COEFFICIENT INEQUALITIES FOR CERTAIN UNIVALENT FUNCTIONS

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Abstract. We introduce certain subclasses of univalent functions that are starlike of order α and convex of order α. We obtain coefficient inequalities for the functions belong to the subclasses. Using the coefficient inequalities we give some example s of such univalent functions. Also distortion inequalities and convolution properties are determined for the subclasses.

1. Introduction

Let A be the class of functions $f(z)$ of the form:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n,$$

that are analytic in the open unit disk $U = \{z : z \in \mathbf{C} \text{ and } |z| < 1\}$.

Let $A(n)$ denote the subclass of A consisting of all functions of the form:

$$f(z) = z - \sum_{k=n+1}^{\infty} a_k z^k \quad (a_k \geqq 0; \ n \in \mathbf{N} = \{1, 2, 3, \cdots\}).$$

$A(n)$ is said to be the subclass of analytic functions with negative coefficients. Let $T(n)$ be the subclass of $A(n)$ of univalent functions in U. Further, $T_\alpha(n)$ and $C_\alpha(n)$ denote the subclasses of $T(n)$ consisting of functions which are starlike of order $\alpha(0 \leqq \alpha < 1)$ and convex of order $\alpha(0 \leqq \alpha < 1)$, respectively. These subclasses $T(n), T_\alpha(n)$ and $C_\alpha(n)$ were introduced by Chatterjea[1]. For $n = 1$, these notations are usually used as $T(1) = T, T_\alpha(1) = T^*(\alpha)$ and $C_\alpha(1) = C(\alpha)$, which were introduced earlier by Silverman[3]. Silverman[3] determined coefficient inequalities, distortion and covering theorems for univalent functions belong to the subclasses $T^*(\alpha)$ and $C(\alpha)$. Chatterjea[1] determined the coefficient inequalities for functions in the subclasses $T_\alpha(n)$ and $C_\alpha(n)$. Srivastava, Owa and Chatterjea[6] subsequently investigated distortion theorems and convolution properties for the subclasses $T_\alpha(n)$ and $C_\alpha(n)$. For the definitions of these subclasses, coefficient and distortion inequalities, and convolutions,

one may refer to Srivastava and Owa[5]

We introduce a subclass $A(n, \theta)$ of A in the following manner.
Let $A(n, \theta)$ denote the subclass of A consisting of all functions of the form:

$$f(z) = z - \sum_{k=n+1}^{\infty} e^{i(k-1)\theta} a_k z^k \quad (a_k \geqq 0; \ n \in \mathbf{N}).$$

We note that when $\theta = 0, A(n, 0) = A(n)$. Further, we define the subclasses $T(n, \theta)$, $T_{\alpha}^{*}(n, \theta)$ and $C_{\alpha}(n, \theta)$ of $A(n, \theta)$ in the same way as $T(n), T_{\alpha}(n)$ and $C_{\alpha}(n)$. Thereby, when $\theta = 0$, $T(n, 0) = T(n), T_{\alpha}^{*}(n, 0) = T_{\alpha}(n)$ and $C_{\alpha}(n, 0) = C_{\alpha}(n)$. We obtain the coefficient inequalities for the functions belong to the subclasses $T_{\alpha}^{*}(n, \theta)$ and $C_{\alpha}(n, \theta)$. We note that the coefficient inequalities above do not contain θ and coincide with the coefficient inequalities for $T_{\alpha}(n)$ and $C_{\alpha}(n)$ of Chatterjea[1], and also for $T^{*}(\alpha)$ and $C(\alpha)$ of Silverman[3]. By using the coefficient inequalities we show some examples of the functions belong to $T_{\alpha}^{*}(n, \theta)$ and $C_{\alpha}(n, \theta)$. Further, we determine distortion ineqalities and convolution properties for the subclasses $T_{\alpha}^{*}(n, \theta)$ and $C_{\alpha}(n, \theta)$.

2. Coefficients inequalities

THEOREM 2.1. *A function $f(z)$ in $A(n, \theta)$ is in $T_{\alpha}^{*}(n, \theta)$ if and only if*

$$\sum_{k=n+1}^{\infty} (k - \alpha)a_k \leqq 1 - \alpha. \tag{2.1}$$

Proof. If $f(z)$ is in $A(n, \theta)$ and coefficient inequality $\sum_{k=n+1}^{\infty}(k - \alpha)a_k \leqslant 1 - \alpha$ hold true,
then

$$\left| \frac{zf'(z)}{f(z)} - 1 \right| = \left| \frac{\displaystyle\sum_{k=n+1}^{\infty} (k - 1)e^{i(k-1)\theta} a_k z^{k-1}}{1 - \displaystyle\sum_{k=n+1}^{\infty} e^{i(k-1)\theta} a_k z^{k-1}} \right| \leqq \frac{\displaystyle\sum_{k=n+1}^{\infty} (k - 1)a_k |z|^{k-1}}{1 - \displaystyle\sum_{k=n+1}^{\infty} a_k |z|^{k-1}}$$

$$\leqq \frac{\displaystyle\sum_{k=n+1}^{\infty} (k - 1)a_k}{1 - \displaystyle\sum_{k=n+1}^{\infty} a_k} \leqq 1 - \alpha.$$

Thereby the values for $\dfrac{zf'(z)}{f(z)}$ lie in a circle centered at $w = 1$ whose radius is $1 - \alpha$.
Hence we have

$$\mathrm{Re}\left\{ \frac{zf'(z)}{f(z)} \right\} > \alpha.$$

The sufficiency of the condition is proved.

We shall prove the necessity of the condition.

If $f(z)$ is in $T_\alpha^*(n, \theta)$, then we have

$$\text{Re}\left\{\frac{zf'(z)}{f(z)}\right\} = \text{Re}\left\{\frac{z - \sum\limits_{k=n+1}^{\infty} e^{i(k-1)\theta}ka_k z^k}{z - \sum\limits_{k=n+1}^{\infty} e^{i(k-1)\theta}a_k z^k}\right\} > \alpha$$

for all $z \in U$. Choose the values of z on half line $z = re^{-i\theta}(0 \leqq r < 1)$, then

$$\text{Re}\left\{\frac{z - \sum\limits_{k=n+1}^{\infty} e^{i(k-1)\theta}ka_k z^k}{z - \sum\limits_{k=n+1}^{\infty} e^{i(k-1)\theta}a_k z^k}\right\} = \frac{1 - \sum\limits_{k=n+1}^{\infty} ka_k r^{k-1}}{1 - \sum\limits_{k=n+1}^{\infty} a_k r^{k-1}} > \alpha. \qquad (2.2)$$

Since $1 - \sum_{k=n+1}^{\infty} ka_k r^{k-1} > 0$, we have by the inequality (2.2) that

$$1 - \sum_{k=n+1}^{\infty} ka_k r^{k-1} > \alpha\left(1 - \sum_{k=n+1}^{\infty} a_k r^{k-1}\right). \qquad (2.3)$$

By letting $r \longrightarrow 1$ through half line $z = re^{-i\theta}(0 \leqq r < 1)$ in (2.3), we have

$$1 - \sum_{k=n+1}^{\infty} ka_k \geq \alpha\left(1 - \sum_{k=n+1}^{\infty} a_k\right).$$

That is,

$$\sum_{k=n+1}^{\infty} (k - \alpha)a_k \leqq 1 - \alpha$$

and the proof of the theorem is completed. $\qquad\qquad\square$

THEOREM 2.2. *A function $f(z)$ in $A(n, \theta)$ is in $C_\alpha(n, \theta)$ if and only if*

$$\sum_{k=n+1}^{\infty} k(k - \alpha)a_k \leqq 1 - \alpha. \qquad (2.4)$$

Proof. It is clear that

$$f(z) \in C_\alpha(n, \theta) \text{ if and only if } zf'(z) \in T_\alpha^*(n, \theta)$$

for $0 \leqq \alpha < 1$.

Since

$$zf'(z) = z - \sum_{k=n+1}^{\infty} e^{i(k-1)\theta}ka_k z^k,$$

if we put ka_k instead of a_k in Theorem 2.1, we obtain Theorem 2.2.   □

We again note that the coefficient inequalities (2.1) and (2.4) do not contain θ and these coefficient inequalities coincide with those of Chatterjea[1] and Silverman[3].

Now, we shall give some examples of functions $f(z)$ in $T_\alpha^*(2, \theta)$ and $C_\alpha(2, \theta)$. We refer to Sekine and Yamanaka [4] for the proof of the following theorem.

THEOREM 2.3. *The function*

$$f(z) = \frac{1}{2e^{i\theta}}(1 - e^{i\theta}z)^2 \log(1 - e^{i\theta}z) + \frac{3}{2}z - \frac{3e^{i\theta}}{4}z^2$$

belongs to $T_0^*(2, \theta)$.

Proof. We note that

$$1 + e^{i\theta}z + (e^{i\theta}z)^2 + (e^{i\theta}z)^3 + (e^{i\theta}z)^4 + \cdots = \frac{1}{1 - e^{i\theta}z} \quad (|z| < 1). \qquad (2.5)$$

Let integrate both sides of the equation (2.5) from 0 to z three times. Then we get

$$\frac{z^3}{2\cdot 3} + \frac{e^{i\theta}z^4}{2\cdot 3\cdot 4} + \frac{e^{i2\theta}z^5}{3\cdot 4\cdot 5}\cdots = -\frac{1}{2e^{i3\theta}}(1 - e^{i\theta}z)^2\log(1 - e^{i\theta}z) - \frac{1}{2e^{i2\theta}}z + \frac{3}{4e^{i\theta}}z^2. \quad (2.6)$$

By multiplying $e^{i2\theta}$ to both sides of (2.6), we have

$$\frac{e^{i2\theta}z^3}{2\cdot 3} + \frac{e^{i3\theta}z^4}{2\cdot 3\cdot 4} + \frac{e^{i4\theta}z^5}{3\cdot 4\cdot 5}\cdots = -\frac{1}{2e^{i\theta}}(1 - e^{i\theta}z)^2\log(1 - e^{i\theta}z) - \frac{1}{2}z + \frac{3e^{i\theta}}{4}z^2.$$

We therefore define a function $f(z)$ as follows.

$$\begin{aligned}
f(z) &= \frac{1}{2e^{i\theta}}(1 - e^{i\theta}z)^2 \log(1 - e^{i\theta}z) + \frac{3}{2}z - \frac{3e^{i\theta}}{4}z^2 \\
&= z - \frac{e^{i2\theta}}{1\cdot 2\cdot 3}z^3 - \frac{e^{i3\theta}}{2\cdot 3\cdot 4}z^4 - \frac{e^{i4\theta}}{3\cdot 4\cdot 5}z^5 - \cdots.
\end{aligned}$$

That is,

$$f(z) = z - \sum_{k=3}^{\infty} e^{i(k-1)\theta}a_k z^k \quad (|z| < 1) \qquad (2.7)$$

where $a_k = \dfrac{1}{(k - 2)(k - 1)k}$.

In this case, we have

$$\begin{aligned}
\sum_{k=3}^{\infty} ka_k &= \frac{3}{1\cdot 2\cdot 3} + \frac{4}{2\cdot 3\cdot 4} + \frac{5}{3\cdot 4\cdot 5} + \cdots = \frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \cdots \\
&= 1 - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} + \frac{1}{3} - \frac{1}{4} + \cdots = 1.
\end{aligned}$$

Hence we have that $f(z)$ belongs to $T_0^*(2,\theta)$ by Theorem 2.1. $\square$

By virtue of (2.7) of Theorem 2.3, and Theorem 2.2, we have the following corollary.

COROLLARY 2.1. *The function*

$$f(z) = z - \sum_{k=3}^{\infty} e^{i(k-1)\theta} \frac{1}{(k-2)(k-1)k^2} z^k$$

belongs to $C_0(2,\theta)$.

If $\theta = 0$ or $\theta = \pi$ in the theorem 2.3, we have the following examples.

EXAMPLE 2.1. (T. Sekine and T. Yamanaka [4]) *The function*

$$
\begin{aligned}
f(z) &= \frac{1}{2}(1-z)^2 \log(1-z) + \frac{3}{2}z - \frac{3}{4}z^2 \\
&= z - \frac{z^3}{2\cdot 3} - \frac{z^4}{2\cdot 3\cdot 4} - \frac{z^5}{3\cdot 4\cdot 5} - \cdots
\end{aligned}
$$

belongs to $T_0^*(2,0)$.

EXAMPLE 2.2. *The function*

$$
\begin{aligned}
f(z) &= -\frac{1}{2}(1+z)^2 \log(1+z) + \frac{3}{2}z + \frac{3}{4}z^2 \\
&= z - \frac{z^3}{2\cdot 3} + \frac{z^4}{2\cdot 3\cdot 4} - \frac{z^5}{3\cdot 4\cdot 5} + \cdots
\end{aligned}
$$

belongs to $T_0^*(2,\pi)$.

We note that the function above is a starlike function with not all negative coefficients of order α.

3. Distortion inequalities for $T_\alpha^*(n,\theta)$ and $C_\alpha(n,\theta)$

THEOREM 3.1. *If $f(z)$ is in $T_\alpha^*(n,\theta)$, then*

$$|z| - \frac{1-\alpha}{n+1-\alpha}|z|^{n+1} \leqq |f(z)| \leqq |z| + \frac{1-\alpha}{n+1-\alpha}|z|^{n+1}$$

and

$$1 - \frac{(n+1)(1-\alpha)}{n+1-\alpha}|z|^n \leqq |f'(z)| \leqq 1 + \frac{(n+1)(1-\alpha)}{n+1-\alpha}|z|^n.$$

The right-hand equality holds for the function

$$f(z) = z - e^{in\theta}\frac{1-\alpha}{n+1-\alpha}z^{n+1} \quad \left(z = re^{-i(\theta+\frac{\pi}{n})}\right)$$

and the left-hand equality holds for the function

$$f(z) = z - e^{in\theta}\frac{1-\alpha}{n+1-\alpha}z^{n+1} \quad (z = re^{-i\theta}).$$

Proof. By virtue of Theorem 2.1, note that

$$(n+1-\alpha)\sum_{k=n+1}^{\infty} a_k \leqq \sum_{k=n+1}^{\infty} (k-\alpha)a_k \leqq 1-\alpha.$$

Hence we have

$$\sum_{k=n+1}^{\infty} a_k \leqq \frac{1-\alpha}{n+1-\alpha}. \tag{3.1}$$

By using the coefficient inequality (3.1), we have

$$|f(z)| \leqq |z| + \sum_{k=n+1}^{\infty} a_k |z|^k \quad \leqq \quad |z| + |z|^{n+1} \sum_{k=n+1}^{\infty} a_k$$

$$\leqq \quad |z| + \frac{1-\alpha}{n+1-\alpha}|z|^{n+1}.$$

Similarly, we have

$$|f(z)| \geqq |z| - \frac{1-\alpha}{n+1-\alpha}|z|^{n+1}.$$

By means of (3.1), we have

$$\sum_{k=n+1}^{\infty} ka_k \leqq \alpha \sum_{k=n+1}^{\infty} a_k + (1-\alpha) \quad \leqq \quad \alpha \cdot \frac{1-\alpha}{n+1-\alpha} + (1-\alpha)$$

$$= \quad \frac{(n+1)(1-\alpha)}{n+1-\alpha}.$$

Hence

$$|f'(z)| \leqq 1 + \sum_{k=n+1}^{\infty} ka_k |z|^{k-1} \quad \leqq \quad 1 + |z|^n \sum_{k=n+1}^{\infty} ka_k$$

$$\leqq \quad 1 + \frac{(n+1)(1-\alpha)}{n+1-\alpha}|z|^n.$$

Similarly, we have

$$|f'(z)| \geqq 1 - \frac{(n+1)(1-\alpha)}{n+1-\alpha}|z|^n.$$

$\square$

By using the coefficient inequality (2.4) in Theorem 2.2, we can similarly prove the following theorem.

THEOREM 3.2. *If $f(z)$ is in $C_\alpha(n,\theta)$, then*

$$|z| - \frac{1-\alpha}{(n+1)(n+1-\alpha)}|z|^{n+1} \leqq |f(z)| \leqq |z| + \frac{1-\alpha}{(n+1)(n+1-\alpha)}|z|^{n+1}$$

and

$$1 - \frac{1-\alpha}{n+1-\alpha}|z|^n \leqq |f'(z)| \leqq 1 + \frac{1-\alpha}{n+1-\alpha}|z|^n.$$

The right-hand equality holds for the function

$$f(z) = z - e^{in\theta} \frac{1-\alpha}{(n+1)(n+1-\alpha)} z^{n+1} \quad \left(z = re^{-i(\theta + \frac{\pi}{n})} \right)$$

and the left-hand equality holds for the function

$$f(z) = z - e^{in\theta} \frac{1-\alpha}{(n+1)(n+1-\alpha)} z^{n+1} \quad (z = re^{-i\theta}).$$

Remark 3.1. When $\theta = 0$, Theorem 3.1 and Theorem 3.2 coincide with Theorem 1 and Theorem 2 of Srivastava, Owa and Chatterjea[6], respectively. Further, when $n = 1$ and $\theta = 0$, Theorem 3.1 coincides with the connection of Theorem 4 and Theorem 6 of Silverman[3], and also Theorem 3.2 coincides with the connection of two Corollarys of Silverman[3].

4. Convolutions of functions from subclasses of $A(n, \theta)$

Let $f(z)$ and $g(z)$ be in $A(n, \theta_1)$ and $A(n, \theta_2)$, respectively. That is,

$$f(z) = z - \sum_{k=n+1}^{\infty} e^{i(k-1)\theta_1} a_k z^k \text{ and } g(z) = z - \sum_{k=n+1}^{\infty} e^{i(k-1)\theta_2} b_k z^k.$$

Then we define by $(f * g)(z)$ the convolution of $f(z)$ and $g(z)$, that is,

$$(f * g)(z) = z - \sum_{k=n+1}^{\infty} e^{i(k-1)(\theta_1+\theta_2)} a_k b_k z^k.$$

We shall prove the following theorem employing the technique used earlier by Schild and Silverman[2].

THEOREM 4.1. *If $f(z)$ is in $T_\alpha^*(n, \theta_1)$ and $g(z)$ is in $T_\beta^*(n, \theta_2)$, then $(f * g)(z)$ is an element of $T_\gamma^*(n, \theta_1 + \theta_2)$, where $\gamma = \dfrac{n+1-\alpha\beta}{n+2-\alpha-\beta}$.*
The result is sharp for the functions

$$f(z) = z - e^{in\theta_1} \frac{1-\alpha}{n+1-\alpha} z^{n+1} \ (\in T_\alpha^*(n, \theta_1))$$

and

$$g(z) = z - e^{in\theta_2} \frac{1-\beta}{n+1-\beta} z^{n+1} \ (\in T_\beta^*(n, \theta_2)).$$

Proof. By virtue of Theorem 2.1, we need to find the largest $\gamma = \gamma(\alpha, \beta)$ such that

$$\sum_{k=n+1}^{\infty} (k - \gamma) a_k b_k \leqslant 1 - \gamma.$$

Hence it is equivalent to show that

$$\sum_{k=n+1}^{\infty} (k - \alpha)a_k \leqslant 1 - \alpha$$

and

$$\sum_{k=n+1}^{\infty} (k - \beta)b_k \leqslant 1 - \beta$$

imply that

$$\sum_{k=n+1}^{\infty} (k - \gamma)a_k b_k \leqslant 1 - \gamma$$

for all

$$\gamma = \gamma(\alpha, \beta) \leqslant \frac{n + 1 - \alpha\beta}{n + 2 - \alpha - \beta}.$$

By using Cauchy-Schwarz inequality, we have the following inequality.

$$\sum_{k=n+1}^{\infty} \frac{\sqrt{k - \alpha}\sqrt{k - \beta}}{\sqrt{1 - \alpha}\sqrt{1 - \beta}} \sqrt{a_k b_k} \leqslant 1. \tag{4.1}$$

Therefore it suffices to show that

$$\sqrt{a_k b_k} \leqslant \left(\frac{\sqrt{k - \alpha}}{\sqrt{1 - \alpha}}\right) \left(\frac{\sqrt{k - \beta}}{\sqrt{1 - \beta}}\right) \left(\frac{1 - \gamma}{k - \gamma}\right) \quad (k = n + 1,\ n + 2,\ \cdots).$$

Since we have

$$\sqrt{a_k b_k} \leqslant \left(\frac{\sqrt{1 - \alpha}}{\sqrt{k - \alpha}}\right) \left(\frac{\sqrt{1 - \beta}}{\sqrt{k - \beta}}\right)$$

for each k by (4.1), it will suffice to show that

$$\left(\frac{\sqrt{1 - \alpha}}{\sqrt{k - \alpha}}\right) \left(\frac{\sqrt{1 - \beta}}{\sqrt{k - \beta}}\right) \leqslant \left(\frac{\sqrt{k - \alpha}}{\sqrt{k - \alpha}}\right) \left(\frac{\sqrt{1 - \beta}}{\sqrt{1 - \beta}}\right) \left(\frac{1 - \gamma}{k - \gamma}\right) \tag{4.2}$$

for all k.

Inequality (4.2) is equivalent to

$$\gamma \leqslant \gamma(\alpha, \beta) = \frac{1 - k\left(\dfrac{1 - \alpha}{k - \alpha}\right)\left(\dfrac{1 - \beta}{k - \beta}\right)}{1 - \left(\dfrac{1 - \alpha}{k - \alpha}\right)\left(\dfrac{1 - \alpha}{k - \alpha}\right)}. \tag{4.3}$$

Because the right hand side of (4.3) is an increasing function of $k(k = n+1,\ n+2,\ \cdots)$, we have

$$\gamma \leqslant \frac{1 - (n + 1)\left(\dfrac{1 - \alpha}{n + 1 - \alpha}\right)\left(\dfrac{1 - \beta}{n + 1 - \beta}\right)}{1 - \left(\dfrac{1 - \alpha}{n + 1 - \alpha}\right)\left(\dfrac{1 - \beta}{n + 1 - \beta}\right)} = \frac{n + 1 - \alpha\beta}{n + 2 - \alpha - \beta}.$$

□

Remark 4.1. When $\alpha = \beta$ and $\theta_1 = \theta_2 = 0$, Theorem 4.1 coincides with Throrem 3 of Srivastava, Owa and Chatterjea[6]. By supposing further $n = 1$, Theorem 4.1 coincides with the theorem 1 of Schild and Silverman [2].

THEOREM 4.2. *If $f(z)$ and $g(z)$ are in $C_\alpha(n, \theta_1)$ and $C_\beta(n, \theta_2)$, respectively. Then $(f * g)(z)$ is an element of $C_\gamma(n, \theta_1 + \theta_2)$, where*

$$\gamma = \frac{(n+1)(n+2-\alpha-\beta)}{n^2 + 3n + 3 - (n+2)\alpha - (n+3)\beta + \alpha\beta}.$$

The result is sharp for the functions

$$f(z) = z - e^{in\theta_1} \frac{1 - \alpha}{(n+1)(n+1-\alpha)} z^{n+1} \ \ (\in C_\alpha(n, \theta_1))$$

and

$$g(z) = z - e^{in\theta_2} \frac{1 - \beta}{(n+1)(n+1-\beta)} z^{n+1} \ \ (\in C_\beta(n, \theta_2)).$$

The proof is similar to that of Theorem 4.1, and we omit the proof.

Remark 4.2. When $\alpha = \beta$ and $\theta_1 = \theta_2 = 0$, Theorem 4.2 coincides with Throrem 4 of Srivastava, Owa and Chatterjea[6]. When $\theta_1 = \theta_2 = 0$ and $n = 1$, Theorem 4.2 coincides with the theorem 3 of Schild and Silverman [2].

THEOREM 4.3. *If $f(z)$ and $g(z)$ are in $T_\alpha^*(n, \theta_1)$ and $T_\beta^*(n, \theta_2)$, respectively. Then $(f * g)(z)$ is an element of $C_\gamma(n, \theta_1 + \theta_2)$, where*

$$\gamma = \frac{(n+1)\alpha + (n+1)\beta - (n+2)\alpha\beta}{(n+1) - \alpha\beta}.$$

The result is sharp for the functions

$$f(z) = z - e^{in\theta_1} \frac{1 - \alpha}{n+1-\alpha} z^{n+1} \ \ (\in T_\alpha^*(n, \theta_1))$$

and

$$g(z) = z - e^{in\theta_2} \frac{1 - \beta}{n+1-\beta} z^{n+1} \ \ (\in T_\alpha^*(n, \theta_2)).$$

Also we omit the proof.

Remark 4.3. When $n = 1$ and $\theta_1 = \theta_2 = 0$, Theorem 4.3 coincides with the theorem 5 of Schild and Silverman [2].

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