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STABILITY THEORY OF FUZZY DIFFERENTIAL EQUATIONS VIA DIFFERENTIAL INEQUALITIES

V. LAKSHMIKANTHAM AND S. G. LEELA

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Abstract. Lyapunov theory of stability is developed for fuzzy differential equations employing suitable theory of differential inequalities. Method of perturbing Lyapunov function is utilized which provides weaker assumptions for discussing stability theory.

1. Introduction

The industrial interest in fuzzy control and logic [1, 14] has dramatically increased the study of fuzzy systems. The calculus of fuzzy valued functions has recently been developed [2, 3, 4] and the investigation of fuzzy differential equations has been initiated [5, 6, 8, 12, 13].

In this paper, we shall develop stability theory which corresponds to Lyapunov theory of stability for fuzzy differential systems. For this purpose, one needs to discuss comparison results in terms of Lyapunov-like functions employing the theory of differential inequalities. Once such a result is available, it is comparatively eaisier to systematically develop a theory parallel to Lyapunov's stability theory. To avoid monotony, we shall once for all consider the stability criteria by the method of perturbing Lyapunov functions [11] so that the standard theorems on stability criteria result as a consequence.

2. Preliminaries

Let $P_k(\mathbf{R}^n)$ denote the family of all nonempty compact, convex subsets of $\mathbf{R}^n$. If $\alpha, \beta \in \mathbf{R}$ and $A, B \in P_k(\mathbf{R}^n)$, then

$$\alpha(A+B) = \alpha A + \alpha B, \quad \alpha(\beta A) = (\alpha\beta)A, \quad 1A = A$$

and if $\alpha, \beta \geqslant 0$, then $(\alpha+\beta)A = \alpha A + \beta A$. Let $I = [t_0, t_0 + \alpha]$, $t_0 \geqslant 0$ and $a > 0$ and denote by $E^n = [u : \mathbf{R}^n \to [0,1]$ such that u satisfies (i) to (iv) mentioned bellow]:

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(i) u is normal, that is, there exists an $x_0 \in \mathbf{R}^n$ such that $u(x_0) = 1$;

(ii) u is fuzzy convex, that is, for $x, y \in \mathbf{R}^n$ and $0 \leqslant \lambda \leqslant 1$,

$$u(\lambda x + (1-\lambda)y) \geqslant \min[u(x), u(y)];$$

(iii) u is upper semicontinuous;
(iv) $[u]^0 = [x \in \mathbf{R}^n : u(x) > 0]$ is compact.

For $0 < \alpha < 1$, we denote $[u]^\alpha = [x \in \mathbf{R}^n : u(x) \geqslant \alpha]$. Then from (i) to (iv), it follows that the α-level sets $[u]^\alpha \in P_k(\mathbf{R}^n)$ for $0 \leqslant \alpha \leqslant 1$.

For later purposes, we define $\hat{o} \in E^n$ as $\hat{o}(x) = 1$ if $x = 0$ and $\hat{o}(x) = 0$ if $x \neq 0$.

Let $d_H(A, B)$ be the Hausdorff distance between the sets $A, B \in P_k(\mathbf{R}^n)$. Then we define

$$d[u, v] = \sup_{0 \leqslant \alpha \leqslant 1} d_H[[u]^\alpha, [v]^\alpha],$$

which defines a metric in E^n and (E^n, d) is a comlete metric space. We list the following properties of $d[u, v]$ (see [6]):

$$d[u+w, v+w] = d[u, v] \quad \text{and} \quad d[u, v] = d[v, u],$$
$$d[\lambda u, \lambda v] = |\lambda|\, d[u, v],$$
$$d[u, v] \leqslant d[u, w] + d[w, v],$$

for all $u, v, w \in E^n$ and $\lambda \in \mathbf{R}$.

For $x, y \in E^n$ if there exists a $z \in E^n$ such that $x = y + z$, then z is called the H-difference of x and y and is denoted by $x - y$. A mapping $F : I \to E^n$ is differentiable at $t \in I$ if there exists a $F'(t) \in E^n$ such that the limits

$$\lim_{h \to 0^+} \frac{F(t+h) - F(t)}{h} \quad \text{and} \quad \lim_{h \to 0^+} \frac{F(t) - F(t+h)}{h}$$

exist and are equal to $F'(t)$. Here the limits are taken in the metric space (E^n, d).

Moreover, if $F : I \to E^n$ is continuous, then it is integrable and

$$\int_a^b F = \int_a^c F + \int_c^b F.$$

Also, the following properties of the integral are valid (see [3, 4, 5, 6]). If $F, G : I \to E^n$ are integrable, $\lambda \in \mathbf{R}$, then the following hold:

$$\int (F + G) = \int F + \int G;$$
$$\int \lambda F = \lambda \int F,\ \lambda \in \mathbf{R};$$
$$d[F, G] \text{ is integrable;}$$
$$d\Big[\int F, \int G\Big] \leqslant \int d[F, G].$$

Finally, let $F : I \to E^n$ be continuous. Then the integral $G(t) = \int_{t_0}^{t} F$ is differentiable and $G'(t) = F(t)$. Furthermore,

$$F(t) - F(t_0) = \int_a^t F'(t)$$

(see [2, 3, 4, 5, 6] for details).

3. Comparison result

Consider the fuzzy differential equation

$$u' = f(t,u), \quad u(t_0) = u_0, \tag{3.1}$$

where $f \in C[\mathbf{R}_+ \times S(\rho), E^n]$ and $S(\rho) = [u \in E^n : d[u,\hat{o}] < \rho]$. We assume that $f(t,\hat{o}) = \hat{o}$ so that we have the trivial solution for (3.1).

To investigate stability criteria, the follwing comparison result in terms of a Lyapunov function is very important which can be proved via the theory of differential inequalities [9]. Here Lyapunov function serves as a vehicle to transform the fuzzy differential equation into a scalar comparison differential equation and therefiore, it is enough to consider the stability properties of the simpler comparison equation.

THEOREM 3.1. *Assume that*

(i) $V \in C[\mathbf{R}_+ \times S(\rho), \mathbf{R}_+]$, $|V(t,u_1) - V(t,u_2)| \leqslant Ld[u_1,u_2]$, $L > 0$ *and* $D^+V(t,u) \equiv \lim\sup_{h\to 0^+} \frac{1}{h}[V(t+h, u+hf(t,u)) - V(t,u)] \leqslant g(t,V(t,u))$, *where* $g \in C[\mathbf{R}_+^2, \mathbf{R}]$. *Then, if* $u(t)$ *is any solution of* (3.1) *existing on* $[t_0,\infty)$ *such that* $V(t_0,u_0) \leqslant w_0$, *we have*

$$V(t,u(t)) \leqslant r(t,t_0,w_0), \quad t \geqslant t_0,$$

where $r(t,t_0,w_0)$ *is the maximal solution of the scalar differential equation*

$$w' = g(t,w), \quad w(t_0) = w_0 \geqslant 0, \tag{3.2}$$

existing on $[t_0,\infty)$.

Proof. Let $u(t)$ be any solution of (3.1) existing on $[t_0,\infty)$. Define $m(t) = V(t,u(t))$ so that $m(t_0) = V(t_0,u_0) \leqslant w_0$. Now for small $h > 0$,

$$\begin{aligned} m(t+h) - m(t) &= V(t+h, u(t+h)) - V(t,u(t)) \\ &= V(t+h, u(t+h)) - V(t+h, u(t) + hf(t,u(t))) \\ &\qquad + V(t+h, u(t)+hf(t,u(t)) - V(t,u(t)), \\ &\leqslant Ld[u(t+h), u(t)+hf(t,u(t))] \\ &\qquad + V(t+h, u(t)+hf(t,u(t))) - V(t,u(t)), \end{aligned}$$

using the Lipschitz condition given in (i). Thus

$$D^+m(t) = \lim_{h\to 0^+} \sup \frac{1}{h}[m(t+h) - m(t)] \leqslant D^+V(t,u(t)) + L \lim_{h\to 0^+} \sup \frac{1}{h}[d[u(t+h), u(t) + hf(t,u(t))]].$$

Let $u(t+h) = u(t) + z(t)$, where $z(t)$ is the H-difference for small $h > 0$ which is assumed to exist. Hence employing the properties of $d[u,v]$, we see that

$$\begin{aligned} d[u(t+h), u(t) + hf(t,u(t)) &= d[u(t) + z(t), u(t) + hf(t,u(t))] \\ &= d[z(t), hf(t,u(t)] \\ &= d[u(t+h) - u(t), hf(t,u(t))]. \end{aligned}$$

Consequently

$$\frac{1}{h}d[u(t+h), u(t) + hf(t,u(t))] = d\Big[\frac{u(t+h) - u(t)}{h}, f(t,u(t))\Big]$$

and therefore

$$\begin{aligned} &\lim_{h\to 0^+} \sup \frac{1}{h}[d[u(t+h), u(t) + hf(t,u(t))]] \\ &\quad = \lim_{h\to 0^+} \sup \frac{1}{h}\Big[d\Big[\frac{u(t+h) - u(t)}{h}, f(t,u(t))\Big]\Big] \\ &\quad = d[u'(t), f(t,u(t))] = 0, \end{aligned}$$

since $u(t)$ is the solution of (3.1). We therefore have the scalar differential inequality

$$D^+m(t) \leqslant g(t,m(t)), m(t_0) \leqslant w_0,\ t \geqslant t_0,$$

which by the theory of differential inequalities [9] implies

$$m(t) \leqslant r(t,t_0,w_0),\ t \geqslant t_0.$$

This proves the claimed estimate of the theorem.

The following corollaries are useful.

COROLLARY 3.1. *The function $g(t,w) \equiv 0$ is admissible in Theorem 3.1 to yield the estimate*

$$V(t,u(t)) \leqslant V(t_0,u_0),\ t \geqslant t_0.$$

COROLLARY 3.2. *If, in Theorem 3.1, we strengthen the assumption on $D^+V(t,u)$ to*

$$D^+V(t,u) \leqslant -C[w(t,u)] + g(t,V(t,u)),$$

where $w \in C[\mathbf{R}_+ \times S(\rho), \mathbf{R}_+]$, $C \in \mathscr{K} = [a \in C[[o,\rho), \mathbf{R}_+] : a(w)$ is increasing in w and $a(0) = 0]$, and $g(t,w)$ is nondecreasing in w for each $t \in \mathbf{R}_+$, then we get the estimate

$$V(t,u(t)) + \int_{t_0}^{t} C[w(s,u(s))]ds \leqslant r(t,t_0,w_0), \quad t \geqslant t_0,$$

whenever $V(t_0, u_0) \leqslant w_0$.

Proof. Set $L(t,u(t)) = V(t,u(t)) + \int_{t_0}^{t} C[w(s,u(s))]ds$ and note that

$$D^+L(t,u(t)) \leqslant D^+V(t,u(t)) + C[w(t,u(t))] \leqslant g(t,V(t,u(t))) \leqslant g(t,L(t,u(t))),$$

using the monotone character of $g(t,w)$. We then get immediately by Theorem 3.1, the estimate

$$L(t,u(t)) \leqslant r(t,t_0,w_0), \quad t \geqslant t_0,$$

where $u(t)$ is any solution of (3.1). This implies the stated estimate.

A simple example of $V(t,u)$ is $d[u,\hat{o}]$ so that

$$D^+V(t,u) = \lim_{h\to 0^+} \sup \frac{1}{h}[d[u+hf(t,u),\hat{o}] - d[u,\hat{o}]].$$

4. Stability criteria

Let $u(t)$ be any solution of (3.1) existing on $[t_0,\infty)$.

DEFINITION 4.1. We say that the trivial solution of (3.1) is equi-stable, if given $0 < \varepsilon < \rho$ and $t_0 \in \mathbf{R}_+$, there exists a $\delta = \delta(t_0,\varepsilon) > 0$ such that

$$d[u_0,\hat{o}] < \delta \quad \text{implies} \quad d[u(t),\hat{o}] < \varepsilon,\ t \geqslant t_0.$$

If δ is independent of t_0, then the stability is uniform. Based on this definition, the other notions of stability can be formulated. See [9, 10] for details.

We begin with the following result which provides nonuniform stability criteria under weaker assumption. See [10].

THEOREM 4.1. *Assume that*

(A_1) $V_1 \in C[\mathbf{R}_+ \times S(\rho), \mathbf{R}_+]$, $|V_1(t,u_1) - V_1(t,u_2)| \leqslant L_1 d[u_1,u_2]$, $L_1 > 0$, $V_1(t,u) \leqslant a_0(t,d[u,\hat{o}])$, *where* $a \in C[\mathbf{R}_+ \times [0,p), \mathbf{R}_+]$ *and* $a_0(t,\cdot) \in \mathscr{K}$ *for each* $t \in \mathbf{R}_+$*;*

(A_2) $D^+V_1(t,u) \leqslant g_1(t,V_1(t,u))$, $(t,u) \in \mathbf{R}^+ \times S(\rho)$, *where* $g_1 \in C[\mathbf{R}_+^2, \mathbf{R}]$ *and* $g_1(t,0) \equiv 0$*;*

(A_3) *for every* $\eta > 0$, *there exists a* $V_\eta \in C[\mathbf{R}_+ \times S(\rho) \cap S^c(\eta), \mathbf{R}_+]$,

$$|V_\eta(t,u_1) - V_\eta(t,u_2)| \leqslant L_\eta d[u_1,u_2],$$
$$b(d[u,\hat{o}]) \leqslant V(t,u) \leqslant a(d[u,\hat{o}])\ a,b \in \mathscr{K},$$

and

$$D^+V_1(t,u) + D^+V_\eta(t,u) \leqslant g_2(t,V_1(t,u) + V_\eta(t,u))$$

for $(t,u) \in \mathbf{R}_+ \times S(\rho) \cap S^c(\eta)$.

(A_4) *the trivial solution* $w_1 \equiv 0$ *of*

$$w_1' = g_1(t,w_1), w_1(t_0) = w_{10} \geqslant 0, \tag{4.1}$$

is equi-stable;

(A_5) *the trivial solution* $w_2 = 0$ *of*

$$w_2' = g_2(t, w_2), w_2(t_0) = w_{20} \geqslant 0, \tag{4.2}$$

is uniformly stable.

Then the trivial solution of (3.1) *is equi-stable.*

Proof. Let $0 < \varepsilon < \rho$ and $t_0 \in \mathbf{R}_+$ be given. Since the trivial solution of (4.2) is uniformly stable, given $b(\varepsilon) > 0$ and $t_0 \in \mathbf{R}_+$, there exists a $\delta^0 = \delta^0(\varepsilon) > 0$ satisfying

$$0 \leqslant w_{20} < \delta^0 \quad \text{implies} \quad w_2(t, t_0, w_{20}) < b(\varepsilon), \quad t \geqslant t_0, \tag{4.3}$$

where $w_2(t, t_0, w_{20})$ is any solution of (4.2). In view of the hypothesis on $a(w)$, there is a $\delta_2 = \delta_2(\varepsilon) > 0$ such that

$$a(\delta_2) < \frac{\delta^0}{2}. \tag{4.4}$$

Since the trivial solution of (4.1) is equi-stable, given $\frac{\delta^0}{2} > 0$ and $t_0 \in \mathbf{R}_+$, we can find a $\delta^* = \delta^*(t_0, \varepsilon) > 0$ such that

$$0 \leqslant w_{10} < \delta^* \quad \text{implies} \quad w_1(t, t_0, w_{10}) < \frac{\delta^0}{2}, \quad t \geqslant t_0, \tag{4.5}$$

where $w_1(t, t_0, w_{10})$ is any solution of (4.1).

Choose $w_{10} = V_1(t_0, u_0)$. Since $V_1(t, u) \leqslant a_0(t, d[u, \hat{0}])$, we see that there exists a $\delta_1 = \delta_1(t_0, \varepsilon) > 0$ satisfying

$$d[u_0, \hat{0}] < \delta_1 \quad \text{and} \quad a_0, (t_0, d[u_0, \hat{o}]) < \delta^*, \tag{4.6}$$

hold simultaneously. Define $\delta = \min(\delta_1, \delta_2)$. Then we claim that

$$d[u_0, \hat{o}] < \delta \quad \text{implies} \quad d[u(t), \hat{o}] < \varepsilon, \quad t \geqslant t_0, \tag{4.7}$$

for any solution $u(t)$ of (3.1). If this is false, there would exist a solution $u(t)$ of (3.1) with $d[u_0, \hat{o}] < \delta$ and $t_1, t_2 > t_0$ such that

$$d[u(t_1), \hat{o}] = \delta_2, \quad d[u(t_2), \hat{o}] = \varepsilon \quad \text{and} \quad \delta_2 \leqslant d[u(t), \hat{o}] \leqslant \varepsilon \leqslant \rho \tag{4.8}$$

for $t_1 \leqslant t \leqslant t_2$. We let $\eta = \delta_2$ so that the existence of a V_η satisfying hypothesis (A_3) is assured. Hence, setting

$$m(t) = V_1(t, u(t)) + V_\eta(t, u(t)), \quad t \in [t_1, t_2],$$

we obtain the differential inequality

$$D^+ m(t) \leqslant g_2(t, m(t)), \quad t_1 \leqslant t \leqslant t_2,$$

which yields

$$V_1(t_2, u(t_2)) + V_\eta(t_2, u(t_2)) \leqslant r_2(t_2, t_1, w_{20}), \tag{4.9}$$

with $w_{20} = V_1(t_1, u(t_1)) + V_\eta(t_1, u(t_1))$, $r_2(t, t_1, w_{20})$ is the maximal solution of (4.2). We also have, because od (A_1) and (A_2),

$$V_1(t_1, u(t_1)) \leqslant r_1(t_1, t_0, w_{10}),$$

with $w_{10} = V_1(t_0, u_0)$, where $r_1(t, t_0, w_{10})$ is the maximal solution of (4.1). By (4.5) and (4.6), we get

$$V_1(t_1, u(t_1)) < \frac{\delta_0}{2}. \tag{4.10}$$

Also, by (4.4), (4.8) and (A_3), we arrive at

$$V_\eta(t_1, u(t_1)) \leqslant a(\delta_2) < \frac{\delta_0}{2}. \tag{4.11}$$

Thus (4.10) and (4.11) and the definition of w_{20} shows that $w_{20} < \delta_0$ which, in view of (4.3) shows that $w_2(t_2, t_1, w_{20}) < b(\varepsilon)$. If then follows from (4.9), $V_1(t, u) \geqslant 0$ and (A_3),

$$b(\varepsilon) = b(d[u(t_2), \hat{o}]) \leqslant V_\eta(t_2, u(t_2)) \leqslant r_2(t_2, t_1, w_{20}) < b(\varepsilon).$$

This contradiction proves equi-stability of the trivial solution of (3.1) since (4.7) is then true.

The proof is complete

The next result offers conditions for equi-asymptotic stability.

THEOREM 4.2. *Let the assumptions of Theorem 4.1 hold except that condition* (A_2) *is strengthened to*

(A_2^*) $D^+V_1(t, u) \leqslant -c(w(t, u)) + g_1(t, V_1(t, (u)),\ c \in \mathscr{K},\ w \in C[\mathbf{R}_+ \times S(\rho), \mathbf{R}_+]$, $|w(t, u_1) - w(t, u_2)| \leqslant Nd[u_1, u_2],\ N > 0$ *and* $D^+w(t, u)$ *is bounded above or below.*

Then the trivial solution of (3.1) *is equi-asymptotically stable, if* $g_1(t, w)$ *is monotone nondecreasing in* w *and* $w(t, u) \geqslant b_0(d[u, \hat{o}]),\ b_0 \in \mathscr{K}$.

Proof. By Theorem 4.1, the trivial solution of (3.1) is equi stable. Hence letting $\varepsilon = \rho$ so that $\delta_0 = \delta(\rho, t_0)$, we get, by equi-stability

$$d[u_0, \hat{o}] < \delta_0 \quad \text{implies} \quad d[u(t), \hat{o}] < \rho, t \geqslant t_0.$$

We shall show that, for any solution $u(t)$ of (3.1) with $d[u_0, \hat{o}] < \delta_0$, it follows that $\lim_{t\to\infty} w(t, u(t)) = 0$, which implies by the property of $w(t, u)$, $\lim_{t\to\infty} d[u(t), \hat{o}] = 0$ and we are done.

Suppose that $\limsup_{t\to\infty} w(t, u(t)) \neq 0$. Then there would exist two divergent sequences $\{t_i'\}, \{t_i''\}$ and a $\sigma > 0$ satisfying

(a) $w(t_i', u(t_i')) = \frac{\sigma}{2}$, $w(t_i'', u(t_i'')) = \sigma$ and $w(t, u(t)) \geqslant \frac{\sigma}{2}$, $t \in (t_i'', t_i')$, or

(b) $w(t_i', u(t_i')) = \sigma$, $w(t_i'', u(t_i'')) = \frac{\sigma}{2}$ and $w(t, u(t)) \geqslant \frac{\sigma}{2}$, $t \in (t_i'', t_i')$. Suppose that $D^+w(t, u(t)) \leqslant M$. Then using (a) we obtain

$$\frac{\sigma}{2} = \sigma - \frac{\sigma}{2} = w(t_i'', u(t_i'')) - w(t_i', u(t_i')) \leqslant M(t_i'' - t_i'),$$

which shows that $t_i'' - t_i' \geqslant \frac{\sigma}{2M}$ for each i. Hence by (A_2^*) and Corollary 3.2, we have

$$V_1(t, u(t)) \leqslant r_1(t, t_0, w_{10}) = \sum_{i=1}^{n} \int_{t_i'}^{t_i''} C[w(s, u(s))]\, ds,\ t \geqslant t_0.$$

Since $w_{10} = V_1(t_0, u_0) \leqslant a_0(t_0, d[u_0, \hat{o}]) \leqslant a_0(t_0, \delta_0) < \delta^*(\rho)$, we get from (4.5) $w_1(t, t_0, w_{10}) < \frac{\delta^0(\rho)}{2}$, $t \geqslant t_0$. We thus obtain

$$0 \leqslant V_1(t, u(t)) \leqslant \frac{\delta^0(\rho)}{2} - C\Big(\frac{\sigma}{2}\Big)\frac{\sigma}{2M}n.$$

For sufficiently larger n, we get a contradiction and therefore $\lim\limits_{t\to\infty} \sup w(t, u(t)) = 0$. Since $w(t, u) \geqslant b_0(d[u, \hat{o}])$ by assumption, it follows that $\lim\limits_{t\to\infty} d[u(t), \hat{o}] = 0$ and the proof is complete.

The following remarks are in order.

REMARK 4.1. The functions $g_1(t, w) = g_2(t, w) \equiv 0$ are admissible in Theorem 4.1 to imply the same conclusion. If $V_1(t, u) \equiv 0$ and $g_1(t, w) \equiv 0$, then we get uniform stability from Theorem 4.1. If, on the other hand, $V_\eta(t, u) \equiv 0$, $g_2(t, w) \equiv 0$ and $V_1(t, u) \geqslant b(d[u, \hat{o}])$, $b \in \mathscr{K}$, then Theorem 4.1 yields equi-stability. We note that known results on equi-stability require the assumption to hold everywhere in $S(\rho)$ and Theorem 4.1 relaxes such a requirement considerably by the method of perturbing Lyapunov functions. See [9,10].

REMARK 4.2. The functions $g_1(t, w) \equiv g_2(t, w) \equiv 0$ are admissible in Theorem 4.2 to yield equi-asymptotic stability. Similarly, if $V_\eta(t, u) \equiv 0$, $g_2(t, w) \equiv 0$ with $V_1(t, u) \geqslant b(d[u, \hat{o}])$, $b \in \mathscr{K}$ implies the same conclusion. If $V_1(t, u) \equiv 0$ and $g_1(t, w) \equiv 0$ in Theorem 4.1, to get uniform asymptotic stability, one needs strenghten the estimate on $D^+V_\eta(t, u)$. This we state as a corollary.

COROLLARY 4.1. *Assume that the assumptions of Theorem* 4.1 *hold with* $V_1(t, u) \equiv 0$, $g_1(t, w) \equiv 0$. *Suppose further that*

$$D^+V_\eta(t, u) \leqslant -C[w(t, u)] + g_2(t, V_\eta(t, u)), (t, u) \in \mathbf{R}_+ \times S(\rho) \cap S^c(\eta), \tag{4.12}$$

where $w \in C[\mathbf{R}_+ \times S(\rho), \mathbf{R}_+]$, $w(t, u) \geqslant b(d[u, \hat{o}])$, $c, b \in \mathscr{K}$ *and* $g_2(t, w)$ *is nondecreasing in* w. *Then the trivial solution of* (3.1) *is uniformly asymptotically stable.*

Proof. The trivial solution of (3.1) is uniformly stable by Remark 4.1 in the present case. Hence taking $\varepsilon = \rho$ and designating by $\delta_0 = \delta(\rho)$, we have

$$d[u_o, \hat{o}] < \delta_0 \quad \text{implies} \quad d[u(t), \hat{o}] < \rho,\ t \geqslant t_0.$$

To provr uniform attractivity, let $o < \varepsilon < \rho$ be given. Let $\delta = \delta(\varepsilon) > 0$ be the number relative to ε in uniform stability. Choose $T = \frac{b(\rho)}{C(\delta)} + 1$. Then we shall show that there exists a $t^* \in [t_0, t_0 + T]$ such that $w(t^*, u(t^*)) < b(\delta)$ for any solution $u(t)$ of (3.1) with $d[u_0, \hat{o}] < \delta_0$. If this is not true, then $w(t, u(t)) \geqslant b(\delta)$, $t \in [t_0, t_0 + T]$. Now using the assumption (4.12) and arguing as in Corollary 3.2, we get

$$0 \leqslant V_\eta(t_0 + T, u(t_0 + T)) \leqslant r_2(t_0 + T, t_0, w_{20}) - \int_{t_0}^{t_0+T} w(s, u(s))\, ds.$$

This yields, since $r_2(t,t_0,w_{20}) < b(\rho)$ and the choice of T,

$$0 \leqslant V_\eta(t_0+T, u(t_0+T)) \leqslant b(\rho) - c(\delta)T < 0,$$

which is a contradiction. Hence there exists a $t^* \in [t_0, t_0+T]$ satisfying $w(t^*, u(t^*)) < b(\delta)$, which implies $d[u(t^*), \hat{o}] < \delta$. Consequently, it follows, by uniform stability that

$$d[u_0, \hat{o}] < \delta_0 \quad \text{implies} \quad d[u(t), \hat{o}] < \varepsilon,\ t \geqslant t_0 + T,$$

and the proof is complete

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V. Lakshmikantham
Florida Institute of Technology
Department of Mathematical Sciences
Melbourne, FL 32901
USA
e-mail: lakshmik@winnie.fit.edu
e-mail: dkermani@winnie.fit.edu

S. G. Leela
SUNY at Geneseo
Department of Mathematics
Geneseo, NY 14454
USA