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SEVERAL PROPERTIES ON CLASS A INCLUDING p–HYPONORMAL AND LOG–HYPONORMAL OPERATORS

MASATOSHI ITO

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Abstract. In this paper, we shall show that Yamazaki's result in [23] remains valid for class A introduced in [13] as a class of operators including p-hyponormal and log-hyponormal operators. Moreover we shall show several properties on class A which correspond to the properties on paranormal in [8][12][18] and [19].

1. Introduction

A capital letter means a bounded linear operator on a complex Hilbert space H. An operator T is said to be positive (denoted by $T \geqslant 0$) if $(Tx, x) \geqslant 0$ for all $x \in H$.

An operator T is said to be p-*hyponormal* for $p > 0$ if $(T^*T)^p \geqslant (TT^*)^p$ and an operator T is said to be *log-hyponormal* if T is invertible and $\log T^*T \geqslant \log TT^*$. p-Hyponormal and log-hyponormal operators are defined as extensions of hyponormal ones, i.e., $T^*T \geqslant TT^*$, and also they have been studied by many authors, for instance, [1][2][4][7][15][16][17][22][23] and [24]. It is easily obtained that every p-hyponormal operator is q-hyponormal for $p \geqslant q > 0$ by Löwner-Heinz theorem "*$A \geqslant B \geqslant 0$ ensures $A^\alpha \geqslant B^\alpha$ for any $\alpha \in [0, 1]$*", and every p-hyponormal operator is log-hyponormal since $\log t$ is an operator monotone function.

Very recently, Aluthge and Wang [2] showed the following Theorem A.1 on powers of p-hyponormal operators.

THEOREM A.1. ([2]) *Let T be a p-hyponormal operator for $p \in (0, 1]$. The inequalities*

$$(T^{n^*}T^n)^{\frac{p}{n}} \geqslant (T^*T)^p \geqslant (TT^*)^p \geqslant (T^nT^{n^*})^{\frac{p}{n}}$$

hold for all positive integer n.

Theorem A.1 asserts that if T is a p-hyponormal operator for $p \in (0, 1]$, then T^n is $\frac{p}{n}$-hyponormal for any positive integer n.

As generalizations of Theorem A.1, Furuta-Yanagida [16] and Yamazaki [23] obtained the following results.

THEOREM A.2. ([16, Theorem 1]) *Let T be a p-hyponormal operator for $p \in (0, 1]$. Then*

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(1) $(T^*T)^{p+1} \leqslant (T^{2^*}T^2)^{\frac{p+1}{2}} \leqslant \cdots \leqslant (T^{n^*}T^n)^{\frac{p+1}{n}}$,
(2) $(TT^*)^{p+1} \geqslant (T^2T^{2^*})^{\frac{p+1}{2}} \geqslant \cdots \geqslant (T^nT^{n^*})^{\frac{p+1}{n}}$,
hold for all positive integer n.

THEOREM A.3. ([23, Theorem 2]) *Let* T *be a log-hyponormal operator. Then*
(1) $T^*T \leqslant (T^{2^*}T^2)^{\frac{1}{2}} \leqslant \cdots \leqslant (T^{n^*}T^n)^{\frac{1}{n}}$,
(2) $TT^* \geqslant (T^2T^{2^*})^{\frac{1}{2}} \geqslant \cdots \geqslant (T^nT^{n^*})^{\frac{1}{n}}$,
hold for all positive integer n.

We remark that Theorem A.3 is a parallel result to Theorem A.2. In fact, Theorem A.3 corresponds to Theorem A.2 in the case $p \to +0$ since p-hyponormality of T (i.e., $(T^*T)^p \geqslant (TT^*)^p$) approaches log-hyponormality of T (i.e., $\log T^*T \geqslant \log TT^*$) as $p \to +0$.

An operator T is *paranormal* if $\|T^2x\| \geqslant \|Tx\|^2$ for every unit vector $x \in H$. It has been studied by many authors, so there are too many to cite their references, for instance, [3][8][12] and [19].

An operator T belongs to *class* A if $|T^2| \geqslant |T|^2$. We call an operator T "class A operator" briefly if T belongs to class A. In [13], we showed that *every log-hyponormal operator is a class* A *operator* and *every class* A *operator is a paranormal operator*. It turns out that these results contain another proof of Ando's result [3] which *every log-hyponormal operator is a paranormal operator.*

Moreover, in [13], we introduced new classes of operators as follows: An operator T belongs to *class* $A(k)$ for $k > 0$ if $(T^*|T|^{2k}T)^{\frac{1}{k+1}} \geqslant |T|^2$, and also an operator T is *absolute-k-paranormal* for $k > 0$ if $\||T|^kTx\| \geqslant \|Tx\|^{k+1}$ for every unit vector $x \in H$. Particularly an operator T is a class A (resp. paranormal) operator if and only if T is a class $A(1)$ (resp. absolute-1-paranormal) operator. On class $A(k)$ operators and absolute-k-paranormal operators, we proved the following result.

THEOREM B. ([13, Theorem 2])
(1) *Every log-hyponormal operator is a class* $A(k)$ *operator for* $k > 0$.
(2) *For each* $k > 0$, *every invertible class* $A(k)$ *operator is a class* $A(l)$ *operator for* $l \geqslant k$.
(3) *For each* $k > 0$, *every absolute-k-paranormal operator is an absolute-l-paranormal operator for* $l \geqslant k$.
(4) *For each* $k > 0$, *every class* $A(k)$ *operator is an absolute-k-paranormal operator.*

Theorem B states that invertible class $A(k)$ operators determined by operator inequalities and absolute-k-paranormal operators determined by norm inequalities have monotonicity on $k > 0$, namely they constitute clearly parallel and increasing lines.

In this paper, we shall show that Theorem A.3 remains valid for class A operators which are extensions of log-hyponormal operators. Moreover we shall show several properties on class A which correspond to the properties on paranormal in [8][12][18] and [19].

2. Results

LEMMA 1. *Let T be an invertible operator such that*

$$(T^*|T^{n-1}|^{2k}T)^{\frac{1}{(n-1)k+1}} \geqslant |T|^2$$

for some $k > 0$ and $n = 2, 3, \cdots$. Then for any fixed $\delta \geqslant -1$,

$$f_{n,\delta}(l) = (T^*|T^{n-1}|^{2l}T)^{\frac{\delta+1}{(n-1)l+1}} \tag{2.1}$$

is increasing for $l \geqslant \max\{k, \frac{\delta}{n-1}\}$.

By using Lemma 1, we obtain the following results.

THEOREM 2. *Let T be an invertible and class A operator. Then the following assertions hold;*

(1) $|T^n|^{\frac{2}{n}} \geqslant (T^*|T^{n-1}|^{\frac{2}{n-1}}T)^{\frac{1}{2}} \geqslant |T|^2$ *for $n = 2, 3, \cdots$.*
(2) $|T^{n+1}|^{\frac{2n}{n+1}} \geqslant |T^n|^2$ *for all positive integer n.*
(3) $|T^{2n}| \geqslant |T^n|^2$ *for all positive integer n.*
(4) $|T|^2 \leqslant |T^2| \leqslant \cdots \leqslant |T^n|^{\frac{2}{n}}$ *for all positive integer n.*
(5) $|T^{-2}| \geqslant |T^{-1}|^2$.

THEOREM 3. *Let T be an invertible and class A operator. Then the following assertions hold;*

(1) $|T^*|^2 \geqslant (T|T^{n-1^*}|^{\frac{2}{n-1}}T^*)^{\frac{1}{2}} \geqslant |T^{n^*}|^{\frac{2}{n}}$ *for $n = 2, 3, \cdots$.*
(2) $|T^{n^*}|^2 \geqslant |T^{n+1^*}|^{\frac{2n}{n+1}}$ *for all positive integer n.*
(3) $|T^{n^*}|^2 \geqslant |T^{2n^*}|$ *for all positive integer n.*
(4) $|T^*|^2 \geqslant |T^{2^*}| \geqslant \cdots \geqslant |T^{n^*}|^{\frac{2}{n}}$ *for all positive integer n.*

(4) of Theorem 2 (resp. (4) of Theorem 3) coincides with (1) of Theorem A.3 (resp. (2) of Theorem A.3), that is, Theorem A.3 holds even if T is an invertible and class A operator.

Theorem 2 can be rewritten in the following form.

COROLLARY 4.

(1) *If T is an invertible and class A operator, then $|T^n|^{\frac{2}{n}} \geqslant |T|^2$ holds for all positive integer n.*
(2) *If T is an invertible and class A operator, then T^n is also a class A operator for all positive integer n.*
(3) *If T is an invertible and class A operator, then T^{-1} is also a class A operator.*
(4) *If T is an invertible and class A operator, then*

$$|T|^2 \leqslant |T^2| \leqslant \cdots \leqslant |T^n|^{\frac{2}{n}}$$

hold for all positive integer n.

(1), (2), (3) and (4) of Corollary 4 follows from (1), (3), (5) and (4) of Theorem 2, respectively.

3. Proofs of the results

We need the following Theorem C' in order to give a proof of Lemma 1.

THEOREM C'. *Let A and B be positive invertible operators such that $(B^{\frac{1}{2}}AB^{\frac{1}{2}})^{\frac{\beta_0}{\alpha_0+\beta_0}} \geqslant B$ holds for fixed $\alpha_0 \geqslant 0$ and $\beta_0 \geqslant 0$ with $\alpha_0+\beta_0 > 0$. Then for any fixed $\delta \geqslant -\beta_0$,*

$$g(\lambda,\mu) = B^{\frac{-\mu}{2}}(B^{\frac{\mu}{2}}A^{\lambda}B^{\frac{\mu}{2}})^{\frac{\delta+\beta_0\mu}{\alpha_0\lambda+\beta_0\mu}}B^{\frac{-\mu}{2}}$$

is an increasing function of both λ and μ for $\lambda \geqslant 1$ and $\mu \geqslant 1$ such that $\alpha_0\lambda \geqslant \delta$.

By replacing A with B^{-1} and B with A^{-1} in the following Theorem C which has been proved by Furuta inequality [9] (see also [5][10] and [20]), we easily obtain Theorem C'.

THEOREM C. ([14, Theorem 1]) *Let A and B be positive invertible operators such that*
$A \geqslant (A^{\frac{1}{2}}BA^{\frac{1}{2}})^{\frac{\beta_0}{\alpha_0+\beta_0}}$ holds for fixed $\alpha_0 \geqslant 0$ and $\beta_0 \geqslant 0$ with $\alpha_0+\beta_0 > 0$. Then for any fixed $\delta \geqslant -\beta_0$,

$$f(\lambda,\mu) = A^{\frac{-\mu}{2}}(A^{\frac{\mu}{2}}B^{\lambda}A^{\frac{\mu}{2}})^{\frac{\delta+\beta_0\mu}{\alpha_0\lambda+\beta_0\mu}}A^{\frac{-\mu}{2}}$$

is a decreasing function of both λ and μ for $\lambda \geqslant 1$ and $\mu \geqslant 1$ such that $\alpha_0\lambda \geqslant \delta$.

Proof of Lemma 1. Let $T = U|T|$ be the polar decomposition of T. We remark that U is unitary since T is invertible. Suppose that

$$(T^*|T^{n-1}|^{2k}T)^{\frac{1}{(n-1)k+1}} \geqslant |T|^2. \tag{3.1}$$

Since

$$(T^*|T^{n-1}|^{2k}T)^{\frac{1}{(n-1)k+1}} = (U^*|T^*||T^{n-1}|^{2k}|T^*|U)^{\frac{1}{(n-1)k+1}} = U^*(|T^*||T^{n-1}|^{2k}|T^*|)^{\frac{1}{(n-1)k+1}}U,$$

(3.1) holds if and only if

$$U^*(|T^*||T^{n-1}|^{2k}|T^*|)^{\frac{1}{(n-1)k+1}}U \geqslant |T|^2$$

if and only if

$$(|T^*||T^{n-1}|^{2k}|T^*|)^{\frac{1}{(n-1)k+1}} \geqslant U|T|^2U^* = |T^*|^2. \tag{3.2}$$

Let $A = |T^{n-1}|^{2k}$ and $B = |T^*|^2$. Then (3.2) is equivalent to the following (3.3):

$$(B^{\frac{1}{2}}AB^{\frac{1}{2}})^{\frac{1}{(n-1)k+1}} \geqslant B. \tag{3.3}$$

By applying Theorem C' to (3.3), for any fixed $\delta \geqslant -1$,

$$\begin{aligned} g(\lambda) &= B^{\frac{-1}{2}}(B^{\frac{1}{2}}A^{\lambda}B^{\frac{1}{2}})^{\frac{\delta+1}{(n-1)k\lambda+1}}B^{\frac{-1}{2}} \\ &= |T^*|^{-1}(|T^*||T^{n-1}|^{2k\lambda}|T^*|)^{\frac{\delta+1}{(n-1)k\lambda+1}}|T^*|^{-1} \end{aligned}$$

is increasing for $\lambda \geqslant 1$ such that $(n-1)k\lambda \geqslant \delta$, and we have that

$$\begin{aligned} g(\tfrac{l}{k}) &= |T^*|^{-1}(|T^*||T^{n-1}|^{2l}|T^*|)^{\frac{\delta+1}{(n-1)l+1}}|T^*|^{-1} \\ &= |T^*|^{-1}(UU^*|T^*||T^{n-1}|^{2l}|T^*|UU^*)^{\frac{\delta+1}{(n-1)l+1}}|T^*|^{-1} \\ &= |T^*|^{-1}U(T^*|T^{n-1}|^{2l}T)^{\frac{\delta+1}{(n-1)l+1}}U^*|T^*|^{-1} \qquad \text{since } U \text{ is unitary} \\ &= {T^*}^{-1}f_{n,\delta}(l)T^{-1} \end{aligned}$$

is increasing for $l \geqslant k$ such that $(n-1)l \geqslant \delta$. Hence $f_{n,\delta}(l)$ is increasing for $l \geqslant \max\{k, \frac{\delta}{n-1}\}$, that is, the proof of Lemma 1 is complete. □

Proof of Theorem 2. Define $f_{n,\delta}(l)$ as (2.1) in Lemma 1.
Proof of (1). We will use induction to establish the inequalities

$$|T^n|^{\frac{2}{n}} \geqslant (T^*|T^{n-1}|^{\frac{2}{n-1}}T)^{\frac{1}{2}} \geqslant |T|^2 \quad \text{for } n = 2,3,\cdots. \tag{3.4}$$

In case $n=2$, $|T^2| = (T^*|T|^2T)^{\frac{1}{2}} \geqslant |T|^2$ hold since T is a class A operator.
Assume that (3.4) holds for some $n \geqslant 2$. Then

$$\begin{aligned} |T|^2 &\leqslant |T^2| \qquad \text{since } T \text{ is a class } A \text{ operator} \\ &= (T^*|T|^2T)^{\frac{1}{2}} \\ &\leqslant (T^*|T^n|^{\frac{2}{n}}T)^{\frac{1}{2}} \quad \text{by (3.4) and Löwner-Heinz theorem.} \end{aligned} \tag{3.5}$$

Then (3.5) and Lemma 1 ensure that

$$f_{n+1,0}(l) = (T^*|T^n|^{2l}T)^{\frac{1}{nl+1}} \text{ is increasing for } l \geqslant \max\{\tfrac{1}{n},0\} = \tfrac{1}{n}, \tag{3.6}$$

and we have

$$\begin{aligned} (T^*|T^n|^{\frac{2}{n}}T)^{\frac{1}{2}} &= f_{n+1,0}(\tfrac{1}{n}) \\ &\leqslant f_{n+1,0}(1) \quad \text{by (3.6)} \\ &= (T^*|T^n|^2T)^{\frac{1}{n+1}} \\ &= |T^{n+1}|^{\frac{2}{n+1}}. \end{aligned} \tag{3.7}$$

Hence (3.5) and (3.7) ensure $|T^{n+1}|^{\frac{2}{n+1}} \geqslant (T^*|T^n|^{\frac{2}{n}}T)^{\frac{1}{2}} \geqslant |T|^2$, so that (3.4) hold for $n = 2,3,\cdots$ by induction, that is, the proof of (1) is complete.

Proof of (2). We will use induction to establish the inequality

$$|T^{n+1}|^{\frac{2n}{n+1}} \geqslant |T^n|^2 \quad \text{for all positive integer } n. \tag{3.8}$$

In case $n=1$, $|T^2| \geqslant |T|^2$ holds since T is a class A operator.
Assume that (3.8) holds for some n. We remark the following:
Since $(T^*|T^{n+1}|^{\frac{2}{n+1}}T)^{\frac{1}{2}} \geqslant |T|^2$ holds by (1), Lemma 1 ensures that

$$f_{n+2,n}(l) = (T^*|T^{n+1}|^{2l}T)^{\frac{n+1}{(n+1)l+1}} \text{ is increasing for } l \geqslant \max\{\tfrac{1}{n+1},\tfrac{n}{n+1}\} = \tfrac{n}{n+1}. \tag{3.9}$$

Then we have

$$\begin{aligned}
|T^{n+1}|^2 &= T^*|T^n|^2 T \\
&\leqslant T^*|T^{n+1}|^{\frac{2n}{n+1}} T \qquad \text{by (3.8)} \\
&= f_{n+2,n}(\tfrac{n}{n+1}) \\
&\leqslant f_{n+2,n}(1) \qquad \text{by (3.9)} \\
&= (T^*|T^{n+1}|^2 T)^{\frac{n+1}{n+2}} \\
&= |T^{n+2}|^{\frac{2(n+1)}{n+2}}.
\end{aligned}$$

Hence (3.8) holds for all positive integer n by induction, that is, the proof of (2) is complete.

Proof of (3). By (2) and Löwner-Heinz theorem, we obtain

$$\begin{aligned}
|T^n|^2 \leqslant |T^{n+1}|^{\frac{2n}{n+1}} &= |T^{n+1}|^{2\cdot\frac{n}{n+1}} \\
\leqslant |T^{n+2}|^{\frac{2(n+1)}{n+2}\cdot\frac{n}{n+1}} &= |T^{n+2}|^{2\cdot\frac{n}{n+2}} \\
\leqslant \cdots & \\
\leqslant |T^{2n}|^{\frac{2(2n-1)}{2n}\cdot\frac{n}{2n-1}} &= |T^{2n}|^{2\cdot\frac{n}{2n}} = |T^{2n}|,
\end{aligned}$$

so that we have (3).

Proof of (4). Applying Löwner-Heinz theorem to (2), $|T^{n+1}|^{\frac{2}{n+1}} \geqslant |T^n|^{\frac{2}{n}}$ holds for all positive integer n. Therefore we obtain

$$|T|^2 \leqslant |T^2| \leqslant \cdots \leqslant |T^n|^{\frac{2}{n}} \quad \text{for all positive integer } n.$$

Proof of (5). We cite the following obvious result (see also [11][13]): *Let S be an invertible operator. Then*

$$(S^*S)^{\lambda} = S^*(SS^*)^{\lambda-1}S \quad \textit{holds for any real number } \lambda. \tag{3.10}$$

Suppose that T is an invertible class A operator. Then

$$T^*T = |T|^2 \leqslant |T^2| = (T^{2^*}T^2)^{\frac{1}{2}} = T^{2^*}(T^2T^{2^*})^{\frac{-1}{2}}T^2 \tag{3.11}$$

holds by (3.10). (3.11) holds if and only if

$$T^{-1^*}T^{-1} \leqslant (T^2T^{2^*})^{\frac{-1}{2}} = (T^{-2^*}T^{-2})^{\frac{1}{2}}$$

if and only if

$$|T^{-2}| \geqslant |T^{-1}|^2,$$

so that the proof of (5) is complete.

Whence the proof of Theorem 2 is complete. □

Proof of Theorem 3. First of all, we remark that

$$|S^{-1}| = (S^{-1^*}S^{-1})^{\frac{1}{2}} = (SS^*)^{\frac{-1}{2}} = |S^*|^{-1} \quad \text{for any invertible operator } S. \tag{3.12}$$

Suppose that T is an invertible and class A operator. Then T^{-1} is also a class A operator by (5) of Theorem 2.

Proof of (1). Since T^{-1} is a class A operator, applying (1) of Theorem 2, we have

$$|T^{-n}|^{\frac{2}{n}} \geqslant (T^{-1^*}|T^{-(n-1)}|^{\frac{2}{n-1}}T^{-1})^{\frac{1}{2}} \geqslant |T^{-1}|^2. \tag{3.13}$$

By (3.12), (3.13) hold if and only if

$$|T^{n^*}|^{\frac{-2}{n}} \geqslant (T^{-1^*}|T^{n-1^*}|^{\frac{-2}{n-1}}T^{-1})^{\frac{1}{2}} \geqslant |T^*|^{-2}$$

if and only if

$$|T^*|^2 \geqslant (T|T^{n-1^*}|^{\frac{2}{n-1}}T^*)^{\frac{1}{2}} \geqslant |T^{n^*}|^{\frac{2}{n}}.$$

Proof of (2). Since T^{-1} is a class A operator, applying (2) of Theorem 2, we have

$$|T^{-(n+1)}|^{\frac{2n}{n+1}} \geqslant |T^{-n}|^2. \tag{3.14}$$

By (3.12), (3.14) holds if and only if

$$|T^{n^*}|^2 = |T^{-n}|^{-2} \geqslant |T^{-(n+1)}|^{\frac{-2n}{n+1}} = |T^{n+1^*}|^{\frac{2n}{n+1}}.$$

Proof of (3). Since T^{-1} is a class A operator, applying (3) of Theorem 2, we have

$$|T^{-2n}| \geqslant |T^{-n}|^2. \tag{3.15}$$

By (3.12), (3.15) holds if and only if

$$|T^{n^*}|^2 = |T^{-n}|^{-2} \geqslant |T^{-2n}|^{-1} = |T^{2n^*}|.$$

Proof of (4). Since T^{-1} is a class A operator, applying (4) of Theorem 2, we have

$$|T^{-1}|^2 \leqslant |T^{-2}| \leqslant \cdots \leqslant |T^{-n}|^{\frac{2}{n}}. \tag{3.16}$$

By (3.12), (3.16) hold if and only if

$$|T^*|^{-2} \leqslant |T^{2^*}|^{-1} \leqslant \cdots \leqslant |T^{n^*}|^{\frac{-2}{n}}$$

if and only if

$$|T^*|^2 \geqslant |T^{2^*}| \geqslant \cdots \geqslant |T^{n^*}|^{\frac{2}{n}}.$$

Hence the proof of Theorem 3 is complete. □

4. Properties on paranormal which correspond to Corollary 4

On paranormal operators, the following Theorem D is obtained in [8][12][18] and [19] except (4). We recognize that Corollary 4 on class A corresponds to Theorem D on paranormal.

THEOREM D. ([8][12][18][19])

(1) *If T is a paranormal operator, then $\|T^n x\|^{\frac{1}{n}} \geqslant \|Tx\|$ holds for every unit vector $x \in H$ and all positive integer n.*

(2) *If T is a paranormal operator, then T^n is also a paranormal operator for all positive integer n.*

(3) *If T is an invertible and paranormal operator, then T^{-1} is also a paranormal operator.*

(4) *If T is a paranormal operator, then*

$$\|Tx\| \leqslant \|T^2 x\|^{\frac{1}{2}} \leqslant \cdots \leqslant \|T^n x\|^{\frac{1}{n}} \tag{4.1}$$

hold for every unit vector $x \in H$ and all positive integer n.

(4) of Theorem D follows from the following fundamental inequalities on paranormal operators in [8]: *If T is a paranormal operator, then*

$$\|T\| \geqslant \cdots \geqslant \frac{\|T^{n+2}x\|}{\|T^{n+1}x\|} \geqslant \frac{\|T^{n+1}x\|}{\|T^n x\|} \geqslant \frac{\|T^n x\|}{\|T^{n-1}x\|} \geqslant \cdots \geqslant \frac{\|T^2 x\|}{\|Tx\|} \geqslant \frac{\|Tx\|}{\|x\|} \tag{4.2}$$

hold for all $x \in H$ and all positive integer n. For the sake of convenience, we give a proof of (4) of Theorem D.

Proof of (4) of Theorem D. We will use induction to establish the inequality

$$\|T^n x\|^{\frac{1}{n}} \leqslant \|T^{n+1}x\|^{\frac{1}{n+1}} \tag{4.3}$$

for every unit vector $x \in H$ and all positive integer n. In case $n = 1$, $\|Tx\| \leqslant \|T^2 x\|^{\frac{1}{2}}$ holds since T is a paranormal operator.

Assume that (4.3) holds for some n. Then for every unit vector $x \in H$, we have the following (4.4).

$$\begin{aligned} \|T^{n+1}x\|^2 &\leqslant \|T^{n+2}x\|\|T^n x\| && \text{by (4.2)} \\ &\leqslant \|T^{n+2}x\|\|T^{n+1}x\|^{\frac{n}{n+1}} && \text{by (4.3).} \end{aligned} \tag{4.4}$$

Therefore

$$\|T^{n+1}x\|^{\frac{1}{n+1}} \leqslant \|T^{n+2}x\|^{\frac{1}{n+2}}$$

holds for every unit vector $x \in H$, so that (4.3) holds for all positive integer n by induction.

Consequently we have

$$\|Tx\| \leqslant \|T^2 x\|^{\frac{1}{2}} \leqslant \cdots \leqslant \|T^n x\|^{\frac{1}{n}} \tag{4.1}$$

hold for every unit vector $x \in H$ and all positive integer n, that is, the proof of (4) of Theorem D is complete. □

An operator T is *n-paranormal* for positive integer n such that $n \geqslant 2$ if $\|T^n x\| \geqslant \|Tx\|^n$ for every unit vector $x \in H$. It has been studied in [6][12] and [18]. (1) of Theorem D states the following well-known result that every paranormal operator is an n-paranormal operator for $n = 2, 3, \cdots$, and also we have the following Proposition 5.

PROPOSITION 5. *If T satisfies $|T^n|^{\frac{2}{n}} \geqslant |T|^2$ for some positive integer n such that $n \geqslant 2$, then T is an n-paranormal operator.*

In case $n = 2$, Proposition 5 means that every class A operator is a paranormal operator [13]. We need the following theorem in order to give a proof of Proposition 5.

THEOREM E. (Hölder-McCarthy inequality [21]) *Let A be a positive operator. Then the following inequalities hold for all $x \in H$:*

(i) $(A^r x, x) \leqslant (Ax, x)^r \|x\|^{2(1-r)}$ *for* $0 < r \leqslant 1$.

(ii) $(A^r x, x) \geqslant (Ax, x)^r \|x\|^{2(1-r)}$ *for* $r \geqslant 1$.

Proof of Proposition 5. Suppose that T satisfies

$$|T^n|^{\frac{2}{n}} \geqslant |T|^2 \tag{4.5}$$

for some positive integer n such that $n \geqslant 2$. Then for every unit vector $x \in H$,

$$\begin{aligned} \|T^n x\|^2 &= (|T^n|^2 x, x) \\ &\geqslant (|T^n|^{\frac{2}{n}} x, x)^n \quad \text{by (ii) of Theorem E} \\ &\geqslant (|T|^2 x, x)^n \quad \text{by (4.5)} \\ &= \|Tx\|^{2n}. \end{aligned}$$

Hence we have

$$\|T^n x\| \geqslant \|Tx\|^n \quad \text{for every unit vector } x \in H,$$

so that T is n-paranormal for positive integer n such that $n \geqslant 2$. □

Remark. An operator T is *n-perinormal* for positive integer n such that $n \geqslant 2$ if $T^{*n}T^n \geqslant (T^*T)^n$. n-Perinormal operators are introduced by Fujii, Izumino and Nakamoto [6]. Particularly the class of 2-perinormal operators coincides with the class of quasihyponormal one, i.e., $T^*(T^*T)T \geqslant T^*(TT^*)T$. We easily obtain the following result by Löwner-Heinz theorem: *For each positive integer n such that $n \geqslant 2$, every n-perinormal operator satisfies $|T^n|^{\frac{2}{n}} \geqslant |T|^2$.*

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Masatoshi Ito
Department of Applied Mathematics
Faculty of Science
Science University of Tokyo
1-3 Kagurazaka, Shinjuku
Tokyo 162-8601
Japan
e-mail: m-ito@am.kagu.sut.ac.jp