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AN OSTROWSKI TYPE INEQUALITY FOR A RANDOM VARIABLE WHOSE PROBABILITY DENSITY FUNCTION BELONGS TO $L_p[a,b], p > 1$

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(*communicated by B. Mond*)

Abstract. An inequality of Ostrowski's type for a random variable whose probability density function is in $L_p[a,b], p > 1$, in terms of the cumulative distribution function and expectation is given. An application for a Beta random variable is also given.

1. Introduction

The following theorem contains the integral inequality which is known in the literature as Ostrowski's inequality [1, p. 469]:

THEOREM 1.1. *Let $f : I \subseteq \mathbf{R} \to \mathbf{R}$ be a differentiable mapping in $\overset{\circ}{I}$ ($\overset{\circ}{I}$ is the interior of I), and let $a, b \in \overset{\circ}{I}$ with $a < b$. If $f' : (a,b) \to \mathbf{R}$ is bounded on (a,b), i.e., $\|f'\|_\infty := \sup_{t\in(a,b)} |f'(t)| < \infty$, then we have*

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(t)\,dt \right| \leqslant \left[\frac{1}{4} + \frac{\left(x - \frac{a+b}{2}\right)^2}{(b-a)^2}\right](b-a)\|f'\|_\infty \tag{1.1}$$

for all $x \in [a,b]$.
The constant $\frac{1}{4}$ is sharp in the sense that it can not be replaced by a smaller one.

In [2], S.S Dragomir and S. Wang applied Ostrowski's inequality in Numerical Analysis obtaining an estimation of the error bound for the quadrature rules of Riemann type in terms of the infinity norm. Application for special means: logarithmic mean, identric mean, $p-$ logarithmic mean etc... were also given.

In [3], N.S. Barnett and S.S. Dragomir established the following version of Ostrowski's inequality for cumulative distribution functions

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THEOREM 1.2. *Let X be a random variable with the probability density function $f : [a,b] \subset \mathbf{R} \to \mathbf{R}_+$ and with cumulative distribution function $F(x) = \Pr(X \leqslant x)$. If $f \in L_\infty [a,b]$ and $\|f\|_\infty := \sup_{t\in(a,b)} f(t) < \infty$, then*

$$\left|\Pr(X \leqslant x) - \frac{b-E(X)}{b-a}\right| \leqslant \left[\frac{1}{4} + \frac{\left(x - \frac{a+b}{2}\right)^2}{(b-a)^2}\right](b-a)\|f\|_\infty \tag{1.2}$$

for all $x \in [a,b]$.
The constant $\frac{1}{4}$ in (1.2) is sharp.

The main aim of this paper is to give an Ostrowski type inequality for a random variable whose probability density functions are in $L_p[a,b]\,(p > 1)$. An application for a Beta Random Variable is also given.

2. An Ostrowski type inequality

The following theorem holds

THEOREM 2.1. *Let X be a random variable with the probability density function $f : [a,b] \subset \mathbf{R} \to \mathbf{R}_+$ and with cumulative distribution function $F(x) = \Pr(X \leqslant x)$. If $f \in L_p[a,b], p > 1$, then we have the inequality:*

$$\left|\Pr(X \leqslant x) - \frac{b-E(X)}{b-a}\right| \tag{2.1}$$

$$\leqslant \frac{q}{q+1}\|f\|_p (b-a)^{\frac{1}{q}} \left[\left(\frac{x-a}{b-a}\right)^{\frac{1+q}{q}} + \left(\frac{b-x}{b-a}\right)^{\frac{1+q}{q}}\right]$$

$$\leqslant \frac{q}{q+1}\|f\|_p (b-a)^{\frac{1}{q}}$$

for all $x \in [a,b]$, where $\frac{1}{p} + \frac{1}{q} = 1$.

Proof. By Hölder's integral inequality we have

$$|F(x) - F(y)| = \left|\int_x^y f(t)\,dt\right| \tag{2.2}$$

$$\leqslant \left|\int_x^y dt\right|^{\frac{1}{q}} \left|\int_x^y |f(t)|^p\,dt\right|^{\frac{1}{p}} \leqslant |x-y|^{\frac{1}{q}} \|f\|_p$$

for all $x, y \in [a,b]$, where $p > 1, \frac{1}{p} + \frac{1}{q} = 1$ and

$$\|f\|_p := \left(\int_a^b |f(t)|^p\,dt\right)^{\frac{1}{p}}$$

is the usual $p-$ norm on $L_p[a,b]$.

The inequality (2.2) shows in fact that the mapping $F(\cdot)$ is of $r-H-$ Hölder type, i.e.,

$$|F(x)-F(y)| \leqslant H|x-y|^r, \qquad (\forall)\, x,y \in [a,b] \tag{2.3}$$

with $0 < H = \|f\|_p$ and $r = \frac{1}{q} \in (0,1)$.

Integrating the inequality (2.2) over $y \in [a,b]$ we get successively

$$\left|F(x) - \frac{1}{b-a}\int_a^b F(y)\,dy\right| \tag{2.4}$$

$$\begin{aligned}
&\leqslant \frac{1}{b-a}\int_a^b |F(x)-F(y)|\,dy \leqslant \frac{1}{b-a}\|f\|_p \int_a^b |x-y|^{\frac{1}{q}}\,dy \\
&= \frac{1}{b-a}\|f\|_p\left[\int_a^x (x-y)^{\frac{1}{q}}\,dy + \int_x^b (y-x)^{\frac{1}{q}}\,dy\right] \\
&= \frac{1}{b-a}\|f\|_p\left[\frac{(x-a)^{\frac{1}{q}+1}}{\frac{1}{q}+1} + \frac{(b-x)^{\frac{1}{q}+1}}{\frac{1}{q}+1}\right] \\
&= \frac{q}{q+1}\cdot\frac{1}{b-a}\|f\|_p\left[(x-a)^{\frac{1}{q}+1} + (b-x)^{\frac{1}{q}+1}\right] \\
&= \frac{q}{q+1}\|f\|_p (b-a)^{\frac{1}{q}}\left[\left(\frac{x-a}{b-a}\right)^{\frac{1}{q}+1} + \left(\frac{b-x}{b-a}\right)^{\frac{1}{q}+1}\right]
\end{aligned}$$

for all $x \in [a,b]$.

It is well known that

$$E(X) = b - \int_a^b F(t)\,dt$$

then, by (2.4), we get the first inequality in (2.1).

For the second inequality, we observe that

$$\left(\frac{x-a}{b-a}\right)^{\frac{1}{q}+1} + \left(\frac{b-x}{b-a}\right)^{\frac{1}{q}+1} \leqslant 1, \qquad (\forall)\, x \in [a,b]$$

and the theorem is completely proved. $\square$

REMARK 2.1. The inequality (2.1) is equivalent to

$$\left|\Pr(X \geqslant x) - \frac{E(X)-a}{b-a}\right| \tag{2.5}$$

$$\leqslant \frac{q}{q+1}\|f\|_p (b-a)^{\frac{1}{q}}\left[\left(\frac{x-a}{b-a}\right)^{\frac{1+q}{q}}+\left(\frac{b-x}{b-a}\right)^{\frac{1+q}{q}}\right]$$
$$\leqslant \frac{q}{q+1}\|f\|_p (b-a)^{\frac{1}{q}}, \qquad (\forall)\, x\in[a,b].$$

COROLLARY 2.2. *Under the above assumptions, we have the double inequality*

$$b-\frac{q}{q+1}\|f\|_p (b-a)^{1+\frac{1}{q}} \leqslant E(X) \leqslant a+\frac{q}{q+1}\|f\|_p (b-a)^{\frac{1}{q}+1} \tag{2.6}$$

Proof. We know that $a \leqslant E(X) \leqslant b$.

Now, choose in (2.1) $x=a$ to get

$$\left|\frac{b-E(X)}{b-a}\right| \leqslant \frac{q}{q+1}\|f\|_p (b-a)^{\frac{1}{q}}$$

i.e.,

$$b-E(X) \leqslant \frac{q}{q+1}\|f\|_p (b-a)^{1+\frac{1}{q}}$$

which is equivalent to the first inequality in (2.6).

Also, choosing $x=b$ in (2.1), we get

$$\left|1-\frac{b-E(X)}{b-a}\right| \leqslant \frac{q}{q+1}\|f\|_p (b-a)^{\frac{1}{q}}$$

i.e.,

$$E(X)-a \leqslant \frac{q}{q+1}\|f\|_p (b-a)^{\frac{1}{q}+1}$$

which is equivalent to the second inequality in (2.6). □

REMARK 2.2. We know that by Hölder's integral inequality

$$1=\int_a^b f(t)\,dt \leqslant (b-a)^{\frac{1}{q}}\|f\|_p$$

which gives

$$\|f\|_p \geqslant \frac{1}{(b-a)^{\frac{1}{q}}}.$$

Now, if we assume that $\|f\|_p$ is not too large, i.e.,

$$\|f\|_p \leqslant \frac{q+1}{q}\cdot\frac{1}{(b-a)^{\frac{1}{q}}} \tag{2.7}$$

then we get

$$a+\frac{q}{q+1}\|f\|_p (b-a)^{\frac{1}{q}+1} \leqslant b$$

and

$$b-\frac{q}{q+1}\|f\|_p (b-a)^{1+\frac{1}{q}} \geqslant a$$

which shows that the inequality (2.6) is a tighter inequality than $a \leqslant E(X) \leqslant b$ when (2.7) holds.

Another equivalent inequality to (2.6) which can be more useful in practice is the following one:

COROLLARY 2.3. *With the above assumptions, we have the inequality:*

$$\left|E(X)-\frac{a+b}{2}\right| \leqslant (b-a)\left[\frac{q}{q+1}\|f\|_p (b-a)^{\frac{1}{q}}-\frac{1}{2}\right] \tag{2.8}$$

Proof. From the inequality (2.6) we have:

$$\begin{aligned} b-\frac{a+b}{2}-\frac{q}{q+1}\|f\|_p (b-a)^{1+\frac{1}{q}} &\leqslant E(X)-\frac{a+b}{2} \\ &\leqslant a-\frac{a+b}{2}+\frac{q}{q+1}\|f\|_p (b-a)^{1+\frac{1}{q}} \end{aligned}$$

i.e.,

$$\begin{aligned} \frac{b-a}{2}-\frac{q}{q+1}\|f\|_p (b-a)^{1+\frac{1}{q}} &\leqslant E(X)-\frac{a+b}{2} \\ &\leqslant -\frac{b-a}{2}+\frac{q}{q+1}\|f\|_p (b-a)^{1+\frac{1}{q}} \end{aligned}$$

which is equivalent to

$$\begin{aligned} \left|E(X)-\frac{a+b}{2}\right| &\leqslant \frac{q}{q+1}\|f\|_p (b-a)^{1+\frac{1}{q}}-\frac{b-a}{2} \\ &= (b-a)\left[\frac{q}{q+1}\|f\|_p (b-a)^{\frac{1}{q}}-\frac{1}{2}\right] \end{aligned}$$

and the inequality (2.8) is proved. □

This Corollary provides the possibility of finding a sufficient condition in terms of $\|f\|_p$ $(p>1)$ for the expectation $E(X)$ to be close to the mean value $\frac{a+b}{2}$.

COROLLARY 2.4. *Let X and f be as above and $\varepsilon>0$. If*

$$\|f\|_p \leqslant \frac{q+1}{2q}\cdot\frac{1}{(b-a)^{\frac{1}{q}}}+\frac{\varepsilon(q+1)}{q(b-a)^{1+\frac{1}{q}}}$$

then

$$\left|E(X)-\frac{a+b}{2}\right| \leqslant \varepsilon.$$

The proof is similar, and we omit the details.

The following corollary of Theorem 2.1 also holds:

COROLLARY 2.5. *Let X and f be as above. Then we have the inequality:*

$$\left|\Pr\left(X \leqslant \frac{a+b}{2}\right)-\frac{1}{2}\right| \leqslant \frac{q}{2^{\frac{1}{q}}(q+1)}\|f\|_p (b-a)^{\frac{1}{q}}+\frac{1}{b-a}\left|E(X)-\frac{a+b}{2}\right|.$$

Proof. If we choose in (2.1) $x=\frac{a+b}{2}$, we get

$$\left|\Pr\left(X\leqslant\frac{a+b}{2}\right)-\frac{b-E(X)}{b-a}\right|\leqslant\frac{q}{2^{\frac{1}{q}}(q+1)}\|f\|_{p}(b-a)^{\frac{1}{q}}$$

which is clearly equivalent to:

$$\left|\Pr\left(X\leqslant\frac{a+b}{2}\right)-\frac{1}{2}+\frac{1}{b-a}\left(E(X)-\frac{a+b}{2}\right)\right|\leqslant\frac{q}{2^{\frac{1}{q}}(q+1)}\|f\|_{p}(b-a)^{\frac{1}{q}}.$$

Now, using the triangle inequality, we get

$$\begin{aligned}
&\left|\Pr\left(X\leqslant\frac{a+b}{2}\right)-\frac{1}{2}\right|\\
&\quad=\left|\Pr\left(X\leqslant\frac{a+b}{2}\right)-\frac{1}{2}+\frac{1}{b-a}\left(E(X)-\frac{a+b}{2}\right)-\frac{1}{b-a}\left(E(X)-\frac{a+b}{2}\right)\right|\\
&\quad\leqslant\left|\Pr\left(X\leqslant\frac{a+b}{2}\right)-\frac{1}{2}+\frac{1}{b-a}\left(E(X)-\frac{a+b}{2}\right)\right|+\frac{1}{b-a}\left|E(X)-\frac{a+b}{2}\right|\\
&\quad\leqslant\frac{q}{2^{\frac{1}{q}}(q+1)}\|f\|_{p}(b-a)^{\frac{1}{q}}++\frac{1}{b-a}\left|E(X)-\frac{a+b}{2}\right|
\end{aligned}$$

and the corollary is proved. □

Finally, the following result also holds:

COROLLARY 2.6. *With the above assumptions, we have:*

$$\begin{aligned}
&\left|E(X)-\frac{a+b}{2}\right|\\
&\qquad\leqslant\frac{q}{2^{\frac{1}{q}}(q+1)}\|f\|_{p}(b-a)^{1+\frac{1}{q}}+(b-a)\left|\Pr\left(X\leqslant\frac{a+b}{2}\right)-\frac{1}{2}\right|.
\end{aligned}$$

The proof is similar and we omit the details.

3. Applications for a beta random variable

A Beta Random Variable X with parameters $(s,t)\in\Omega$ has the probability density function

$$f\ (x;s,t):=\frac{x^{s-1}(1-x)^{t-1}}{B(s,t)};\quad 0<x<1$$

where

$$\Omega:=\{(s,t):s,t>0\}$$

and

$$B(s,t) := \int_0^1 \tau^{s-1}(1-\tau)^{t-1}\,d\tau.$$

We observe that, for $p > 1$,

$$\begin{aligned}\|f(\cdot;s,t)\|_p &= \frac{1}{B(s,t)}\left(\int_0^1 \tau^{p(s-1)}(1-\tau)^{p(t-1)}\,d\tau\right)^{\frac{1}{p}}\\ &= \frac{1}{B(s,t)}\left(\int_0^1 \tau^{p(s-1)+1-1}(1-\tau)^{p(t-1)+1-1}\,d\tau\right)^{\frac{1}{p}}\\ &= \frac{1}{B(s,t)}\left[B\left(p(s-1)+1,p(t-1)+1\right)\right]^{\frac{1}{p}}\end{aligned}$$

provided

$$p(s-1)+1, \quad p(t-1)+1 > 0$$

i.e.,

$$s > 1-\frac{1}{p} \quad \text{and} \quad t > 1-\frac{1}{p}.$$

Now, using Theorem 2.1, we can state the following proposition:

PROPOSITION 3.1. *Let $p > 1$ and X be a Beta random variable with the parameters (s,t), $s > 1-\frac{1}{p}, t > 1-\frac{1}{p}$. Then we have the inequality:*

$$\left|\Pr(X \leqslant x) - \frac{t}{s+t}\right| \leqslant \frac{q}{q+1}\frac{\left[x^{\frac{1+q}{q}}+(1-x)^{\frac{1+q}{q}}\right]\left[B\left(p(s-1)+1,p(t-1)+1\right)\right]^{\frac{1}{p}}}{B(s,t)} \tag{3.1}$$

for all $x \in [0,1]$.
Particularly, we have

$$\left|\Pr\left(X \leqslant \frac{1}{2}\right) - \frac{t}{s+t}\right| \leqslant \frac{q}{2^{\frac{1}{q}}(q+1)}\frac{\left[B\left(p(s-1)+1,p(t-1)+1\right)\right]^{\frac{1}{p}}}{B(s,t)}.$$

The proof follows by Theorem 2.1 choosing $f(x) = f(x;s,t)$, $x \in [0,1]$ and taking into account that $E(X) = \frac{s}{s+t}$.

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