A NEW REFINEMENT OF THE KY FAN INEQUALITY

JÓZSEF SÁNDOR AND TIBERIU TRIF

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Abstract. In this note we obtain a new refinement, involving the identric mean of several variables, of the inequality \( \frac{G_n}{G'_n} \leq \frac{A_n}{A'_n} \), due to Ky Fan.

1. Introduction

Let \( x_1, \ldots, x_n \) be a sequence of positive real numbers lying in the open interval \( ]0, 1[ \), and let \( A_n \), \( G_n \), and \( H_n \) denote their arithmetic, geometric, and harmonic mean, respectively, i.e.

\[
A_n = \frac{1}{n} \sum_{i=1}^{n} x_i, \quad G_n = \left( \prod_{i=1}^{n} x_i \right)^{1/n}, \quad H_n = \frac{n}{\sum_{i=1}^{n} \frac{1}{x_i}}.
\]

Further, let \( A'_n \), \( G'_n \), and \( H'_n \) denote the arithmetic, geometric, and harmonic mean, respectively, of \( 1-x_1, \ldots, 1-x_n \), i.e.

\[
A'_n = \frac{1}{n} \sum_{i=1}^{n} (1-x_i), \quad G'_n = \left( \prod_{i=1}^{n} (1-x_i) \right)^{1/n}, \quad H'_n = \frac{n}{\sum_{i=1}^{n} \frac{1}{1-x_i}}.
\]

The arithmetic-geometric mean inequality \( G_n \leq A_n \) (and its weighted variant) played an important role in the development of the theory of inequalities. Because of its importance, many proofs and refinements have been published. In 1961, a remarkable new counterpart of the AM-GM inequality was published in the famous book [7]:

THEOREM 1. If \( x_i \in ]0, 1/2] \) for all \( i \in \{ 1, \ldots, n \} \), then

\[
\frac{G_n}{G'_n} \leq \frac{A_n}{A'_n},
\]

with equality holding if and only if \( x_1 = \cdots = x_n \).

Inequality (1), which is due to Ky Fan, has evoked the interest of several mathematicians, and different proofs as well as many extensions, sharpenings, and variants...
have been published. For proofs of (1) the reader is referred to [3], [6], [16], [17]. Refinements of (1) are proved in [1], [5], [6], [18], while generalizations can be found in [9], [11], [19], [21]. For converses and related results see [2], [4], [14]. See also the survey paper [6].

In 1984, Wang and Wang [20] established the following counterpart of (1):

\[
\frac{H_n}{H'_n} \leq \frac{G_n}{G'_n}.
\] (2)

For extension to weighted means and other proofs of (2) see, for instance, [6] and [18].

In 1990, J. Sándor [15, relation (33)] proved the following refinement of (1) in the case of two arguments (i.e. \(n = 2\)):

\[
\frac{G}{G'} \leq \frac{I}{I'} \leq \frac{A}{A'},
\] (3)

where \(G = G_2\), \(G' = G'_2\) etc., and \(I\) denotes the so-called identric mean of two numbers:

\[
I(x_1, x_2) = \frac{1}{e} \left( \frac{x_2^{x_2} - x_1^{x_1}}{x_2 - x_1} \right)^{1/(x_2 - x_1)}, \quad \text{if } x_1 \neq x_2
\]

\[
I(x, x) = x.
\]

Here \(I'(x_1, x_2) = I(1 - x_1, 1 - x_2)\) and \(x_1, x_2 \in ]0, 1/2]\).

In what follows, inequality (3) will be extended to the case of \(n\) arguments, thus giving a new refinement of the Ky Fan inequality (1).

2. Main result

Let \(n \geq 2\) be a given integer, and let

\[
A_{n-1} = \{ (\lambda_1, \ldots, \lambda_{n-1}) \mid \lambda_i \geq 0, \ i = 1, \ldots, n-1, \ \lambda_1 + \cdots + \lambda_{n-1} \leq 1 \}
\]

be the Euclidean simplex. Given \(X = (x_1, \ldots, x_n) \ (x_i > 0 \ \text{for all} \ i \in \{1, \ldots, n\})\), and a probability measure \(\mu\) on \(A_{n-1}\), for a continuous strictly monotone function \(f : ]0, \infty[ \to \mathbb{R}\), the following functional mean of \(n\) arguments can be introduced:

\[
M_f (X; \mu) = f^{-1} \left( \int_{A_{n-1}} f(X \cdot \lambda) d\mu(\lambda) \right),
\] (4)

where \(X \cdot \lambda = \sum_{i=1}^n x_i \lambda_i\) denotes the scalar product, \(\lambda = (\lambda_1, \ldots, \lambda_{n-1}) \in A_{n-1}\), and \(\lambda_n = 1 - \lambda_1 - \cdots - \lambda_{n-1}\).

For \(\mu = (n-1)!\) and \(f(t) = 1/t\), the unweighted logarithmic mean

\[
L(x_1, \ldots, x_n) = \left( (n-1)! \int_{A_{n-1}} \frac{1}{X \cdot \lambda} d\lambda_1 \cdots d\lambda_{n-1} \right)^{-1}
\] (5)
is obtained. For properties and an explicit form of this mean, the reader is referred to [13].

For \( f(t) = \log t \) we obtain a mean, which can be considered as a generalization of the identric mean

\[
I(X; \mu) = \exp \left( \int_{A_{n-1}} \log(X \cdot \lambda) d\mu(\lambda) \right).
\]

(6)

Indeed, it is immediately seen that for the classical identric mean of two arguments one has

\[
I(x_1, x_2) = \exp \left( \int_0^1 \log(tx_1 + (1-t)x_2) dt \right).
\]

For \( \mu = (n-1)! \) we obtain the unweighted (and symmetric) identric mean of \( n \) variables

\[
I(x_1, \ldots, x_n) = \exp \left( (n-1)! \int_{A_{n-1}} \log(X \cdot \lambda) d\lambda_1 \cdots d\lambda_{n-1} \right),
\]

(7)

in analogy with (5). It should be noted that (7) is a special case of (4), which has been considered in [13]. The mean (4) even is a special case of the B. C. Carlson’s function \( M \) (see [8, p. 33]). For an explicit form of \( I(x_1, \ldots, x_n) \) see [12].

Let \( n \geq 2 \), let \( \mu \) be a probability measure on \( A_{n-1} \), and let \( i \in \{1, \ldots, n\} \). The \( i \)th weight \( w_i \) associated to \( \mu \) is defined by

\[
w_i = \int_{A_{n-1}} \lambda_i d\mu(\lambda), \quad \text{if} \ 1 \leq i \leq n-1,
\]

(8)

\[
w_n = \int_{A_{n-1}} (1 - \lambda_1 - \cdots - \lambda_{n-1}) d\mu(\lambda),
\]

where \( \lambda = (\lambda_1, \ldots, \lambda_{n-1}) \in A_{n-1} \). Obviously, \( w_i > 0 \) for all \( i \in \{1, \ldots, n\} \), and \( w_1 + \cdots + w_n = 1 \). Moreover, if \( \mu = (n-1)! \), then \( w_i = 1/n \) for all \( i \in \{1, \ldots, n\} \).

We are now in a position to state the main result of the paper, a weighted improvement of the Ky Fan inequality.

**Theorem 2.** Let \( n \geq 2 \), let \( \mu \) be a probability measure on \( A_{n-1} \) whose weights \( w_1, \ldots, w_n \) are given by (8), and let \( x_i \in [0, 1/2] \ (i = 1, \ldots, n) \). Then

\[
\frac{\prod_{i=1}^n x_i^{w_i}}{\prod_{i=1}^n (1 - x_i)^{w_i}} \leq \frac{I(x_1, \ldots, x_n; \mu)}{I(1 - x_1, \ldots, 1 - x_n; \mu)} \leq \frac{\sum_{i=1}^n w_i x_i}{\sum_{i=1}^n w_i (1 - x_i)}.
\]

(9)

**Proof.** First remark that the function \( \phi : [0, 1/2] \rightarrow \mathbb{R} \) defined by \( \phi(t) = \log t - \log(1-t) \) is concave. Consequently

\[
\sum_{i=1}^n w_i \phi(x_i) \leq \int_{A_{n-1}} \phi(X \cdot \lambda) d\mu(\lambda) \leq \phi \left( \sum_{i=1}^n w_i x_i \right).
\]

(10)
This inequality has been established in [10]. From (10), after a simple computation we deduce (9). □

**REMARK.** For \( \mu = (n - 1)! \), inequality (9) reduces to the following unweighted improvement of the Ky Fan inequality, which generalizes (3):

\[
\frac{G_n}{G'_n} \leq \frac{I_n}{I'_n} \leq \frac{A_n}{A'_n}.
\]

Here \( I_n = I(x_1, \ldots, x_n) \), while \( I'_n = I(1 - x_1, \ldots, 1 - x_n) \).

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**REFERENCES**


József Sándor
Universitatea Babeş-Bolyai
Fac. de Matematică şi Informatică
Str. Kogălniceanu Nr. 1
RO-3400 Cluj-Napoca
România
e-mail: jsandor@math.ubbcluj.ro

Tiberiu Trif
Universitatea Babeş-Bolyai
Fac. de Matematică şi Informatică
Str. Kogălniceanu Nr. 1
RO-3400 Cluj-Napoca
România
e-mail: ttrif@math.ubbcluj.ro