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UNIFYING APPROACH TO THE STUDY OF p–HYPONORMAL OPERATORS VIA FURUTA INEQUALITY

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Abstract. In this paper we propose a unifying approach to the study of the operators $|T|^g U|T|^h$ and $|T^*|^g U|T^*|^h$ being p-hyponormal. This approach is based on the direct use of combined Furuta inequalities, and, consequently, characterizations of a normal operator are presented.

1. Notation and terminology

Throughout this note it is to be understood that the capital letters mean bounded linear operators acting on a Hilbert space H. T is positive (written $T \geqslant O$) in case $(Tx, x) \geqslant 0$ for all $x \in$ H. If S and T are Hermitian, we write $T \geqslant S$ in case $T - S \geqslant O$. $T = U|T|$ is the polar decomposition of T with U the partial isometry, and $|T|$ the positive square root of the positive operator T^*T such that $N(U) = N(|T|)$, where $N(T)$ denotes the kernel of T.

In [10] Xia first introduced the semi-hyponormal operator, i.e., T is such an operator if $(T^*T)^{1/2} \geqslant (TT^*)^{1/2}$. Its natural generalization is the p-hyponormal operator [1, 11], viz. $(T^*T)^p \geqslant (TT^*)^p$ holds for $0 < p \leqslant 1$, and T is hyponormal when $p = 1$ in particular. If $0 < p' \leqslant p$ and T is p-hyponormal, then T is p'-hyponormal by the Löwner-Heinz formula ($A^\alpha \geqslant B^\alpha$ if $A \geqslant B \geqslant O$ and $\alpha \in [0, 1]$).

2. Characterization of a p-hyponormal operator by Furuta inequalities

We recall first the following celebrated Furuta inequality [3, 5], which is a remarkable generalization of the Löwner-Heinz formula.

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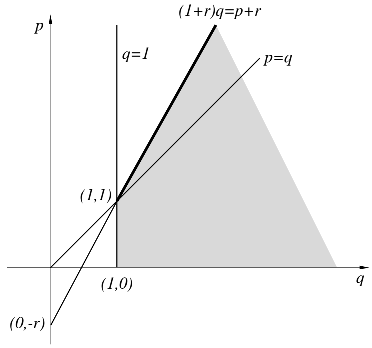
THEOREM F (Furuta inequality). *If* $A \geqslant B \geqslant O$, *then for each* $r \geqslant 0$,

(*i*) $(B^{\frac{r}{2}}A^pB^{\frac{r}{2}})^{\frac{1}{q}} \geqslant (B^{\frac{r}{2}}B^pB^{\frac{r}{2}})^{\frac{1}{q}}$

and

(*ii*) $(A^{\frac{r}{2}}A^pA^{\frac{r}{2}})^{\frac{1}{q}} \geqslant (A^{\frac{r}{2}}B^pA^{\frac{r}{2}})^{\frac{1}{q}}$

hold for $p \geqslant 0$ *and* $q \geqslant 1$ *with* $(1+r)q \geqslant p+r$.



Figure

The domain drawn for $p,\ q$ and r in the Figure is best possible one for the Furuta inequality by K. Tanahashi [9]. For $r \geqslant 0,\ k \geqslant 0$ and $q \geqslant 1$ with $(1+2r)q \geqslant k+2r$ and suppose that $A \geqslant B \geqslant C \geqslant O$, we may put the two inequalities (i) and (ii) into one expression, i.e., the combined Furuta inequalities, viz.

$(*)\ (B^rA^kB^r)^{\frac{1}{q}} \geqslant B^{\frac{k+2r}{q}} \geqslant (B^rC^kB^r)^{\frac{1}{q}}$.

We notice first that the following well-known basic relations (cf. [2, 4, 8]) are crucial in the proofs of results. Let $T = U \mid T \mid$ be the polar decomposition. For $m,n > 0$, $(U \mid T \mid^m U^*)^n = U \mid T \mid^{mn} U^*$ holds in general, but $(U^* \mid T \mid^m U)^n = U^* \mid T \mid^{mn} U$ holds if $N(T) = N(T^*)$. U^*U and UU^* are the initial projection and the final projection, respectively. $\mid T^* \mid^m = U \mid T \mid^m U^*$ holds in general, and $UU^* = U^*U$ if $N(T) = N(T^*)$. We shall cite the following obvious result.

PROPOSITION. *Let* $A \geqslant O$ *with* $N(A) = N(T)$, *and* $a,b > 0$. *Then* $\mid T \mid^a A \mid T \mid^a = \mid T \mid^{b+2a}$ *if and only if* $A = \mid T \mid^b$. *(Recall that without* $N(A) = N(T)$, *there exists an easy counterexample to Proposition.)*

It is clear now that $T = U \mid T \mid$ is p-hyponormal if and only if $U \mid T \mid^p$ is hyponormal, a well-known result. The next result is a characterization of a p-hyponormal operator by using (i) and (ii) in above, which is required in the main result.

LEMMA. *Let* $T = U \mid T \mid$ *be the polar decomposition and* $0 < p \leqslant 1$. *Then the following are equivalent.*

(1) T *is a* p*-hyponormal operator;*

(2) $U^*\{\mid T^* \mid^{2pr} \mid T \mid^{2pk} \mid T^* \mid^{2pr}\}^{\frac{1}{q}} U \geqslant \mid T \mid^{\frac{2p(k+2r)}{q}} \geqslant \{\mid T \mid^{2pr} \mid T^* \mid^{2pk} \mid T \mid^{2pr}\}^{\frac{1}{q}}$,

or

$\{\mid T^* \mid^{2pr} \mid T \mid^{2pk} \mid T^* \mid^{2pr}\}^{\frac{1}{q}} \geqslant \mid T^* \mid^{\frac{2p(k+2r)}{q}} \geqslant U\{\mid T \mid^{2pr} \mid T^* \mid^{2pk} \mid T \mid^{2pr}\}^{\frac{1}{q}} U^*$

for $k,r \geqslant 0$ *and* $q \geqslant 1$ *such that* $(1+2r)q \geqslant k+2r$;

(3) $U^* \mid T \mid^{2p} U \geqslant \mid T \mid^{2p} \geqslant U \mid T \mid^{2p} U^*$,

or

$U^* \mid T^* \mid^{2p} U \geqslant \mid T^* \mid^{2p} \geqslant U \mid T^* \mid^{2p} U^*.$

Proof. (1) implies (2). Notice first that T is p-hyponormal if and only if $\mid T^* \mid^{2p} \leqslant \mid T \mid^{2p}$. Let $A = \mid T \mid^{2p}$ and $B = \mid T^* \mid^{2p}$ in (i) and (ii) above, respectively. Then we have

$$\{\mid T^* \mid^{2pr} \mid T \mid^{2pk} \mid T^* \mid^{2pr}\}^{\frac{1}{q}} \geqslant \mid T^* \mid^{\frac{2p(k+2r)}{q}} = U \mid T \mid^{\frac{2p(k+2r)}{q}} U^*$$

and

$$U^* \mid T^* \mid^{\frac{2p(k+2r)}{q}} U = \mid T \mid^{\frac{2p(k+2r)}{q}} \geqslant \{\mid T \mid^{2pr} \mid T^* \mid^{2pk} \mid T \mid^{2pr}\}^{\frac{1}{q}}.$$

(2) implies (3). Let $k = q = 1$ and $r = 0$ in (2).

Clearly, (3) implies that T is p-hyponormal since $U \mid T \mid^{2p} U^* = \mid T^* \mid^{2p}$ and $U^* \mid T^* \mid^{2p} U = \mid T \mid^{2p}$.

3. Unifying approach to the study of p-hyponormal operators via combined Furuta inequalities

Recently a p-hyponormal operator of the type $\mid T \mid^g U \mid T \mid^h$ for certain $g, h > 0$ has been intensively studied by several authors and the results obtained have been found useful in the operator theory [1, 6, 7, 8, and the references cited therein]. By using the combined Furuta inequalities (*) our next main result shows such type of p-hyponormal operator in a more general form, and operator of the type $\mid T^* \mid^g U \mid T^* \mid^h$ is equally discussed.

THEOREM. *Let $T = U \mid T \mid$ be a p-hyponormal operator, $0 < p \leqslant 1$, $N(T) = N(T^*)$, and let $\widetilde{T} = \mid T \mid^{kp} U \mid T \mid^{2pr}$, $\widehat{T} = \mid T \mid^{2pr} U \mid T \mid^{kp}$, $\widetilde{S} = \mid T^* \mid^{kp} U \mid T^* \mid^{2pr}$, and $\widehat{S} = \mid T^* \mid^{2pr} U \mid T^* \mid^{kp}$ for $k, r \geqslant 0$ and $q \geqslant 1$. Then we have*

(1) $(\widetilde{T}^*\widetilde{T})^{\frac{1}{q}} \geqslant \mid T \mid^{\frac{2p(k+2r)}{q}} \geqslant (\widehat{T}\widehat{T}^*)^{\frac{1}{q}}$ *with* $(1+2r)q \geqslant k+2r$;

(2) $(\widehat{T}^*\widehat{T})^{\frac{1}{q}} \geqslant \mid T \mid^{\frac{2p(k+2r)}{q}} \geqslant (\widetilde{T}\widetilde{T}^*)^{\frac{1}{q}}$ *with* $(1+k)q \geqslant k+2r$;

(3) $(\widetilde{S}^*\widetilde{S})^{\frac{1}{q}} \geqslant \mid T^* \mid^{\frac{2p(k+2r)}{q}} \geqslant (\widehat{S}\widehat{S}^*)^{\frac{1}{q}}$ *with* $(1+2r)q \geqslant k+2r$;

(4) $(\widehat{S}^*\widehat{S})^{\frac{1}{q}} \geqslant \mid T^* \mid^{\frac{2p(k+2r)}{q}} \geqslant (\widetilde{S}\widetilde{S}^*)^{\frac{1}{q}}$ *with* $(1+k)q \geqslant k+2r$.

Moreover, all four statements are equivalent. In particular, the operators $\widetilde{T}$, $\widehat{T}$, $\widetilde{S}$, and $\widehat{S}$ are all α-hyponormal with $\alpha \leqslant min\ \{\frac{1+2r}{k+2r}, \frac{1+k}{k+2r}, 1\}$.

Proof. To prove (1), due to Lemma we may let $A = U^* \mid T \mid^{2p} U$, $B = \mid T \mid^{2p}$, and $C = U \mid T \mid^{2p} U^*$ in the inequalities (*). Then,

$$\{\mid T \mid^{2pr} (U^* \mid T \mid^{2p} U)^k \mid T \mid^{2pr}\}^{\frac{1}{q}} \geqslant \mid T \mid^{\frac{2p(k+2r)}{q}} \geqslant \{\mid T \mid^{2pr} (U \mid T \mid^{2p} U^*)^k \mid T \mid^{2pr}\}^{\frac{1}{q}},$$

so that since $N(T) = N(T^*)$

$$\{\mid T \mid^{2pr} U^* \mid T \mid^{2kp} U \mid T \mid^{2pr}\}^{\frac{1}{q}} \geqslant \mid T \mid^{\frac{2p(k+2r)}{q}} \geqslant \{\mid T \mid^{2pr} U \mid T \mid^{2kp} U^* \mid T \mid^{2pr}\}^{\frac{1}{q}}$$

for $k, r \geqslant 0$ and $q \geqslant 1$ with $(1+2r)q \geqslant k+2r$, and (1) follows.

(1) implies (2). Since $k, r \geqslant 0$, we may replace k by $2r$, and vice versa, in the last inequalities of the above proof without affecting the inequalities themself. Thus,

$$\{| T |^{kp} U^* | T |^{4pr} U | T |^{kp}\}^{\frac{1}{q}} \geqslant | T |^{\frac{2p(k+2r)}{q}} \geqslant \{| T |^{kp} U | T |^{4pr} U^* | T |^{kp}\}^{\frac{1}{q}}$$

for $k, r \geqslant 0$ and $q \geqslant 1$ with $(1+k)q \geqslant k+2r$, and (2) follows.

(2) implies (1). Use the same method as the proof (1) implies (2) in above.

To prove (3), due to Lemma we may put $A = U^* \mid T^* \mid^{2p} U,\ B = \mid T^* \mid^{2p}$, and $C = U \mid T^* \mid^{2p} U^*$ in the inequalities (*), and the process goes exactly the same as above.

Similarly, we can show the two inequalities (3) and (4) are equivalent. In fact, all four inequalities are equivalent.

It is readily seen that the last statement of Theorem follows immediately by combining inequalities (1) and (2), and (3) and (4), respectively.

Let us point out three most recent results in this area. Firstly, it was mentioned in [6, Theorem 1′], and was proved in [7, Theorem 1] that both $\mid T \mid^s U \mid T \mid^t$ and $\mid T \mid^t U \mid T \mid^s$ are $(\frac{p+s}{s+t})$-hyponormal operators for $s \geqslant 0$ and $t \geqslant \max\ \{p, s\}$. A simple calculation shows that this is the case when $kp = s$ and $2pr = t$ in Theorem , then min $\{\frac{p+t}{s+t}, \frac{p+s}{s+t}, 1\} = \frac{p+s}{s+t}$ if $t \geqslant \max\ \{p, s\}$. Moreover, the two operators $\mid T^* \mid^s U \mid T^* \mid^t$ and $\mid T^* \mid^t U \mid T^* \mid^s$ have exactly the same properties as the above two. Secondly, in [7, Theorem 2] it was shown that $\mid T \mid^t U \mid T \mid^{s-t}$ is a j-hyponormal operator for $s > 0,\ s \geqslant t \geqslant 0$, and $j = \min\ \{\frac{p+t}{s}, \frac{p+s-t}{s}, 1\}$ without the assumption $N(T) = N(T^*)$. This is another case when $kp = t$ and $2pr = s - t$ in Theorem, then $\alpha \leqslant \min\ \{\frac{p+t}{s}, \frac{p+s-t}{s}, 1\}$. In fact, the operators $\mid T \mid^{s-t} U \mid T \mid^t$, $\mid T^* \mid^t U \mid T^* \mid^{s-t}$, and $\mid T^* \mid^{s-t} U \mid T^* \mid^t$ are all α-hyponormal. Thirdly, the operator $\mid T \mid^{1/2} U \mid T \mid^{1/2}$ was proved to be $(p+\frac{1}{2})$-hyponormal if $0 < p \leqslant \frac{1}{2}$ [1, Theorem 2]. This follows easily if we put $kp = 2pr = \frac{1}{2}$ in Theorem so that $\min\{p+\frac{1}{2}, 1\} = p+\frac{1}{2}$. Besides, the operator $\mid T^* \mid^{1/2} U \mid T^* \mid^{1/2}$ is $(p+\frac{1}{2})$-hyponormal, too. Thus we see that our Theorem is indeed in a more general setting.

4. Applications

As easy consequences of Theorem we obtain the following two results. The first one is a generalization of [1, Theorem 1] and [6, Theorem 2], and the second one is new characterizations of a normal operator.

COROLLARY 1. *Let $T = U \mid T \mid$ be a p-hyponormal operator, $0 < p \leqslant 1$, and $N(T) = N(T^*)$. Then the operators $\mid T \mid^{kp} U \mid T \mid^{2pr}$, $\mid T \mid^{2pr} U \mid T \mid^{kp}$, $\mid T^* \mid^{kp} U \mid T^* \mid^{2pr}$, and $\mid T^* \mid^{2pr} U \mid T^* \mid^{kp}$ are all hyponormal for $k, 2r \in (0, 1]$ (hence, α-hyponormal for $0 < \alpha \leqslant 1$).*

Proof. Let $\frac{1+2r}{k+2r}$ and $\frac{1+k}{k+2r} \geqslant 1$ in Theorem. Then $k, 2r \in (0, 1]$ and $\alpha \leqslant 1$.

It may be noted that two particular cases of Corollary 1 are as follows. Let $T = U \mid T \mid$ be p-hyponormal. Then: (1) $\mid T \mid^{1/2} U \mid T \mid^{1/2}$ is hyponormal for

$\frac{1}{2} \leqslant p < 1$ [1, Theorem 1], and (2) $|T|^q U |T|^q$ is hyponormal for $1 \geqslant p \geqslant q > 0$ [6, Theorem 2].

COROLLARY 2. *Let $T = U|T|$ be the polar decomposition, $0 < p \leqslant 1$, $N(T) = N(T^*)$, and let $\widetilde{T} = |T|^{kp} U |T|^{2pr}$, $\widehat{T} = |T|^{2pr} U |T|^{kp}$, $\widetilde{S} = |T^*|^{kp} U |T^*|^{2pr}$, and $\widehat{S} = |T^*|^{2pr} U |T^*|^{kp}$ for $k, r \geqslant 0$ with $k + r > 0$. Then the following are equivalent.*

(1) *T is a normal operator;*
(2) *$\widetilde{T}$ is a normal operator, and $\widetilde{T}^*\widetilde{T} = \widetilde{T}\widetilde{T}^* = |T|^{2p(k+2r)}$;*
(3) *$\widehat{T}$ is a normal operator, and $\widehat{T}^*\widehat{T} = \widehat{T}\widehat{T}^* = |T|^{2p(k+2r)}$;*
(4) *$\widetilde{S}$ is a normal operator, and $\widetilde{S}^*\widetilde{S} = \widetilde{S}\widetilde{S}^* = |T^*|^{2p(k+2r)}$;*
(5) *$\widehat{S}$ is a normal operator, and $\widehat{S}^*\widehat{S} = \widehat{S}\widehat{S}^* = |T^*|^{2p(k+2r)}$.*

Proof. (1) implies (2) and (3). Notice that T is normal if and only if $\| T^*x \| = \| Tx \|$ for all $x \in$ H, equivalently, $|T|^2 = |T^*|^2$, or $U^* |T|^2 U = |T|^2 = U |T|^2 U^*$. Hence, $U^* |T|^{2p} U = |T|^{2p} = U |T|^{2p} U^*$ since $N(T) = N(T^*)$. Let $q = 1$. Then from the proof of Theorem we have

$$\widetilde{T}^*\widetilde{T} = |T|^{2p(k+2r)} = \widehat{T}\widehat{T}^*;$$

and

$$\widehat{T}^*\widehat{T} = |T|^{2p(k+2r)} = \widetilde{T}\widetilde{T}^*.$$

(2) implies (1). Suppose that $\widetilde{T}$ is normal and $\widetilde{T}^*\widetilde{T} = \widetilde{T}\widetilde{T}^* = |T|^{2p(k+2r)}$, then

$$|T|^{2pr} U^* |T|^{2kp} U |T|^{2pr} = |T|^{2p(k+2r)} = |T|^{kp} U |T|^{4pr} U^* |T|^{kp}.$$

Now, consider the following two cases: (a) If $k \neq 0$, then the first equality in above becomes $U^* |T|^{2kp} U = |T|^{2kp}$ by Proposition, so that $|T|^{2kp} = UU^* |T|^{2kp} UU^* = U |T|^{2kp} U^* = |T^*|^{2kp}$ since $N(T) = N(T^*)$. Hence, $|T|^2 = |T^*|^2$. (b) If $r \neq 0$, then the second equality in above becomes $|T|^{4pr} = U |T|^{4pr} U^*$ by Proposition, i.e., $|T|^{4pr} = |T^*|^{4pr}$ so that $|T|^2 = |T^*|^2$.

That (3) implies (1) may be carried out similarly.

Notice that $|T^*|^a = U |T|^a U^*$ for $a > 0$, and so we may change the relation in (4) to that in (2), and (5) to (3).

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