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A NEW REFINEMENT OF THE KY FAN INEQUALITY

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Abstract. In this note we obtain a new refinement, involving the identric mean of several variables, of the inequality $G_n/G'_n \leqslant A_n/A'_n$, due to Ky Fan.

1. Introduction

Let $x_1, \ldots, x_n$ be a sequence of positive real numbers lying in the open interval $]0,1[$, and let A_n, G_n, and H_n denote their arithmetic, geometric, and harmonic mean, respectively, i. e.

$$A_n = \frac{1}{n}\sum_{i=1}^{n} x_i, \qquad G_n = \left(\prod_{i=1}^{n} x_i\right)^{1/n}, \qquad H_n = \frac{n}{\sum_{i=1}^{n}\frac{1}{x_i}}.$$

Further, let A'_n, G'_n, and H'_n denote the arithmetic, geometric, and harmonic mean, respectively, of $1-x_1, \ldots, 1-x_n$, i. e.

$$A'_n = \frac{1}{n}\sum_{i=1}^{n}(1-x_i), \qquad G_n = \left(\prod_{i=1}^{n}(1-x_i)\right)^{1/n}, \qquad H_n = \frac{n}{\sum_{i=1}^{n}\frac{1}{1-x_i}}.$$

The arithmetic-geometric mean inequality $G_n \leqslant A_n$ (and its weighted variant) played an important role in the development of the theory of inequalities. Because of its importance, many proofs and refinements have been published. In 1961, a remarkable new counterpart of the AM-GM inequality was published in the famous book [7]:

THEOREM 1. *If $x_i \in]0,1/2]$ for all $i \in \{1,\ldots,n\}$, then*

$$\frac{G_n}{G'_n} \leqslant \frac{A_n}{A'_n}, \tag{1}$$

with equality holding if and only if $x_1 = \cdots = x_n$.

Inequality (1), which is due to Ky Fan, has evoked the interest of several mathematicians, and different proofs as well as many extensions, sharpenings, and variants

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have been published. For proofs of (1) the reader is referred to [3], [6], [16], [17]. Refinements of (1) are proved in [1], [5], [6], [18], while generalizations can be found in [9], [11], [19], [21]. For converses and related results see [2], [4], [14]. See also the survey paper [6].

In 1984, Wang and Wang [20] established the following counterpart of (1):

$$\frac{H_n}{H_n'} \leqslant \frac{G_n}{G_n'}. \tag{2}$$

For extension to weighted means and other proofs of (2) see, for instance, [6] and [18].

In 1990, J. Sándor [15, relation (33)] proved the following refinement of (1) in the case of two arguments (i. e. $n = 2$):

$$\frac{G}{G'} \leqslant \frac{I}{I'} \leqslant \frac{A}{A'}, \tag{3}$$

where $G = G_2$, $G' = G_2'$ etc., and I denotes the so-called identric mean of two numbers:

$$I(x_1,x_2) = \frac{1}{e}\left(\frac{x_2^{x_2}}{x_1^{x_1}}\right)^{1/(x_2-x_1)}, \qquad \text{if } x_1 \neq x_2$$
$$I(x,x) = x.$$

Here $I'(x_1,x_2) = I(1-x_1, 1-x_2)$ and $x_1,x_2 \in \,]0,1/2]$.

In what follows, inequality (3) will be extended to the case of n arguments, thus giving a new refinement of the Ky Fan inequality (1).

2. Main result

Let $n \geqslant 2$ be a given integer, and let

$$A_{n-1} = \{\,(\lambda_1,\ldots,\lambda_{n-1}) \mid \lambda_i \geqslant 0,\ i = 1,\ldots,n-1,\ \lambda_1 + \cdots + \lambda_{n-1} \leqslant 1\,\}$$

be the Euclidean simplex. Given $X = (x_1,\ldots,x_n)$ ($x_i > 0$ for all $i \in \{1\ldots,n\}$), and a probability measure μ on A_{n-1}, for a continuous strictly monotone function $f :]0,\infty[\to \mathbb{R}$, the following functional mean of n arguments can be introduced:

$$M_f(X;\mu) = f^{-1}\left(\int_{A_{n-1}} f(X\cdot\lambda)d\mu(\lambda)\right), \tag{4}$$

where $X\cdot\lambda = \sum_{i=1}^{n} x_i\lambda_i$ denotes the scalar product, $\lambda = (\lambda_1,\ldots,\lambda_{n-1}) \in A_{n-1}$, and $\lambda_n = 1 - \lambda_1 - \cdots - \lambda_{n-1}$.

For $\mu = (n-1)!$ and $f(t) = 1/t$, the unweighted logarithmic mean

$$L(x_1,\ldots,x_n) = \left((n-1)!\int_{A_{n-1}} \frac{1}{X\cdot\lambda} d\lambda_1\cdots d\lambda_{n-1}\right)^{-1} \tag{5}$$

is obtained. For properties and an explicit form of this mean, the reader is referred to [13].

For $f(t) = \log t$ we obtain a mean, which can be considered as a generalization of the identric mean

$$I(X;\mu) = \exp\left(\int_{A_{n-1}} \log(X\cdot\lambda)d\mu(\lambda)\right). \tag{6}$$

Indeed, it is immediately seen that for the classical identric mean of two arguments one has

$$I(x_1,x_2) = \exp\left(\int_0^1 \log(tx_1 + (1-t)x_2)dt\right).$$

For $\mu = (n-1)!$ we obtain the unweighted (and symmetric) identric mean of n variables

$$I(x_1,\dots,x_n) = \exp\left((n-1)!\int_{A_{n-1}} \log(X\cdot\lambda)d\lambda_1\cdots d\lambda_{n-1}\right), \tag{7}$$

in analogy with (5). It should be noted that (7) is a special case of (4), which has been considered in [13]. The mean (4) even is a special case of the B. C. Carlson's function M (see [8, p. 33]). For an explicit form of $I(x_1,\dots,x_n)$ see [12].

Let $n \geqslant 2$, let μ be a probability measure on A_{n-1}, and let $i \in \{1,\dots,n\}$. The ith weight w_i associated to μ is defined by

$$w_i = \int_{A_{n-1}} \lambda_i d\mu(\lambda), \qquad \text{if } 1 \leqslant i \leqslant n-1, \tag{8}$$

$$w_n = \int_{A_{n-1}} (1-\lambda_1-\cdots-\lambda_{n-1})d\mu(\lambda),$$

where $\lambda = (\lambda_1,\dots,\lambda_{n-1}) \in A_{n-1}$. Obviously, $w_i > 0$ for all $i \in \{1,\dots,n\}$, and $w_1+\cdots+w_n = 1$. Moreover, if $\mu = (n-1)!$, then $w_i = 1/n$ for all $i \in \{1,\dots,n\}$.

We are now in a position to state the main result of the paper, a weighted improvement of the Ky Fan inequality.

THEOREM 2. *Let $n \geqslant 2$, let μ be a probability measure on A_{n-1} whose weights $w_1,\dots,w_n$ are given by (8), and let $x_i \in]0,1/2]$ $(i = 1,\dots,n)$. Then*

$$\frac{\prod_{i=1}^n x_i^{w_i}}{\prod_{i=1}^n (1-x_i)^{w_i}} \leqslant \frac{I(x_1,\dots,x_n;\mu)}{I(1-x_1,\dots,1-x_n;\mu)} \leqslant \frac{\sum_{i=1}^n w_i x_i}{\sum_{i=1}^n w_i(1-x_i)}. \tag{9}$$

Proof. First remark that the function $\phi :]0,1/2] \to \mathbb{R}$ defined by $\phi(t) = \log t - \log(1-t)$ is concave. Consequently

$$\sum_{i=1}^n w_i\phi(x_i) \leqslant \int_{A_{n-1}} \phi(X\cdot\lambda)d\mu(\lambda) \leqslant \phi\left(\sum_{i=1}^n w_i x_i\right). \tag{10}$$

This inequality has been established in [10]. From (10), after a simple computation we deduce (9). □

REMARK. For $\mu = (n-1)!$, inequality (9) reduces to the following unweighted improvement of the Ky Fan inequality, which generalizes (3):

$$\frac{G_n}{G_n'} \leqslant \frac{I_n}{I_n'} \leqslant \frac{A_n}{A_n'}.$$

Here $I_n = I(x_1, \ldots, x_n)$, while $I_n' = I(1-x_1, \ldots, 1-x_n)$.

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