \\ x^t y^{1-t}, & \text{if } r = 0, \end{cases} \tag{1.2}$$

while the integral mean over $[0,1]$ of a positive function f is

$$M_r(f) = \begin{cases} \left[\int_0^1 (f(t))^r dt\right]^{1/r}, & \text{if } r \neq 0, \\ \exp\left(\int_0^1 \ln f(t) dt\right), & \text{if } r = 0. \end{cases}$$

It can be verified readily that

$$E_{r,s}(x,y) = M_{s-r}(m_r),$$

where $m_r(t) := m_r(x,y;t)$.

This suggests that a natural way to generalize the Stolarsky mean is to replace the role of a power mean in this relation by a quasiarithmetic mean. In this paper we develop such a generalization, which is seen to subsume and unify some recently proposed functional means.

In Section 2 we define a general class of weighted functional Stolarsky means and establish a basic comparison theorem. In Section 3 we generalize some Hadamard–type results from [13] for r–convex functions. We conclude in Section 4 by addressing multidimensional generalizations.

2. Functional Stolarsky means

DEFINITION 2.1. Let $g(\cdot)$ be strictly monotone and continuous function on an interval I, and let f be strictly monotone and continuous on the range of g^{-1}. Suppose μ is a probability measure on $[0,1]$. Then the weighted functional Stolarsky mean of two real numbers x, $y \in I$ is given by

$$\phi_{f,g}(x,y;\mu) = f^{-1}\left\{\int_0^1 f\left[g^{-1}(ug(y)+(1-u)g(x))\right] d\mu(u)\right\}.$$

We have trivially that $\phi_{f,g}(x,x;\mu) = x$. Our definition subsumes a number of means extant in the literature. Thus for $g(x) := x$, $f : (0,\infty) \to \mathbf{R}$, we have a functional mean considered in [2]. If $\mu(u) := u$, we suppress μ from the notation for ϕ and write $\phi_{f,g}(x,y)$. For $f(x) = x^{s-r}$ and $g(x) = x^r$, $\phi_{f,g}(x,y)$ reduces to the classical Stolarsky mean $E_{r,s}(x,y)$ given by (1.1).

For $x \neq y$, set $t = u[g(y)-g(x)]+g(x)$. Under this change of variable we derive

$$\phi_{f,g}(x,y) = f^{-1}\left\{\frac{1}{g(y)-g(x)}\int_{g(x)}^{g(y)} f\left(g^{-1}(t)\right) dt\right\}.$$

For $f(x) := x$, this reduces to the mean considered in [17].

The general functional Stolarsky mean admits the following comparison theorem.

THEOREM 2.2. *Suppose f, g satisfy the conditions of Definition 2.1 and similarly for F, G. If*

(i) *$F \circ f^{-1}$ is convex and F is increasing, or $F \circ f^{-1}$ is concave and F is decreasing, and*

(ii) *either $G \circ g^{-1}$ is convex and G is increasing, or $G \circ g^{-1}$ is concave and G is decreasing,*

then

$$\phi_{f,g}(x,y;\mu) \leqslant \phi_{F,G}(x,y;\mu).$$

If

(iii) *either $F \circ f^{-1}$ is convex and F is decreasing, or $F \circ f^{-1}$ is concave and F is increasing, and*

(iv) *either $G \circ g^{-1}$ is convex and G is decreasing, or $G \circ g^{-1}$ is concave and G is increasing,*

then

$$\phi_{f,g}(x,y;\mu) \geqslant \phi_{F,G}(x,y;\mu).$$

Proof. Suppose $G \circ g^{-1}$ is convex. Then the discrete Jensen inequality gives for X, Y in the domain of $G \circ g^{-1}$ that

$$(G \circ g^{-1})(tX + (1-t)Y) \leqslant t((G \circ g^{-1})(X) + (1-t)(G \circ g^{-1})(Y).$$

For $X = g(x)$ and $Y = g(y)$, this is equivalent to

$$G\{g^{-1}[tg(x) + (1-t)g(y)]\} \leqslant tG(x) + (1-t)G(y).$$

If G is increasing, we have consequently that

$$g^{-1}[tg(x) + (1-t)g(y)] \leqslant G^{-1}[tG(x) + (1-t)G(y)]. \tag{2.1}$$

Similarly, we can prove that (2.1) holds if $G \circ g^{-1}$ is concave and G is decreasing, and that the inequality is reversed if either $G \circ g^{-1}$ is convex and G is decreasing, or $G \circ g^{-1}$ is concave and G is increasing.

Moreover, by the integral Jensen inequality for a convex function $F \circ f^{-1}$, we have for H integrable that

$$(F \circ f^{-1}) \left[\int_0^1 H(t) d\mu(t) \right] \leqslant \int_0^1 (F \circ f^{-1})(H(t)) d\mu(t),$$

which for $H(t) = f(h(t))$ becomes

$$F \left\{ f^{-1} \left[\int_0^1 f(h(t)) d\mu(t) \right] \right\} \leqslant \int_0^1 F(h(t)) d\mu(t).$$

If F is also increasing we get

$$f^{-1} \left[\int_0^1 f(h(t)) d\mu(t) \right] \leqslant F^{-1} \left[\int_0^1 F(h(t)) d\mu(t) \right]. \tag{2.2}$$

Similarly, we can prove that (2.2) applies if $F\circ f^{-1}$ is concave and F is decreasing, and that (2.2) is reversed if either $F\circ f^{-1}$ is convex and F is decreasing, or $F\circ f^{-1}$ is concave and F is increasing. We have that f and f^{-1} are either both increasing or both decreasing. Therefore if $h_1(t)\leqslant h_2(t)$, we have

$$f^{-1}\left[\int_0^1 f(h_1(t))d\mu(t)\right]\leqslant f^{-1}\left[\int_0^1 f(h_2(t))d\mu(t)\right].$$

Now let $x\neq y$ and suppose the conditions for (2.1) and (2.2) are satisfied. Then

$$\begin{aligned}
\phi_{f,g}(x,y;\mu) &= f^{-1}\left\{\int_0^1 f\left[g^{-1}(ug(y)+(1-u)g(x))\right]d\mu(u)\right\}\\
&\leqslant f^{-1}\left\{\int_0^1 f[G^{-1}(uG(y)+(1-u)G(x))]d\mu(u)\right\}\\
&\leqslant F^{-1}\left\{\int_0^1 F[G^{-1}(uG(y)+(1-u)G(x))]d\mu(u)\right\}\\
&= \phi_{F,G}(x,y;\mu).
\end{aligned}\tag{2.3}$$

If the conditions apply for the inequalities in (2.1) and (2.2) to be reversed, we have the reverse inequalities in (2.3) too. □

In the special case $g(x)=G(x)=x$ with f and F strictly increasing on $(0,\infty)$, this reduces to [2, Theorem 1.3].

3. Inequalities of Hadamard type for g-convex functions

In [13] the following definition was given.

DEFINITION 3.1. Let f be a real–valued function on an interval $[a,b]$ and g a strictly monotone continuous function on the range of f. We say that f is g–convex if, for all x and $y\in[a,b]$ and $\lambda\in[0,1]$,

$$f(\lambda x+(1-\lambda)y)\leqslant g^{-1}[\lambda(g\circ f)(x)+(1-\lambda)(g\circ f)(y)].$$

We say that f is g–concave if the reverse inequality holds.

THEOREM 3.2. *Suppose f is defined on $[a,b]$ and let F be a strictly monotone continuous function defined on the range of f. If f is G–convex, then*

$$F^{-1}\left[\frac{1}{b-a}\int_a^b F(f(x))dx\right]\leqslant \phi_{F,G}(f(a),f(b)).$$

If f is G–concave then the reverse inequality applies.

Proof. We have for a G–convex function f that

$$\begin{aligned} F^{-1}\left[\frac{1}{b-a}\int_a^b F(f(x))dx\right] &= F^{-1}\left[\int_0^1 F(f(ub+(1-u)a))du\right] \\ &\leqslant F^{-1}\left[\int_0^1 F\circ G^{-1}(uG(f(b))+(1-u)G(f(a)))du\right] \\ &= \phi_{F,G}(f(a),f(b)). \end{aligned}$$

The second part follows similarly. ◻

For $F(x)=x^p$ and $G(x)=x^r$, this reduces to [13, Theorem 3.1] and for $F(x)=x$ to a result from [17].

THEOREM 3.3. *Suppose $f:[a,b]\to\mathbf{R}$ is continuous. Let F be a strictly monotone continuous function defined on the range of f and $w:[a,b]\to\mathbf{R}$ an integrable positive function. If either*

(i) f is g-convex, $F\circ g^{-1}$ is convex and F is decreasing, or

(ii) f is g-concave, $F\circ g^{-1}$ is concave and F is increasing,

then

$$F^{-1}\left\{\frac{\int_a^b w(x)F(f(x))dx}{\int_a^b w(x)dx}\right\}\leqslant g^{-1}\{\alpha^*(g\circ f)(b)+(1-\alpha^*)(g\circ f)(a)\}, \tag{3.1}$$

where

$$\alpha(x)=\frac{x-a}{b-a},\quad \alpha^*=\frac{\int_a^b \alpha(x)w(x)dx}{\int_a^b w(x)dx}.$$

The inequality in (3.1) is reversed if either

(iii) f is g-concave, $F\circ g^{-1}$ is convex and F is increasing, or

(iv) f is g-concave, $F\circ g^{-1}$ is concave and F is decreasing.

Moreover, if either

(v) f is g-convex, $F\circ g^{-1}$ is convex and F is increasing, or

(vi) f is g-convex, $F\circ g^{-1}$ is concave and F is decreasing,

then

$$F^{-1}\left[\frac{\int_a^b w(x)F(f(x))dx}{\int_a^b w(x)dx}\right]\leqslant F^{-1}[\alpha^*(F\circ f)(b)+(1-\alpha^*)(F\circ f)(a)]. \tag{3.2}$$

The inequality in (3.2) is reversed if either

(vii) f is g-concave, $F\circ g^{-1}$ is convex and F is decreasing, or

$(viii)$ f is g-concave, $F\circ g^{-1}$ is concave and F is increasing.

Proof. Let f be g-convex (respectively g-concave). We have

$$
\begin{aligned}
F^{-1}&\left[\frac{\int_a^b w(x)F(f(x))dx}{\int_a^b w(x)dx}\right] \\
&= F^{-1}\left\{\frac{\int_a^b w(x)F[f(\alpha(x)b+(1-\alpha(x))a)]dx}{\int_a^b w(x)dx}\right\} \\
&\underset{(\geqslant)}{\leqslant} F^{-1}\left\{\frac{\int_a^b w(x)F[g^{-1}\{\alpha(x)(g\circ f)(b)+(1-\alpha(x))(g\circ f)(a)\}]dx}{\int_a^b w(x)dx}\right\}.
\end{aligned} \tag{3.3}
$$

On the other hand, by Jensen's integral inequality we have that if $F\circ g^{-1}$ is convex (concave) then

$$
\begin{aligned}
&\frac{\int_a^b w(x)F\{g^{-1}[\alpha(x)(g\circ f)(b)+(1-\alpha(x))(g\circ f)(a)]\}dx}{\int_a^b w(x)dx} \\
&\quad \underset{(\leqslant)}{\geqslant} F\left\{g^{-1}\left[\frac{\int_a^b w(x)[\alpha(x)(g\circ f)(b)+(1-\alpha(x))(g\circ f)(a)]dx}{\int_a^b w(x)dx}\right]\right\} \\
&\quad = F\left\{g^{-1}[\alpha^*(g\circ f)(b)+(1-\alpha^*)(g\circ f)(a)]\right\}.
\end{aligned} \tag{3.4}
$$

From (3.3) and (3.4) we get (3.1) (the reverse inequality).

Moreover, by Jensen's discrete inequality, if $F\circ g^{-1}$ is convex (concave), we have that

$$
\begin{aligned}
&\frac{\int_a^b w(x)F\{g^{-1}[\alpha(x)(g\circ f)(b)+(1-\alpha(x))(g\circ f)(a)]\}dx}{\int_a^b w(x)dx} \\
&\quad \underset{(\geqslant)}{\leqslant} \frac{\int_a^b w(x)\{\alpha(x)(F\circ f)(b)+(1-\alpha(x))(F\circ f)(a)]\}dx}{\int_a^b w(x)dx} \\
&\quad = \alpha^*(F\circ f)(b)+(1-\alpha^*)(F\circ f)(a).
\end{aligned} \tag{3.5}
$$

From (3.3) and (3.5) we get (3.2) (its reverse inequality). □

Let $F(x)=x$ and suppose w is symmetric on $[a,b]$, that is,

$$
w(a+t)=w(b-t), \quad 0\leqslant t\leqslant \frac{1}{2}(b-a).
$$

Then $\alpha^*=1/2$ and we have a result obtained in [14].

DEFINITION 3.4. A positive function f is said to be r–convex on an interval $[a,b]$ if, for all $x,y\in[a,b]$ and $\lambda\in[0,1]$,

$$
f(\lambda x+(1-\lambda)y)\leqslant m_r(f(x),f(y);\lambda),
$$

where m_r is defined by (1.2).

COROLLARY 3.5. *Let* $f : [a,b] \to \mathbf{R}$ *be a positive continuous function and* w *an integrable positive function.*

(a) *If* f *is* r*-convex and* $\ell = \max\{r,p\}$*, then*

$$\left[\frac{\int_a^b w(x)(f(x))^p dx}{\int_a^b w(x)}\right]^{1/p} \leqslant m_\ell(f(b), f(a); \lambda).$$

(b) *If* f *is* r*-concave and* $\ell = \min\{r,p\}$*, then the inequality is reversed.*

For $p = 1$ and $r = 0$, that is, when f is a log-convex function, we have

$$[f(b)]^\lambda [f(a)]^{1-\lambda} \leqslant \frac{\int_a^b w(x)f(x)dx}{\int_a^b w(x)dx} \leqslant \lambda f(b) + (1-\lambda)f(a).$$

(see Fink [4] and Pečarić and Čuljak [14]). For some related results see also [3], [5], [8], [12].

4. Multidimensional functional Stolarsky–Tobey means

DEFINITION 4.1. Let $E_{n-1} \subset \mathbf{R}^{n-1}$ represent the simplex

$$E_{n-1} = \left\{(u_1, \ldots, u_{n-1}) : u_i \geqslant 0 \ (1 \leqslant i \leqslant n-1),\ \sum_{j=1}^{n-1} u_j \leqslant 1\right\}$$

and set $u_n = 1 - \sum_{j=1}^{n-1} u_j$. With $\mathbf{u} = (u_1, \ldots, u_n)$, let $\mu(\mathbf{u})$ be a probability measure on E_{n-1}.

For $\mathbf{u} \in E_{n-1}$, $r \in \mathbf{R}$ and $\mathbf{x} = (x_1, \ldots, x_n) \in \mathbf{R}_+^n$, the power mean of order r of $x_1, \ldots, x_n$ is defined by

$$m_r(\mathbf{x};\mathbf{u}) := \begin{cases} \left(\sum_{i=1}^n u_i x_i^r\right)^{1/r}, & \text{if } r \neq 0, \\ \prod_{i=1}^n x_i^{u_i}, & \text{if } r = 0. \end{cases}$$

The integral power mean $\overline{M}_t$ of order $t \in \mathbf{R}$ of a positive function f on E_{n-1} with probability measure μ is defined by

$$\overline{M}_t(f;\mu) := \begin{cases} \left[\int_{E_{n-1}} \{f(\mathbf{u})\}^t \, d\mu(\mathbf{u})\right]^{1/t}, & \text{if } t \neq 0, \\ \exp\left[\int_{E_{n-1}} \ln(f(\mathbf{u})) d\mu(\mathbf{u})\right], & \text{if } t = 0, \end{cases}$$

assuming that the expressions involved are well–defined (see [6, Ch. 3]).

Let $\mathbf{x} = (x_1, \ldots, x_n) \in \mathbf{R}_+^n$ and $r, t \in \mathbf{R}$. Tobey [19] has studied the two–parameter homogeneous mean

$$L_{r,t}(\mathbf{x};\mu) := \overline{M}_t(m_r(\mathbf{x};\cdot);\mu)$$

of $x_1,\ldots,x_n$.

Now let I be a real interval and $x_i \in I \quad (1 \leqslant i \leqslant n)$ and suppose f,g are two strictly monotone continuous functions on I. We say that $\phi_{f,g}(\mathbf{x};\mu)$ is a functional Stolarsky–Tobey mean if

$$\phi_{f,g}(\mathbf{x};\mu) = f^{-1}\left\{\int_{E_{n-1}} f\left[g^{-1}\left(\sum_{i=1}^{n} u_i g(x_i)\right)\right] d\mu(\mathbf{u})\right\},$$

where $\mathbf{x} = (x_1,x_2,\ldots,x_n)$.

Special cases of the above means are .given in [2], [9], [15], [17]. For example, for $g(x) := x$ we have a functional mean considered in [2]. Tobey's homogeneous mean is subsumed under $f(y) = y^t$, $g(y) = y^r$ and $I = \mathbf{R}$.

The proof of the following theorem follows closely that of Theorem 2.2.

THEOREM 4.2. *If*

(i) *either* $F \circ f^{-1}$ *is convex and* F *is increasing, or* $F \circ f^{-1}$ *is concave and* F *is decreasing, and*

(ii) *either* $G \circ g^{-1}$ *is convex and* G *is increasing, or* $G \circ g^{-1}$ *is concave and* G *is decreasing,*

then

$$\phi_{f,g}(\mathbf{x};\mu) \leqslant \phi_{F,G}(\mathbf{x};\mu).$$

If

(iii) *either* $F \circ f^{-1}$ *is convex and* F *is decreasing, or* $F \circ f^{-1}$ *is concave and* F *is increasing, and*

(iv) *either* $G \circ g^{-1}$ *is convex and* G *is decreasing, or* $G \circ g^{-1}$ *is concave and* G *is increasing,*

then

$$\phi_{f,g}(\mathbf{x};\mu) \geqslant \phi_{F,G}(\mathbf{x};\mu).$$

Denote by

$$w_i = \int_{E_{n-1}} u_i d\mu(\mathbf{u})$$

the i-th weight associated with the probability measure μ on E_{n-1}. Then $w_i > 0$ $(1 \leqslant i \leqslant n)$ and $w_1 + \cdots + w_n = 1$.

We have

$$\phi_{f,f}(\mathbf{x};\mu) = f^{-1}\left\{\sum_{i=1}^{n} w_i f(x_i)\right\},$$

which is just the quasiarithmetic mean of the numbers $\{x_i\}$ with weights $\{w_i\}$ for the function f.

THEOREM 4.3. *If either*

(i) $f \circ g^{-1}$ *is convex and* f *is increasing, or*

(ii) $f \circ g^{-1}$ *is concave and* f *is decreasing,*

then

$$\phi_{g,g}(\mathbf{x};\mu) \leqslant \phi_{f,g}(\mathbf{x};\mu) \leqslant \phi_{f,f}(\mathbf{x};\mu). \tag{4.1}$$

If either

(iii) $f\circ g^{-1}$ *is convex and* f *is decreasing, or*

(iv) $f\circ g^{-1}$ *is concave and* f *is increasing,*

then the inequality is reversed.

Proof. By Jensen's integral inequality we have that if $f\circ g^{-1}$ is convex,

$$\begin{aligned}\int_{E_{n-1}} f\left[g^{-1}\left(\sum_{i=1}^{n} u_i g(x_i)\right)\right] d\mu(\mathbf{u}) &\geqslant f\left\{g^{-1}\left[\int_{E_{n-1}}\left(\sum_{i=1}^{n} u_i g(x_i)\right) d\mu(\mathbf{u})\right]\right\}\\ &= f\left\{g^{-1}\left[\sum_{i=1}^{n} w_i g(x_i)\right]\right\}\\ &= f\left(\phi_{g,g}(\mathbf{x};\mu)\right).\end{aligned} \tag{4.2}$$

By Jensen's discrete inequality we have that if $f\circ g^{-1}$ is convex, then

$$\begin{aligned}\int_{E_{n-1}} f\left[g^{-1}\left(\sum_{i=1}^{n} u_i g(x_i)\right)\right] d\mu(\mathbf{u}) &\leqslant \int_{E_{n-1}} \sum_{i=1}^{n} u_i f(x_i) d\mu(\mathbf{u})\\ &= \sum_{i=1}^{n} w_i f(x_i)\\ &= f\left(\phi_{f,f}(\mathbf{x};\mu)\right).\end{aligned} \tag{4.3}$$

If f is increasing, (4.1) now follows from (4.2) and (4.3). The other cases are derived similarly. □

Our next theorem considers unweighted functional Stolarsky–Tobey means, when μ reduces to Lebesgue measure

$$d\mu(\mathbf{u}) = (n-1)!du_1\ldots du_{n-1} = (n-1)!d\mathbf{u}.$$

An easy calculation gives

$$\int_{E_{n-1}} du_1\ldots du_{n-1} = \frac{1}{(n-1)!}$$

and

$$w_i = \int_{E_{n-1}} u_i d\mu(\mathbf{u}) = \frac{1}{n}.$$

We write $\phi_{f,g}(\mathbf{x})$ for $\phi_{f,g}(\mathbf{x},\mu)$ in this case.

THEOREM 4.4. *Suppose* $x_i\neq x_j$ *for* $i\neq j$ *and let* $H(t)$ *be such that* $H^{(n-1)} = f\circ g^{-1}$. *Then*

$$\phi_{f,g}(\mathbf{x}) = f^{-1}\left[(n-1)!\sum_{i=1}^{n}\frac{(H\circ g)(x_i)}{\prod_{j\in A(i)}(g(x_i)-g(x_j))}\right],$$

where $A(i) := \{1, 2, \ldots, n\} \setminus \{i\}$.

Proof. We use the well–known relation

$$[t_1, \ldots, t_n]f = \sum_{i=1}^{n} \frac{f(t_i)}{\prod_{j \in A(i)} (t_i - t_j)} = \int_{E_{n-1}} f^{(n-1)} \left(\sum_{i=1}^{n} u_i t_i \right) d\mathbf{u},$$

where $[t_1, \ldots, t_n]f$ stands for the divided differences of order $n-1$ of t with knots at $t_1, \ldots, t_n$ and $t \in C^{n-1}(a, b)$, $a = \min(t_i)$, $b = \max(t_i)$, $1 \leqslant i \leqslant n$. So we have

$$\begin{aligned} \phi_{f,g}(\mathbf{x}) &= f^{-1} \left\{ (n-1)! \int_{E_{n-1}} (f \circ g^{-1}) \left(\sum_{i=1}^{n} u_i g(x_i) \right) d\mathbf{u} \right\} \\ &= f^{-1} \left\{ (n-1)! \int_{E_{n-1}} H^{(n-1)} \left(\sum_{i=1}^{n} u_i g(x_i) \right) d\mathbf{u} \right\}, \end{aligned}$$

whence the desired result. □

The above gives as special cases results obtained in [9], [15], [17].

Theorems 4.3 and 4.4 give the following.

COROLLARY 4.5. *If either* (i) *or* (ii) *of Theorem 4.3 holds and* H *is as in Theorem 4.4, then*

$$\begin{aligned} g\left(\frac{1}{n} \sum_{i=1}^{n} g(x_i) \right) &\leqslant f^{-1} \left[(n-1)! \sum_{i=1}^{n} \frac{(H \circ g)(x_i)}{\prod_{j \in A(i)} (g(x_i) - g(x_j))} \right] \\ &\leqslant f^{-1} \left(\frac{1}{n} \sum_{i=1}^{n} f(x_i) \right). \end{aligned}$$

If either (i) *or* (ii) *from Theorem 4.3 applies, then the inequalities are reversed.*

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