Mathematical
Inequalities
& Applications
Volume 2, Number 4 (1999), 509–516

ON THE FACTORIZATION OF INEQUALITIES

L. LEINDLER

(*communicated by R. Mohapatra*)

Abstract. We generalize two theorems pertaining to the factorization of inequalities of infinite numerical series.

1. Introduction

G. Bennett [1] presented several very interesting results concerning the problem of factorization of inequalities. In [1] we can read the motivation of this problem, the precise definition of the factorization of inequalities and an account of its benefits. I do believe that the best way to get acquainted with this interesting research field is to read Bennett's work.

In some recent papers we also studied such problems.

The aim of the present paper is to prove some further theorems of this type.

In order to recall some results to be generalized here we have to recollect some notations and notions.

Let $\mathbf{x} := \{x_n\}$ denote an arbitrary sequence of real (or complex) numbers. Let $\lambda := \{\lambda_n\}$ be a sequence of nonnegative terms having infinitely many positive ones. We shall use the following notations:

$$\Lambda_n := \sum_{k=n}^{\infty} \lambda_k,$$

(whenever Λ_n appears it also signals that $\Lambda_1 < \infty$, except if the contrary is explicitly stated)

$$H_n := \begin{cases} 1, & \text{if} \quad 1 \leqslant n < n_0, \\ 1 + \sum_{k=1}^{n} \lambda_k, & \text{if} \quad n \geqslant n_0, \end{cases}$$

where n_0 is the smallest natural number satisfying $\lambda_{n_0} > 0$.

Given $p, q > 0$, we shall write $\varphi \in \Phi(p)$ if φ is a nonnegative increasing function on $[0, \infty)$, $\varphi(0) = 0$, $\varphi(x)x^{-p}$ is nonincreasing, and $\varphi \in \Psi(q)$ will denote if $\varphi(x)x^{-q}$ is nondecreasing.

Mathematics subject classification (1991): 26D15, 40A05, 40A99.
Key words and phrases: Inequalities for sums, factorization, power-monotone sequences.

Paper MIA-02-41

Furthermore we agree that the constants K, K_i to be appearing in inequalities may vary from occurance to occurance, and they are positive. If we wish to express the dependence explicitly, we write K in the form $K(\alpha,\ldots)$.

A summation sign $\sum$ in which the limits of the summation are omitted will denote summation from 1 to ∞.

We define for $p>1$ and $c\geqslant 0$ the following sets:

$$\ell^p := \Big\{\mathbf{x} : \sum |x_n|^p < \infty\Big\},$$

$$h(p,c,\Lambda) := \Big\{\mathbf{x} : \sum \lambda_n \Lambda_n^{-c}\Big(\sum_{k=1}^{n}|x_k|\Big)^p < \infty\Big\}, \tag{1.1}$$

$$H(p,c,\Lambda) := \Big\{\mathbf{x} : \sum_{k=1}^{n}|x_k|^p = O(\Lambda_n^{(1-p)(1-c)})\Big\}, \tag{1.2}$$

$$h(p,c,H) := \Big\{\mathbf{x} : \sum \lambda_n H_n^{-c}\Big(\sum_{k=1}^{n}|x_k|\Big)^p < \infty\Big\}, \tag{1.3}$$

$$H(p,c,H) := \Big\{\mathbf{x} : \sum_{k=1}^{n}|x_k|^p = O(H_n^{(1-p)(1-c)})\Big\}, \tag{1.4}$$

and the norms:

$$\|\mathbf{x}\|_p := \Big(\sum |x_n|^p\Big)^{1/p},$$

$$\|\mathbf{x}\|_{h(p,c,\Lambda)} := \Big\{\sum \lambda_n \Lambda_n^{-c}\Big(\sum_{k=1}^{n}|x_k|\Big)^p\Big\}^{1/p},$$

$$\|\mathbf{x}\|_{H(p,c,\Lambda)} := \sup_n(\Lambda_n^{(p-1)(1-c)}\sum_{k=1}^{n}|x_k|^p)^{1/p},$$

$$\|\mathbf{x}\|_{h(p,c,H)} := \Big\{\sum \lambda_n H_n^{-c}\Big(\sum_{k=1}^{n}|x_k|\Big)^p\Big\}^{1/p},$$

and

$$\|\mathbf{x}\|_{H(p,c,H)} := \sup_n\Big(H_n^{(p-1)(1-c)}\sum_{k=1}^{n}|x_k|^p\Big)^{1/p}.$$

We underline that if $c=0$ then the sets $h(p,0,\Lambda)$ and $h(p,0,H)$ defined in (1.1) and (1.3) are the same, i.e.

$$h(p,0,\Lambda) \equiv h(p,0,H). \tag{1.5}$$

A sequence $\gamma := \{\gamma_n\}$ of positive terms will be called quasi β-power-monotone decreasing if there exists a constant $K=K(\beta,\gamma)\geqslant 1$ such that

$$n^\beta\gamma_n \leqslant K m^\beta \gamma_m$$

holds for any $n\geqslant m$ and for all m.

Now we recall the results of [2] to be improved here.

THEOREM A. *Let* $0 \leqslant c < 1 < p$ *and let* $\lambda := \{\lambda_n\}$ *be a given sequence of nonnegative terms having infinitely many positive ones.*

(*i*) *If a sequence* $\mathbf{x}$ *belongs to* $h(p,c,\Lambda)$ *then it admits a factorization*

$$\mathbf{x} = \mathbf{y} \cdot \mathbf{z} \quad (x_n = y_n z_n) \tag{1.6}$$

with

$$\mathbf{y} \in \ell^p \quad \text{and} \quad \mathbf{z} \in H(p^*,c,\Lambda), \quad p^* := \frac{p}{p-1}. \tag{1.7}$$

(*ii*) *Conversely, if the sequence* λ *satisfies the additional conditions*

$$\Lambda_n \leqslant K\Lambda_{2n} \tag{1.8}$$

and the sequence $\{\Lambda_n\}$ *is quasi* β*-power-monotone decreasing with some positive* β*, furthermore the sequence* $\mathbf{x}$ *admits a factorization* (1.6) *with* (1.7)*, then* $\mathbf{x}$ *belongs to* $h(p,c,\Lambda)$.

THEOREM B. *Let* $1 < c, p$ *and let* λ *be a given sequence of nonnegative terms having infinitely many positive ones.*

(*i*) *If* $\Lambda_1 = \infty$, $H_{n+1} \leqslant KH_n$ *and a sequence* $\mathbf{x}$ *belongs to* $h(p,c,H)$ *then it admits a factorization* (1.6) *with*

$$\mathbf{y} \in \ell^p \quad \text{and} \quad \mathbf{z} \in H(p^*,c,H). \tag{1.9}$$

(*ii*) *Conversely, if the sequence* λ *satisfies the additional conditions*

$$H_{2n} \leqslant KH_n \tag{1.10}$$

and the sequence $\{H_n^{-1}\}$ *is quasi* β*-power-monotone decreasing with some positive* β*, plus the sequence* $\mathbf{x}$ *admits a factorization* (1.6) *with* (1.9)*, then* $\mathbf{x}$ *belongs to* $h(p,c,H)$.

In [3] and [4] we considered such a problem of factorization where the crucial function x^p, appearing in the previously considered sets $h(p,c,\Lambda)$ and $h(p,c,H)$, is replaced by more general functions $\varphi(x)$ from the sets either $\Phi(p)$ or $\Psi(q)$, given above.

In these papers we generalized only the special case $c = 0$ of Theorem A. Now we intend to prove generalizations with positive c, furthermore not only for Theorem A, however for Theorem B, too.

If in the definition of the sets $h(p,c,\Lambda)$ and $h(p,c,H)$ we replace the function x^p by $\varphi(x)$ the new sets will be denoted by $h(\varphi,c,\Lambda)$ and $h(\varphi,c,H)$, respectively, i.e.

$$h(\varphi,c,\Lambda) := \Big\{\mathbf{x} : \sum \lambda_n \Lambda_n^{-c} \varphi\Big(\sum_{k=1}^{n} |x_k|\Big) < \infty\Big\} \tag{1.11}$$

and

$$h(\varphi,c,H) := \Big\{\mathbf{x} : \sum \lambda_n H_n^{-c} \varphi\Big(\sum_{k=1}^{n} |x_k|\Big) < \infty\Big\}. \tag{1.12}$$

Finally we define the set

$$\ell^\varphi := \Big\{\mathbf{x} : \sum \varphi(|x_n|) < \infty\Big\}.$$

2. Results

We shall prove the following theorems.

THEOREM 1. *Let $0 \leqslant c < 1$ and let $\lambda := \{\lambda_n\}$ be a given sequence of nonnegative terms having infinitely many positive ones.*

(*i*) *If $q > 1$, $\varphi \in \Psi(q)$ and a sequence $\mathbf{x}$ belongs to $h(\varphi, c, \Lambda)$ then it admits a factorization (1.6) with*

$$\mathbf{y} \in \ell^{\varphi} \quad and \quad \mathbf{z} \in H(q^*, c, \Lambda), \quad q^* := \frac{q}{q-1}. \tag{2.1}$$

(*ii*) *Conversely, if $p > 1$, $\varphi \in \Phi(p)$ and the sequence λ satisfies the condition (1.8) and the sequence $\{\Lambda_n\}$ is quasi β-power-monotone decreasing with some positive β, furthermore $\mathbf{x}$ admits a factorization (1.6) with*

$$\mathbf{y} \in \ell^{\varphi} \quad and \quad \mathbf{z} \in H(p^*, c, \Lambda), \tag{2.2}$$

then $\mathbf{x}$ belongs to $h(\varphi, c, \Lambda)$.

THEOREM 2. *Let $1 < c$ and let λ be a given sequence of nonnegative terms having infinitely many positive ones.*

(*i*) *If $\Lambda_1 = \infty$, $H_{n+1} \leqslant KH_n$, $q > 1$, $\varphi \in \Psi(q)$ and a sequence $\mathbf{x}$ belongs to $h(\varphi, c, H)$ then it admits a factorization (1.6) with*

$$\mathbf{y} \in \ell^{\varphi} \quad and \quad \mathbf{z} \in H(q^*, c, H) \tag{2.3}$$

(*ii*) *Conversely, if $p > 1$, $\varphi \in \Phi(p)$ and the sequence λ satisfies the additional conditions (1.10), plus the sequence $\{H_n^{-1}\}$ is quasi β-power-monotone decreasing with some positive β, moreover the sequence $\mathbf{x}$ admits a factorization (1.6) with*

$$\mathbf{y} \in \ell^{\varphi} \quad and \quad \mathbf{z} \in H(p^*, c, H) \tag{2.4}$$

then $\mathbf{x}$ belongs to $h(\varphi, c, H)$.

We mention that the first part of Theorem 1 with $c = 0$ contains the result of [3], and its second part includes the theorem of [4] disregarding the estimates given there. A careful analysis of our present proofs would yield similar estimates in these general cases, too, but the constants to be apprearing in these new estimates would depend on several parameters.

3. Lemmas

We require the following lemmas.

LEMMA 1. *If $p>0$ and $\varphi\in\Phi(p)$ then*

$$t^p\varphi(x)\leqslant\varphi(tx)\quad for\quad 0\leqslant t\leqslant 1,\quad x\geqslant 0, \tag{3.1}$$

and

$$\varphi(tx)\leqslant t^p\varphi(x)\quad for\quad t\geqslant 1,\quad x\geqslant 0. \tag{3.2}$$

If $q>0$ and $\varphi\in\Psi(q)$ then

$$\varphi(tx)\leqslant t^q\varphi(x)\quad for\quad 0\leqslant t\leqslant 1,\quad x\geqslant 0, \tag{3.3}$$

and

$$t^q\varphi(x)\leqslant\varphi(tx)\quad for\quad t\geqslant 1, x\geqslant 0. \tag{3.4}$$

These inequalities are obvious consequences of the definition of the sets $\Phi(p)$ and $\Psi(q)$.

LEMMA 2. *If $c>1$, λ is a sequence of nonnegative terms such that $\Lambda_1=\infty$ and $H_{n+1}\leqslant KH_n$, then*

$$K_1(c)H_n^{1-c}\leqslant\sum_{k=n}^{\infty}\lambda_kH_k^{-c}\leqslant K_2(c)H_n^{1-c}. \tag{3.5}$$

The inqualities (3.5) were proved in [2], see Lemmas 3 and 5 given there.

LEMMA 3. *If $c>1$, λ is a sequence of nonnegative terms such that $H_{n+1}\leqslant KH_n$ and the sequence $\{H_n^{-1}\}$ is quasi β-power-monotone decreasing with some positive β, then the sequence*

$$\widetilde{\Lambda}_n:=\sum_{k=n}^{\infty}\lambda_kH_k^{-c}$$

is quasi $\widetilde{\beta}$-power-monotone decreasing with $\widetilde{\beta}:=\beta(c-1)>0$.

This lemma is known, see Lemma 7 in [2].

4. Proofs

Proof of Theorem 1. First we define a new sequance $\overline{\lambda}:=\{\overline{\lambda}_n\}$ as follows:

$$\overline{\lambda}_n:=\lambda_n\Lambda_n^{-c}.$$

Then the assumption $\mathbf{x}\in h(\varphi,c,\Lambda)$ can be written as follows

$$\sum\overline{\lambda}_n\varphi\Big(\sum_{k=1}^{n}|x_k|\Big)<\infty. \tag{4.1}$$

Since $\varphi \in \Psi(q)$, (4.1) and (3.4) imply that

$$\sum \overline{\lambda}_n \Big(\sum_{k=1}^{n} |x_k|\Big)^q < \infty \tag{4.2}$$

also holds.

If we denote by $h(p,0,\overline{\Lambda})$ and $H(p,0,\overline{\Lambda})$ the sets defined in (1.1) and (1.2) with this new sequence $\overline{\lambda}$ and $c = 0$, respectively, then (4.2) clearly means that $\mathbf{x} \in h(q,0,\overline{\Lambda})$.

Hereafter utilizing the part (i) of Theorem A, we get that $\mathbf{x}$ admits a factization (1.6) with

$$\mathbf{y} \in \ell^q \quad \text{and} \quad \mathbf{z} \in H(q^*,0,\overline{\Lambda}) \quad \Big(q^* = \frac{q}{q-1}\Big). \tag{4.3}$$

Since $\varphi \in \Psi(q)$ thus, by (3.3), $\mathbf{y} \in \ell^q$ implies that $\mathbf{y} \in \ell^\varphi$; and because $\mathbf{z} \in H(q^*,0,\overline{\Lambda})$ is equivalent to $\mathbf{z} \in H(q^*,c,\Lambda)$, thus, by (4.3), we have proved our assertion that $\mathbf{x} \in h(\varphi,c,\Lambda)$ implies the embedding relations given in (2.1).

The second part of Theorem 1 is an easy consequence of the part (ii) of Theorem A.

Namely, since $\varphi \in \Phi(p)$, thus by (3.1) $\mathbf{y} \in \ell^\varphi$ implies $\mathbf{y} \in \ell^p$. On the other hand, $\mathbf{y} \in \ell^p$ and $\mathbf{z} \in H(p^*,c,\Lambda)$, by the part (ii) of Theorem A, imply that $\mathbf{x} \in h(p,c,\Lambda)$, i.e.

$$\sum \lambda_n \Lambda_n^{-c} \Big(\sum_{k=1}^{n} |x_k|\Big)^p < \infty.$$

Hence, $\varphi \in \Phi(p)$ and (3.2), convey

$$\sum \lambda_n \Lambda_n^{-c} \varphi\Big(\sum_{k=1}^{n} |x_k|\Big) < \infty,$$

and this is the required inequality showing that $\mathbf{x} \in h(\varphi,c,\Lambda)$.

The proof of Theorem 1 is complete.

Proof of Theorem 2. The proof is essentially the same as that of Theorem 1. Now we define again a new sequence $\widetilde{\lambda} := \{\widetilde{\lambda}_n\}$ as follows:

$$\widetilde{\lambda}_n := \lambda_n H_n^{-c}.$$

Then the assumption $\mathbf{x} \in h(\varphi,c,H)$ can be rewritten as follows

$$\sum \widetilde{\lambda}_n \varphi\Big(\sum_{k=1}^{n} |x_k|\Big) < \infty. \tag{4.4}$$

Since $\varphi \in \Psi(q)$, (4.4) and (3.4) imply that

$$\sum \widetilde{\lambda}_n \Big(\sum_{k=1}^{n} |x_k|\Big)^q < \infty. \tag{4.5}$$

If we denote by $h(p,0,\widetilde{H})$ and $H(p,0,\widetilde{H})$ the sets defined in (1.3) and (1.4) with this new sequence $\widetilde{\lambda}$ and $c=0$, respectively, then (4.5) plainly means that $\mathbf{x}\in h(q,0,\widetilde{H})\equiv h(q,0,\widetilde{\Lambda})$, see (1.5).

Thus, by the part (i) of Theorem A with $\widetilde{\lambda}$ and $c=0$, we obtain that $\mathbf{x}$ admits a factorization (1.6) with

$$\mathbf{y}\in\ell^q \quad\text{and}\quad \mathbf{z}\in H(q^*,0,\widetilde{\Lambda}). \tag{4.6}$$

By (1.2) $\mathbf{z}\in H(q^*,0,\widetilde{\Lambda})$ means that

$$\sum_{k=1}^{n}|z_k|^{q^*}=O\Big(\Big(\sum_{k=n}^{\infty}\lambda_k H_k^{-c})\Big)^{1-q^*}\Big).$$

Hence, by Lemma 2, it is obvious that $\mathbf{z}\in H(q^*,0,\widetilde{\Lambda})$ implies $\mathbf{z}\in H(q^*,c,H)$. This and (4.6) proves (2.3), namely $\mathbf{y}\in\ell^q$ also implies $\mathbf{y}\in\ell^\varphi$, arguing as in Theorem 1.

To prove the part (ii) of Theorem 2 first we show that the new sequence $\widetilde{\lambda}$ satisfies all of the additional conditions of Theorem A claimed on the sequence λ in part (ii).

Since

$$\widetilde{\Lambda}_n:=\sum_{k=n}^{\infty}\widetilde{\lambda}_k=\sum_{k=n}^{\infty}\lambda_k H_k^{-c},$$

thus, by Lemma 2,

$$K_1(c)H_n^{1-c}\leqslant\widetilde{\Lambda}_n\leqslant K_2(c)H_n^{1-c}, \tag{4.7}$$

whence and from (1.10), because $c>1$, we can see that the condition (1.8) is satisfied with $\widetilde{\Lambda}_n$ in place of Λ_n.

Furthermore Lemma 3 shows that the sequence $\{\widetilde{\Lambda}_n\}$ is quasi β-power-monotone decreasing with $\widetilde{\beta}:=\beta(c-1)>0$.

Herewith we have verified that the sequence $\widetilde{\lambda}$ fulfills all of the additional conditions of Theorem A.

Since $\varphi\in\Phi(p)$ thus $\mathbf{y}\in\ell^\varphi$ implies $\mathbf{y}\in\ell^p$, furthermore by (4.7)

$$H(p^*,c,H)\equiv H(p^*,0,\widetilde{\Lambda}).$$

Taking into account these facts we see that (2.4) yields that

$$\mathbf{y}\in\ell^p \quad\text{and}\quad \mathbf{z}\in H(p^*,0,\widetilde{\Lambda})$$

hold.

Consequently, applying the part (ii) of Theorem A, we obtain that $\mathbf{x}\in h(p,0,\widetilde{\Lambda})$. By the definition of $\widetilde{\lambda}$ it is clear that

$$h(p,0,\widetilde{\Lambda})\equiv h(p,c,H),$$

i.e. we have got that

$$\sum\lambda_n H_n^{-c}\Big(\sum_{k=1}^{n}|x_k|\Big)^p<\infty. \tag{4.8}$$

Since $\varphi \in \Phi(p)$, thus (4.8) and (3.2) imply that

$$\sum \lambda_n H_n^{-c} \varphi\Big(\sum_{k=1}^{n} |x_k|\Big) < \infty$$

also holds.

Herewith we verified that the factorization (1.6) with (2.4) implies that $\mathbf{x}$ belongs to the set $h(\varphi, c, H)$.

The proof is complete.

REFERENCES

[1] G. BENNETT, *Factorizing the Classical Inequalities*, Memoirs of Amer. Math. Soc. **120** (1996, Number 576), 1-130.

[2] L. LEINDLER, *Two Hardy-Bennett-type theorems*, Acta Math. Hung. (to appear).

[3] ———, *An addition to factorization of inequalites*, Math. Pannonica **10** (1999), 61–65.

[4] ———, *An attachment to factorization of inequalities*, Acta Sci. Math. (Szeged), **65** (1999), 85–92.

(Received January 15, 1999)

L. Leindler
Bolyai Institute
University of Szeged
Aradi vértanúk tere 1
6720 Szeged, Hungary
e-mail: leindler@math.u-szeged.hu