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BOUNDEDNESS OF SOLUTIONS OF CERTAIN THIRD ORDER DIFFERENTIAL EQUATION

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Abstract. We study the behavior of the solutions of the differential equation $x''' + a(t)f(x'') + b(t)g(x') + c(t)h(x) = e(t)$. We shall present sufficient conditions on the functions involved under which the solutions of the above differential equation are bounded. Some results on the regularity and asymptotic behavior of the solutions are also obtained.

We consider the differential equation

$$x''' + a(t)f(x'') + b(t)g(x') + c(t)h(x) = e(t) \tag{1}$$

where a, b, c and e are continuous on $(0, \infty)$ and f, g and h are continuous on $\mathbb{R}$. We further assume

i) $a(t) \geqslant 0, b(t) > 0, b'(t) > 0$, for $t > 0$,
ii) $sf(s) \geqslant 0,\ \forall s \in \mathbb{R}$,
iii) $|h(s)| \leqslant K|s|$, for some constant K and $s \in \mathbb{R}$
iv) $e(t)/\sqrt{b(t)} \in \mathscr{L}^1(0, \infty)$
v) $sg(s) > 0,\ \lim_{s\to\pm\infty} G(s) = +\infty,\ (G(s) = \int_0^s g(\tau)d\tau)$

THEOREM 1. *Let β, α_0 and α_1 be arbitrary continuous functions on $(0, \infty)$ such that*

vi) *α_0 and α_1 are positive and decreasing and β is positive and increasing on $(0, \infty)$.*
vii) $(\frac{\alpha_0}{\alpha_1})^{\frac{1}{2}}, (\frac{\alpha_1}{\beta}b)^{\frac{1}{2}}, |c|(\frac{\beta}{\alpha_0 b})^{\frac{1}{2}} \in \mathscr{L}^1(0, \infty)$,

then, for every solution $x(t)$ of (1), *$x/\sqrt{\beta/\alpha_0}$, x', and $x''/\sqrt{b}$ are bounded for $t \geqslant 0$.*

Proof. Let $x' = y, y' = z$, and transform Eq. (1) into the system

$$\begin{aligned} x' &= y \\ y' &= z \\ z' &= e(t) - c(t)h(x) - b(t)g(y) - a(t)f(z) \end{aligned} \tag{2}$$

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Next we define the energy function

$$E := \frac{\alpha_0}{\beta}x^2 + \frac{\alpha_1}{\beta}y^2 + \frac{1}{b}z^2 + 2G(y)\ , \tag{3}$$

Differentiating (3) we obtain

$$\begin{aligned} \frac{dE}{dt} \ &= \ \Big(\frac{\alpha_0'}{\beta} - \frac{\alpha_0\beta'}{\beta^2}\Big)x^2 + \Big(\frac{\alpha_1'}{\beta} - \frac{\alpha_1\beta'}{\beta^2}\Big)y^2 + \frac{2\alpha_0}{\beta}xy + \frac{2\alpha_1}{\beta}yz \\ &\quad -\frac{b'}{b^2}z^2 + 2\frac{1}{b}ze - 2\frac{a}{b}zf(z) - 2\frac{c}{b}zh(x) \end{aligned} \tag{4}$$

Using the assumptions (i), (ii), (iii) and (iv) we get the following inequality

$$\frac{dE}{dt} \leqq \frac{2}{b}|e||z| + 2\frac{\alpha_1}{\beta}|y||z| + 2\frac{\alpha_0}{\beta}|x||y| + 2K\frac{|c|}{b}|z||x| \tag{5}$$

Considering nonnegativeness of G and making use of the inequality $2|AB| \leqslant |A|^2 + |B|^2$, we find

$$\begin{aligned} \frac{2}{b}|e||z| &\leqq \frac{|e|}{\sqrt{b}} + \frac{|e|}{\sqrt{b}}E \\ 2\frac{\alpha_1}{\beta}|y||z| &\leqq \Big(\frac{\alpha_1 b}{\beta}\Big)^{\frac{1}{2}}E \\ 2\frac{\alpha_0}{\beta}|x||y| &\leqslant \Big(\frac{\alpha_0}{\alpha_1}\Big)^{\frac{1}{2}}E \\ 2\frac{|c|}{b}|z||x| &\leqslant |c|\Big(\frac{\beta}{\alpha_0 b}\Big)^{\frac{1}{2}}E \end{aligned}$$

These inequalities together with (5) yield

$$\frac{dE}{dt} \leqslant \frac{|e(t)|}{\sqrt{b(t)}} + \Phi(t)E(t) \tag{6}$$

where

$$\Phi(t) = \frac{|e|}{\sqrt{b}} + \Big(\frac{\alpha_1 b}{\beta}\Big)^{\frac{1}{2}} + \Big(\frac{\alpha_0}{\alpha_1}\Big)^{\frac{1}{2}} + K|c|\Big(\frac{\beta}{\alpha_0 b}\Big)^{\frac{1}{2}} \tag{7}$$

Integrating (6), having the assumption (iv) in mind, and using Gronwall inequality, we finally obtain

$$E(t) \leqslant A\exp\Big(\int_0^t \Phi(s)ds\Big)$$

for some constant A. By assumptions (iv) and (vii) $\Phi \in \mathscr{L}^1(0,\infty)$, which implies the boundedness of E. By (6) we conclude the boundedness of $\frac{\alpha_0}{\beta}x^2, \frac{\alpha_1}{\beta}y^2, \frac{1}{b}z^2$. Now, by (3) we have the boundedness of $G(y)$. This, together with the assumption (vi) imply the boundedness of y.

REMARK 1. The conclusion of Theorem 1 remains valid for the corresponding homogeneous differential equation, that is, when $e \equiv 0$.

THEOREM 2. *Theorem* 1 *remains valid if the assumptions* (i), (ii) *and* (iv) *are replaced by*

$i)'$ $a(t) > 0, b(t) > 0$ *for* $t > 0$

$ii)'$ *There exists a constant* $M > 0$ *such that* $0 < sf(s) \leqq Ms^2$, $s \in \mathbb{R}$ *and* $b'(t) + 2Mb(t)a(t) > 0,\ t \in (0,\infty)$

$iv)'$ $\dfrac{e}{\sqrt{b}}, \dfrac{e^2}{b' + 2Mba} \in \mathscr{L}^1(0,\infty)$

Proof. Using above assumptions and (4) we get

$$\begin{aligned}\frac{dE}{dt} &\leqslant -2M\frac{a}{b}z^2 - \frac{b'}{b^2}z^2 + \frac{2}{b}|z_2||e| + 2\frac{\alpha_0}{\beta}|x||y| \\ &\quad +2K\frac{c}{b}|z_2||z_0| + \frac{2\alpha_1}{\beta}|y||z|\end{aligned}$$

Noting $e \not\equiv 0$, the first three terms are equal to

$$-(b' + 2Mab)\Big(\frac{|z_2|}{b} - \frac{|e|}{b' + 2Mba}\Big)^2 + \frac{e^2}{b' + 2Mba}$$

It follows by $(ii)'$ and (7) that

$$\frac{dE}{dt} \leqslant \frac{e^2}{b' + 2Mba} + \Big(\Phi - \frac{|e|}{\sqrt{b}}\Big)E, \tag{8}$$

with Φ defined by (7).

Integrating (8) and using the assumption $(iv)'$ and Gronwall inequality we conclude the boundedness of E. The rest of the proof follows along the same lines as in the proof of Theorem 1.

The next theorem is concerned with the regularity of solutions of differential equation (1).

THEOREM 3. *Under the assumptions of Theorem* 2, *every solution of Eq.* (1) *satisfies*

$$\Big(\frac{|\alpha_0'|}{\beta}\Big)^{1/2}x, \Big(\frac{|\alpha_1'|}{\beta}\Big)x' \in \mathscr{L}^2(0,\infty)$$

If, in addition, we assume

$$(vii) \qquad \text{l.u.b.}\ \frac{b^2}{b' + 2Mba} = k < \infty, \quad t \geqslant 0$$

then $x'' \in \mathscr{L}^2(0,\infty)$.

Proof. Starting from Eq. (4) we get

$$\Big(\frac{\alpha_0\beta'}{\beta^2} - \frac{\alpha_0'}{\beta}\Big)x^2 + \Big(\frac{\alpha_1\beta'}{\beta^2} - \frac{\alpha_1'}{\beta}\Big)y^2 + \frac{b' + 2Mba}{b^2}z^2 \leqslant -\frac{dE}{dt} + \frac{|e|}{\sqrt{b}} + \Phi E$$

where Φ is given by Eq. (7). Integrating both sides of this inequality we get

$$\int_0^t \Big[\Big(\frac{\alpha_0\beta'}{\beta^2}-\frac{\alpha_0'}{\beta}\Big)x^2+\Big(\frac{\alpha_1\beta'}{\beta^2}-\frac{\alpha_1'}{\beta}\Big)y^2+\frac{b'+2Mba}{b^2}z^2\Big] \leqslant E(0)-E(t)+\int_0^t \frac{|e|}{\sqrt{b}}ds+K_1\int_0^t \Phi(s)ds$$

where, we have assumed $E(t)\leqslant K_1, t\geqslant 0$. Now using the assumptions $(iv)'$ and $(vii)'$ and considering again the boundedness of $E(t)$, we obtain

$$\int_0^t \frac{|\alpha_0'|}{\beta}x^2<\infty,\ \int_0^t \frac{|\alpha_1'|}{\beta}y^2<\infty,\ \int_0^t z^2<\infty;\ t\geqslant 0$$

which proves the Theorem 3.

THEOREM 4. *In addition to the assumptions of previous theorem we further assume that* $a(t), b(t), c(t)$ *and* $e(t)$ *are bounded for* $t\geqslant 0$. *Now if we assume* β/α_0 *is bounded for* $t\geqslant 0$, *then the solutions of* Eq. (1), *for which* x'' *is bounded, satisfy*

$$\lim_{t\to\infty} x''(t)=0$$

Proof. From Eq. (2) we obtain

$$|zz'|\leqslant |z||e(t)|+|c(t)||z||h(x)|+b(t)|z||g(y)|+a(t)|z||f(z)|$$

Using assumptions (iii) and $(ii)'$ and noting that by boundedness of y and continuity of g, the function g is bounded on $(0,\infty)$, we have

$$|zz'|\leqslant |e|\sqrt{b}\frac{|z|}{\sqrt{b}}+k|c|\sqrt{b}\sqrt{\beta/\alpha_0}\frac{|z|}{\sqrt{b}}\frac{|x|}{\sqrt{\beta/\alpha_0}}+K_1b^{3/2}\frac{|z|}{\sqrt{b}}+Mab\frac{z^2}{b}$$

where we have assumed $|g(y)|\leqslant K_1$. Hence the expression on the right is bounded. It follows, by a simple lemma in [6], (p. 291), that

$$\lim_{t\to\infty} z(t)=0$$

Imposing more restrictive conditions on the differential equation (1) we can obtain similar results for x and y. For example, if we take l.u.b. $\beta/\sqrt{\alpha_1'}\leqslant c_1$, then by Theorem 3, $y\in\mathscr{L}^2(0,\infty)$. Since $y'=z$, then y' is also bounded. Now using the same lemma in [8] we get

$$\lim_{t\to\infty} y(t)=0\,.$$

EXAMPLE. We consider the differential equation

$$x'''+ax''+(t^2+1)x'+c(t)x=e(t) \tag{9}$$

where a is a nonneqative constant. Clearly the assumptions (i), (ii), (iii) and (v) are satisfied. Now if one takes $\alpha_0=(1+t^2)^{-2}, \alpha_1=(1+t^2)^{-1}$ and $\beta_0=(1+t^2)^{-4}$, then (vi) and (vii) will also hold. Finally if we choose $e(t)$ such that $e(t)/(1+t^2)^{\frac{1}{2}}\in\mathscr{L}^1(0,\infty)$, then by Theorem 1 we conclude that, for every solution $x(t)$ of Eq. (9), $x(t)/(1+t^2)^{\frac{3}{2}}, x'(t)$ and $x''(t)/(1+t^2)^{\frac{1}{2}}$ are bounded for $t\geqslant 0$.

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