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EXISTENCE THEOREMS OF GENERALIZED QUASI–VARIATIONAL INEQUALITIES WITH UPPER HEMI–CONTINUOUS AND DEMI OPERATORS ON NON–COMPACT SETS

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Abstract. Suppose that E is a topological vector space and X is a non-empty subset of E. Let $S : X \to 2^X$ and $T : X \to 2^{E^*}$ be two maps. Then the generalized quasi-variational inequality problem (GQVI) is to find a point $\hat{y} \in S(\hat{y})$ and a point $\hat{w} \in T(\hat{y})$ such that $Re\langle \hat{w}, \hat{y} - x\rangle \leqslant 0$ for all $x \in S(\hat{y})$. We shall use Chowdhury and Tan's generalized version [4] of Ky Fan's minimax inequality [7] as a tool to obtain some general theorems on solutions of the GQVI in locally convex Hausdorff topological vector spaces. We obtain the existence theorems of GQVI on paracompact sets X where the set-valued operators T are demi operators [3] and are upper hemi-continuous [5] along line segments in X to the weak *-topology on E^*.

1. Introduction

Let X be a non-empty set, and 2^X be the family of all non-empty subsets of X. Let E be a topological vector space. We shall denote by E^* the continuous dual of E, by $\langle w, x\rangle$ the pairing between E^* and E for $w \in E^*$ and $x \in E$ and by $Re\langle w, x\rangle$ the real part of $\langle w, x\rangle$. Given the maps $S : X \to 2^X$ and $T : X \to 2^{E^*}$, the generalized quasi-variational inequality problem (GQVI) is to find a point $\hat{y} \in S(\hat{y})$ and a point $\hat{w} \in T(\hat{y})$ such that $Re\langle \hat{w}, \hat{y} - x\rangle \leqslant 0$ for all $x \in S(\hat{y})$. The GQVI was introduced by Chan and Pang [2] in 1982 if $E = \mathbb{R}^n$ and by Shih and Tan [10] in 1985 if E is infinite dimensional.

In this paper we shall obtain some general theorems on solutions of the GQVI. In obtaining these results we shall mainly use the following generalized version of Ky Fan's minimax inequality [7] due to Chowdhury and Tan [4].

THEOREM 1.1. *Let E be a topological vector space, X be a non-empty convex subset of E and $f : X \times X \to \mathbb{R} \cup \{-\infty, +\infty\}$ be such that*

(a) for each $A \in \mathcal{F}(X)$ and each fixed $x \in co(A)$, $y \longmapsto f(x, y)$ is lower semicontinuous on $co(A)$;

(b) for each $A \in \mathcal{F}(X)$ and each $y \in co(A)$, $\min_{x\in A} f(x, y) \leqslant 0$;

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(c) for each $A \in \mathcal{F}(X)$ *and each* $x, y \in co(A)$ *and every net* $\{y_\alpha\}_{\alpha\in\Gamma}$ *in* X *converging to* y *with* $f(tx + (1-t)y, y_\alpha) \leqslant 0$ *for all* $\alpha \in \Gamma$ *and all* $t \in [0,1]$*, we have* $f(x,y) \leqslant 0$*;*

(d) there exist a non-empty closed and compact subset K *of* X *and* $x_0 \in K$ *such that* $f(x_0, y) > 0$ *for all* $y \in X \backslash K$*.*

Then there exists $\hat{y} \in K$ *such that* $f(x, \hat{y}) \leqslant 0$ *for all* $x \in X$*.*

2. Preliminaries

If X is a topological space and $\{U_\alpha : \alpha \in \mathcal{A}\}$ is an open cover for X, then a partition of unity subordinated to the open cover $\{U_\alpha : \alpha \in \mathcal{A}\}$ is a family $\{\beta_\alpha : \alpha \in \mathcal{A}\}$ of continuous real-valued functions $\beta_\alpha : X \to [0,1]$ such that

(a) $\beta_\alpha(y) = 0$ for all $y \in X \setminus U_\alpha$,

(b) $\{$ support $\beta_\alpha : \alpha \in \mathcal{A}\}$ is locally finite and

(c) $\Sigma_{\alpha\in\mathcal{A}}\beta_\alpha(y) = 1$ for each $y \in X$.

We shall first state the following result which is Lemma 1 of Shih and Tan in [10]:

LEMMA 2.1. *Let* X *be a non-empty subset of a Hausdorff topological vector space* E *and* $S : X \to 2^E$ *be an upper semicontinuous map such that* $S(x)$ *is a bounded subset of* E *for each* $x \in X$*. Then for each continuous linear functional* p *on* E*, the map* $f_p : X \to \mathbb{R}$ *defined by* $f_p(y) = \sup_{x\in S(y)} Re\langle p, x\rangle$ *is upper semicontinuous; i.e., for each* $\lambda \in \mathbb{R}$*, the set* $\{y \in X : f_p(y) = \sup_{x\in S(y)} Re\langle p, x\rangle < \lambda\}$ *is open in* X*.*

The following result is Lemma 3 of Takahashi in [12] (see also Lemma 3 in [11]):

LEMMA 2.2. *Let* X *and* Y *be topological spaces,* $f : X \to \mathbb{R}$ *be non-negative and continuous and* $g : Y \to \mathbb{R}$ *be lower semicontinuous. Then the map* $F : X \times Y \to \mathbb{R}$*, defined by* $F(x,y) = f(x)g(y)$ *for all* $(x,y) \in X \times Y$*, is lower semicontinuous.*

We shall need the following Kneser's minimax theorem in [8] (see also Aubin [1]):

THEOREM 2.1. *Let* X *be a non-empty convex subset of a vector space and* Y *be a non-empty compact convex subset of a Hausdorff topological vector space. Suppose that* f *is a real-valued funtion on* $X \times Y$ *such that for each fixed* $x \in X$*, the map* $y \mapsto f(x,y)$ *is lower semicontinuous and convex on* Y *and for each fixed* $y \in Y$*, the map* $x \mapsto f(x,y)$ *is concave on* X*. Then*

$$\min_{y\in Y}\sup_{x\in X} f(x,y) = \sup_{x\in X}\min_{y\in Y} f(x,y).$$

The following definition is Definition 2 in [3]:

DEFINITION 2.1. Let E be a topological vector space, X be a non-empty subset of E and $T : X \to 2^{E^*}$. If $h : X \to \mathbb{R}$, then T is said to be an *h-demi* (respectively, a *strong h-demi*) operator if for each $y \in X$ and every net $\{y_\alpha\}_{\alpha\in\Gamma}$ in X converging to y (respectively, weakly to y) with

$$\limsup_{\alpha\in\Gamma}[\inf_{u\in T(y)} Re\langle u, y_\alpha - y\rangle + h(y_\alpha) - h(y)] \leqslant 0$$

we have

$$\limsup_{\alpha\in\Gamma}\Big[\inf_{u\in T(x)} Re\langle u, y_\alpha - x\rangle + h(y_\alpha) - h(x)\Big] \geqslant \inf_{u\in T(x)} Re\langle u, y - x\rangle + h(y) - h(x) \quad \text{for all} \quad x\in X.$$

T is said to be a *demi* (respectively, *strong demi*) operator if T is an h-demi (respectively, a strong h-demi) operator with $h \equiv 0$.

The following proposition is Proposition 2 in [3]:

PROPOSITION 2.1. *Let X be a non-empty bounded subset of a topological vector space E, $h : X \to \mathbb{R}$ be weakly lower semicontinuous and $T : X \to 2^{E^*}$ be an operator such that each $T(x)$ is strongly compact. Then T is an h-demi and a strong h-demi operator.*

The following definition is Definition 2.1(b) in [5]:

DEFINITION 2.2. Let E be a topological vector space, X be a non-empty subset of E and $T : X \to 2^{E^*}$. Then T is said to be *upper hemicontinuous* on X if and only if for each $p \in E$, the function $f_p : X \to \mathbb{R}\cup\{+\infty\}$, defined by

$$f_p(z) = \sup_{u\in T(z)} Re\langle u, p\rangle \quad \text{for each } z\in X,$$

is upper semicontinuous on X (if and only if for each $p \in E$, the function $g_p : X \to \mathbb{R}\cup\{-\infty\}$, defined by

$$g_p(z) = \inf_{u\in T(z)} Re\langle u, p\rangle \quad \text{for each } z\in X,$$

is lower semicontinuous on X).

The following proposition is Proposition 2.4 in [5]:

PROPOSITION 2.2. *Let E be a topological vector space and X be a non-empty subset of E. Let $T : X \to 2^{E^*}$ be upper semicontinuous from relative topology on X to the weak* topology $\sigma\langle E^*, E\rangle$ on E^*. Then T is upper hemicontinuous on X.*

Note that there is a typo in Proposition 2.4 in [5]. The convexity of X is not needed.

The following result is Lemma 3 in [3]:

LEMMA 2.3. *Let E be a Hausdorff topological vector space, $A \in \mathcal{F}(E)$, $X = co(A)$ and C be a non-empty weak*-compact subset of E^*. Let $f : X\times X \to \mathbb{R}$ be defined by $f(x,y) = \inf_{w\in C} Re\langle w, y-x\rangle$ for all $x, y \in X$. Then for each fixed $x\in X$, $y \longmapsto f(x,y)$ is continuous on X.*

The following result is Lemma 4.2 in [5]:

LEMMA 2.4. *Let E be a topological vector space, X be a non-empty convex subset of E, $h : X \to \mathbb{R}$ be convex and $T : X \to 2^{E^*}$ be upper hemicontinuous along line segments in X. Suppose $\hat{y} \in X$ is such that $\inf_{u \in T(x)} Re\langle u, \hat{y} - x\rangle \leqslant h(x) - h(\hat{y})$ for all $x \in X$. Then*

$$\inf_{w \in T(\hat{y})} Re\langle w, \hat{y} - x\rangle \leqslant h(x) - h(\hat{y}) \text{ for all } x \in X.$$

3. Generalized quasi-variational inequalities for both upper hemi-continuous and demi operators on non-compact sets.

In this section we shall obtain some general theorems on solutions of the generalized quasi-variational inequalities for both upper hemi-continuous and demi operators on non-compact sets.

We shall first establish the following result:

THEOREM 3.1. *Let E be a locally convex Hausdorff topological vector space, X be a non-empty paracompact convex and bounded subset of E and $h : E \to \mathbb{R}$ be convex with $h(X)$ bounded. Let $S : X \to 2^X$ be upper semicontinuous such that each $S(x)$ is compact convex and $T : X \to 2^{E^*}$ be an h-demi operator and be upper hemi-continuous along line segments in X to the weak*-topology on E^* such that each $T(x)$ is weak*-compact convex. Suppose that the set*

$$\Sigma = \{y \in X : \sup_{x \in S(y)} [\inf_{u \in T(x)} Re\langle u, y - x\rangle + h(y) - h(x)] > 0\}$$

is open in X and for each $A \in \mathcal{F}(X)$ and each $y \in co(A)$ there exist $\bar{x} \in A$ and $\bar{u} \in T(\bar{x})$ such that $\beta_0(y)[Re\langle \bar{u}, y - \bar{x}\rangle + h(y) - h(\bar{x})] + \Sigma_{p \in E^*}\beta_p(y)Re\langle p, y - \bar{x}\rangle \leqslant 0$ for every family $\{\beta_0, \beta_p : p \in E^*\}$ of non-negative real-valued functions from X into [0,1]. Suppose further that there exists a non-empty compact subset K of X and a point $x_0 \in X$ such that $x_0 \in K \cap S(y)$ and $\min_{u \in T(x_0)} Re\langle u, y - x_0\rangle + h(y) - h(x_0) > 0$ for all $y \in X \setminus K$. Then there exists a point $\hat{y} \in K$ such that

(i) $\hat{y} \in S(\hat{y})$ and

(ii) there exist a point $\hat{w} \in T(\hat{y})$ with $Re\langle \hat{w}, \hat{y} - x\rangle \leqslant h(x) - h(\hat{y})$ for all $x \in S(\hat{y})$.

Proof. We divide the proof into three steps:

Step 1. There exists a point $\hat{y} \in X$ such that $\hat{y} \in S(\hat{y})$ and

$$\sup_{x \in S(\hat{y})} [\inf_{u \in T(x)} Re\langle u, \hat{y} - x\rangle + h(\hat{y}) - h(x)] \leqslant 0.$$

Suppose the contrary. Then for each $y \in X$, either $y \notin S(y)$ or there exists $x \in S(y)$ such that $\inf_{u \in T(x)} Re\langle u, y - x\rangle + h(y) - h(x) > 0$; that is, for each $y \in X$, either $y \notin S(y)$ or $y \in \Sigma$. If $y \notin S(y)$, then by a separation theorem for convex sets in locally convex Hausdorff topological vector spaces, there exists $p \in E^*$ such that $Re\langle p, y\rangle - \sup_{x \in S(y)} Re\langle p, x\rangle > 0$. For each $y \in X$, set

$$\gamma(y) := \sup_{x \in S(y)} [\inf_{u \in T(x)} Re\langle u, y - x\rangle + h(y) - h(x)].$$

Let $V_0 := \{y \in X | \gamma(y) > 0\} = \Sigma$ and for each $p \in E^*$, set

$$V_p := \{y \in X : Re\langle p, y\rangle - \sup_{x \in S(y)} Re\langle p, x\rangle > 0\}.$$

Then $X = V_0 \cup \bigcup_{p \in E^*} V_p$. Since each V_p is open in X by Lemma 2.1 and V_0 is open in X by hypothesis, $\{V_0, V_p : p \in E^*\}$ is an open covering for X. Since X is paracompact, there is a continuous partition of unity $\{\beta_0, \beta_p : p \in E^*\}$ for X subordinated to the open cover $\{V_0, V_p : p \in E^*\}$ (see, e.g., Theorem VIII.4.2 of Dugundji in [6]); that is for each $p \in E^*$, $\beta_p : X \to [0,1]$ and $\beta_0 : X \to [0,1]$ are continuous functions such that for each $p \in E^*$, $\beta_p(y) = 0$ for all $y \in X \setminus V_p$ and $\beta_0(y) = 0$ for all $y \in X \setminus V_0$ and $\{$ support β_0, support $\beta_p : p \in E^*\}$ is locally finite and $\beta_0(y) + \Sigma_{p \in E^*} \beta_p(y) = 1$ for each $y \in X$. Note that for each $A \in \mathcal{F}(X)$, h is continuous on $co(A)$ (see e.g. [9], Corollary 10.1.1, p.83)). Define $\phi, \psi : X \times X \to \mathbb{R}$ by

$$\phi(x,y) = \beta_0(y)[\min_{u \in T(x)} Re\langle u, y - x\rangle + h(y) - h(x)] + \Sigma_{p \in E^*} \beta_p(y) Re\langle p, y - x\rangle$$

for each $x, y \in X$. Then we have the following.

(a) Since E is Hausdorff, for each $A \in \mathcal{F}(X)$ and each fixed $x \in co(A)$, the map

$$y \mapsto \min_{u \in T(x)} Re\langle u, y - x\rangle + h(y) - h(x)$$

is continuous on $co(A)$ by Lemma 2.3 and the fact that h is continuous on $co(A)$ and therefore the map

$$y \mapsto \beta_0(y)[\min_{u \in T(x)} Re\langle u, y - x\rangle + h(y) - h(x)]$$

is lower semi-continuous on $co(A)$ by Lemma 2.2. Also for each fixed $x \in X$,

$$y \mapsto \Sigma_{p \in E^*} \beta_p(y) Re\langle p, y - x\rangle$$

is continuous on X. Hence, for each $A \in \mathcal{F}(X)$ and each fixed $x \in co(A)$, the map $y \longmapsto \phi(x,y)$ is lower semicontinuous on $co(A)$.

(b) Since $\{\beta_0, \beta_p : p \in E^*\}$ is a family of non-negative real-valued functions from X into [0,1], by hypothesis, for each $A \in \mathcal{F}(X)$ and each $y \in co(A)$, there exist $\bar{x} \in A$ and $\bar{u} \in T(\bar{x})$ such that $\beta_0(y)[Re\langle \bar{u}, y - \bar{x}\rangle + h(y) - h(\bar{x})] + \Sigma_{p \in E^*} \beta_p(y) Re\langle p, y - \bar{x}\rangle \leqslant 0$. Thus

$$\min_{u \in T(x)} [\beta_0(y)(Re\langle u, y - \bar{x}\rangle + h(y) - h(\bar{x})) + \Sigma_{p \in E^*} \beta_p(y) Re\langle p, y - \bar{x}\rangle] \leqslant 0,$$

i.e.,

$$\beta_0(y)[\min_{u \in T(x)} Re\langle u, y - \bar{x}\rangle + h(y) - h(\bar{x})] + \Sigma_{p \in E^*} \beta_p(y) Re\langle p, y - \bar{x}\rangle \leqslant 0.$$

Therefore

$$\min_{x\in A}[\beta_0(y)(\min_{u\in T(x)} Re\langle u, y-x\rangle + h(y)-h(x)) + \Sigma_{p\in E^*}\beta_p(y)Re\langle p, y-x\rangle] \leqslant 0.$$

Thus we have $\min_{x\in A}\phi(x,y)\leqslant 0$ for each $A\in\mathcal{F}(X)$ and each $y\in co(A)$.

(c) Suppose that $A\in\mathcal{F}(X)$, $x,y\in co(A)$ and $\{y_\alpha\}_{\alpha\in\Gamma}$ is a net in X converging to y with $\phi(tx+(1-t)y, y_\alpha)\leqslant 0$ for all $\alpha\in\Gamma$ and all $t\in[0,1]$.

Case 1: $\beta_0(y)=0$.

Since β_0 is continuous and $y_\alpha\to y$, we have $\beta_0(y_\alpha)\to\beta_0(y)=0$. Note that $\beta_0(y_\alpha)\geqslant 0$ for each $\alpha\in\Gamma$. Since $T(x)$ is strongly bounded and $\{y_\alpha\}_{\alpha\in\Gamma}$ is a bounded net, it follows that

$$\limsup_\alpha[\beta_0(y_\alpha)(\min_{u\in T(x)} Re\langle u, y_\alpha-x\rangle + h(y_\alpha)-h(x))]=0.$$

Also

$$\beta_0(y)[\min_{u\in T(x)} Re\langle u, y-x\rangle + h(y)-h(x)]=0.$$

Thus

$$\limsup_\alpha[\beta_0(y_\alpha)(\min_{u\in T(x)} Re\langle u, y_\alpha-x\rangle + h(y_\alpha)-h(x))] + \Sigma_{p\in E^*}\beta_p(y)Re\langle p, y-x\rangle$$
$$= \beta_0(y)[\min_{u\in T(x)} Re\langle u, y-x\rangle + h(y)-h(x)] + \Sigma_{p\in E^*}\beta_p(y)Re\langle p, y-x\rangle \quad (1)$$

For $t=1$ we have $\phi(x,y_\alpha)\leqslant 0$ for all $\alpha\in\Gamma$, i.e.,

$$\beta_0(y_\alpha)[\min_{u\in T(x)} Re\langle u, y_\alpha-x\rangle + h(y_\alpha)-h(x)] + \Sigma_{p\in E^*}\beta_p(y_\alpha)Re\langle p, y_\alpha-x\rangle \leqslant 0$$

for all $\alpha\in\Gamma$. It follows that

$$\limsup_\alpha[\beta_0(y_\alpha)(\min_{u\in T(x)} Re\langle u, y_\alpha-x\rangle + h(y_\alpha)-h(x))] + \Sigma_{p\in E^*}\beta_p(y)Re\langle p, y-x\rangle \leqslant 0. \quad (2)$$

Hence by (1) and (2), we have $\phi(x,y)\leqslant 0$.

Case 2: $\beta_0(y)>0$.

Since β_0 is continuous, $\beta_0(y_\alpha)\to\beta_0(y)$. Again since $\beta_0(y)>0$, there exists $\lambda\in\Gamma$ such that $\beta_0(y_\alpha)>0$ for all $\alpha\geqslant\lambda$.

Then for $t=0$ we have $\phi(y,y_\alpha)\leqslant 0$ for all $\alpha\in\Gamma$, i.e., $\beta_0(y_\alpha)[\min_{u\in T(y)} Re\langle u, y_\alpha - y\rangle + h(y_\alpha)-h(y)] + \Sigma_{p\in E^*}\beta_p(y_\alpha)Re\langle p, y_\alpha-y\rangle \leqslant 0$ for all $\alpha\in\Gamma$. It follows that $\limsup_\alpha[\beta_0(y_\alpha)(\min_{u\in T(y)} Re\langle u, y_\alpha-y\rangle + h(y_\alpha)-h(y))]\leqslant 0$.

Since $\beta_0(y_\alpha)>0$ for all $\alpha\geqslant\lambda$, it follows that $\beta_0(y)\limsup_\alpha[\min_{u\in T(y)} Re\langle u, y_\alpha - y\rangle + h(y_\alpha)-h(y)])\leqslant 0$. Therefore

$$\limsup_\alpha[\min_{u\in T(y)} Re\langle u, y_\alpha-y\rangle + h(y_\alpha)-h(y)])\leqslant 0.$$

Since T is an h-demi operator on X, we have

$$\limsup_\alpha[\min_{u\in T(x)} Re\langle u, y_\alpha-x\rangle + h(y_\alpha)-h(x)] \geqslant \min_{u\in T(x)} Re\langle u, y-x\rangle + h(y)-h(x). \quad (3)$$

Since $\beta_0(y) > 0$, we have

$$\beta_0(y)[\limsup_{\alpha}(\min_{u\in T(x)} Re\langle u, y_\alpha - x\rangle + h(y_\alpha) - h(x))]$$
$$\geqslant \beta_0(y)[\min_{u\in T(x)} Re\langle u, y - x\rangle + h(y) - h(x)].$$

Thus

$$\beta_0(y)[\limsup_{\alpha}(\min_{u\in T(x)} Re\langle u, y_\alpha - x\rangle + h(y_\alpha) - h(x))] + \Sigma_{p\in E^*}\beta_p(y)Re\langle p, y - x\rangle$$
$$\geqslant \beta_0(y)[\min_{u\in T(x)} Re\langle u, y - x\rangle + h(y) - h(x)] + \Sigma_{p\in E^*}\beta_p(y)Re\langle p, y - x\rangle . \quad (4)$$

For $t = 1$ we also have $\phi(x, y_\alpha) \leqslant 0$ for all $\alpha \in \Gamma$, i.e.,

$$\beta_0(y_\alpha)[\min_{u\in T(x)} Re\langle u, y_\alpha - x\rangle + h(y_\alpha) - h(x)] + \Sigma_{p\in E^*}\beta_p(y_\alpha)Re\langle p, y_\alpha - x\rangle \leqslant 0$$

for all $\alpha \in \Gamma$. It follows that

$$\limsup_{\alpha}[\beta_0(y_\alpha)(\min_{u\in T(x)} Re\langle u, y_\alpha - x\rangle + h(y_\alpha) - h(x))] + \Sigma_{p\in E^*}\beta_p(y)Re\langle p, y - x\rangle \leqslant 0.$$

Thus

$$\beta_0(y)[\limsup_{\alpha}(\min_{u\in T(x)} Re\langle u, y_\alpha - x\rangle + h(y_\alpha) - h(x))] + \Sigma_{p\in E^*}\beta_p(y)Re\langle p, y - x\rangle \leqslant 0. \quad (5)$$

Hence by (4) and (5), we have $\phi(x, y) \leqslant 0$.

(d) By hypothesis, there exists a non-empty compact (and therefore closed) subset K of X and a point $x_0 \in X$ such that $x_0 \in K \cap S(y)$ and $\min_{u\in T(x_0)} Re\langle u, y - x_0\rangle + h(y) - h(x_0) > 0$ for each $y \in X \setminus K$. Thus for each $y \in X \setminus K$, $\sup_{x\in S(y)}[\min_{u\in T(x)} Re\langle u, y - x\rangle + h(y) - h(x)] > 0$. Hence $y \in V_0$ and $\beta_0(y)[\min_{u\in T(x_0)} Re\langle u, y - x_0\rangle + h(y) - h(x_0) > 0$ for all $y \in X \setminus K$; also $Re\langle p, y - x_0\rangle > 0$ whenever $\beta_p(y) > 0$ for $p \in E^*$. Consequently, $\phi(x_0, y) = \beta_0(y)[\min_{u\in T(x_0)} Re\langle u, y - x_0\rangle + h(y) - h(x_0)] + \Sigma_{p\in E^*}\beta_p(y)Re\langle p, y - x_0\rangle > 0$ for all $y \in X \setminus K$.

Then ϕ satisfies all hypotheses of Theorem 1.1. Hence by Theorem 1.1, there exists a point $\hat{y} \in K$ such that $\phi(x, \hat{y}) \leqslant 0$ for all $x \in X$; i.e.,

$$\beta_0(\hat{y})[\min_{u\in T(x)} Re\langle u, \hat{y} - x\rangle + h(\hat{y}) - h(x)] + \Sigma_{p\in E^*}\beta_p(\hat{y})Re\langle p, \hat{y} - x\rangle \leqslant 0 \quad (6)$$

for all $x \in X$.

If $\beta_0(\hat{y}) > 0$, then $\hat{y} \in V_0 = \Sigma$ so that $\gamma(\hat{y}) > 0$. Choose $\hat{x} \in S(\hat{y}) \subset X$ such that

$$\min_{u\in T(\hat{x})} Re\langle u, \hat{y} - \hat{x}\rangle + h(\hat{y}) - h(\hat{x}) \geqslant \frac{\gamma(\hat{y})}{2} > 0;$$

it follows that

$$\beta_0(\hat{y})[\min_{u\in T(\hat{x})} Re\langle u, \hat{y} - \hat{x}\rangle + h(\hat{y}) - h(\hat{x})] > 0.$$

If $\beta_p(\hat{y}) > 0$ for some $p \in E^*$, then $\hat{y} \in V_p$ and hence

$$Re\langle p, \hat{y}\rangle > \sup_{x\in S(\hat{y})} Re\langle p, x\rangle \geqslant Re\langle p, \hat{x}\rangle$$

so that $Re\langle p, \hat{y}-\hat{x}\rangle > 0$. Then note that

$$\beta_p(\hat{y})Re\langle p, \hat{y}-\hat{x}\rangle > 0 \quad \text{whenever} \quad \beta_p(\hat{y}) > 0 \text{ for } p \in E^*.$$

Since $\beta_0(\hat{y}) > 0$ or $\beta_p(\hat{y}) > 0$ for some $p \in E^*$, it follows that

$$\phi(\hat{x},\hat{y}) = \beta_0(\hat{y})[\min_{u\in T(\hat{x})} Re\langle u, \hat{y}-\hat{x}\rangle + h(\hat{y}) - h(\hat{x})] + \Sigma_{p\in E^*}\beta_p(\hat{y})Re\langle p, \hat{y}-\hat{x}\rangle > 0,$$

which contradicts (6). This contradiction proves Step 1.

Step 2.

$$\inf_{w\in T(\hat{y})} Re\langle w, \hat{y}-x\rangle \leqslant h(x) - h(\hat{y}) \text{ for all } x \in S(\hat{y}).$$

Indeed, from Step 1, $\hat{y} \in S(\hat{y})$ which is a convex subset of X, and

$$\inf_{u\in T(x)} Re\langle u, \hat{y}-x\rangle \leqslant h(x) - h(\hat{y}) \quad \text{for all} \quad x \in S(\hat{y}).$$

Hence by Lemma 2.4, we have

$$\inf_{w\in T(\hat{y})} Re\langle w, \hat{y}-x\rangle \leqslant h(x) - h(\hat{y}) \quad \text{for all} \quad x \in S(\hat{y}).$$

Step 3. There exist a point $\hat{w} \in T(\hat{y})$ with $Re\langle \hat{w}, \hat{y}-x\rangle \leqslant h(x) - h(\hat{y})$ for all $x \in S(\hat{y})$.

From Step 2 we have

$$\sup_{x\in S(\hat{y})} [\inf_{w\in T(\hat{y})} Re\langle w, \hat{y}-x\rangle + h(\hat{y}) - h(x)] \leqslant 0, \tag{7}$$

where $T(\hat{y})$ is a weak*-compact convex subset of the Hausdorff topological vector space E^* and $S(\hat{y})$ is a convex subset of X.

Indeed, define $f : S(\hat{y}) \times T(\hat{y}) \to \mathbb{R}$ by $f(x,w) = Re\langle w, \hat{y}-x\rangle + h(\hat{y}) - h(x)$ for each $x \in S(\hat{y})$ and each $w \in T(\hat{y})$. Note that for each fixed $x \in S(\hat{y})$, the map $w \longmapsto f(x,w)$ is convex and continuous on $T(\hat{y})$ and for each fixed $w \in T(\hat{y})$, the map $x \longmapsto f(x,w)$ is concave on $S(\hat{y})$. Thus by Theorem 2.1, we have

$$\min_{w\in T(\hat{y})} \sup_{x\in S(\hat{y})} [Re\langle w, \hat{y}-x\rangle + h(\hat{y}) - h(x)] = \sup_{x\in S(\hat{y})} \min_{w\in T(\hat{y})} [Re\langle w, \hat{y}-x\rangle + h(\hat{y}) - h(x)].$$

Hence

$$\min_{w\in T(\hat{y})} \sup_{x\in S(\hat{y})} [Re\langle w, \hat{y}-x\rangle + h(\hat{y}) - h(x)] \leqslant 0, \quad \text{by } (7).$$

Since $T(\hat{y})$ is compact, there exists $\hat{w} \in T(\hat{y})$ such that

$$Re\langle \hat{w}, \hat{y}-x\rangle \leqslant h(x) - h(\hat{y}) \quad \text{for all } x \in S(\hat{y}). \quad \square$$

If X is compact, we obtain the following immediate consequence of Theorem 3.1:

COROLLARY 3.1. *Let E be a locally convex Hausdorff topological vector space, X be a non-empty compact convex subset of E and $h : E \to \mathbb{R}$ be convex with $h(X)$ bounded. Let $S : X \to 2^X$ be upper semicontinuous such that each $S(x)$ is closed convex and $T : X \to 2^{E^*}$ be an h-demi operator and be upper hemi-continuous along line segments in X to the weak*-topology on E^* such that each $T(x)$ is weak*-compact convex. Suppose that the set*

$$\Sigma = \{y \in X : \sup_{x \in S(y)} [\inf_{u \in T(x)} Re\langle u, y-x\rangle + h(y) - h(x)] > 0\}$$

is open in X and for each $A \in \mathcal{F}(X)$ and each $y \in co(A)$ there exist $\bar{x} \in A$ and $\bar{u} \in T(\bar{x})$ such that $\beta_0(y)[Re\langle \bar{u}, y-\bar{x}\rangle + h(y) - h(\bar{x})] + \Sigma_{p \in E^*} \beta_p(y) Re\langle p, y-\bar{x}\rangle \leqslant 0$ for every family $\{\beta_0, \beta_p : p \in E^*\}$ of non-negative real-valued functions from X into [0,1]. Then there exists $\hat{y} \in X$ such that

(i) $\hat{y} \in S(\hat{y})$ and

(ii) there exists a point $\hat{w} \in T(\hat{y})$ with $Re\langle \hat{w}, \hat{y}-x\rangle \leqslant h(x) - h(\hat{y})$ for all $x \in S(\hat{y})$.

Note that if the map $S : X \to 2^X$ is, in addition, lower semicontinuous and for each $y \in \Sigma$, T is upper semi-continuous at some point x in $S(y)$ with $\inf_{u \in T(x)} Re\langle u, y-x\rangle + h(y) - h(x)] > 0$, then the set Σ in Theorem 3.1 is always open in X as can be seen in the proof of the following:

THEOREM 3.2. *Let E be a locally convex Hausdorff topological vector space, X be a non-empty paracompact convex and bounded subset of E and $h : E \to \mathbb{R}$ be convex with $h(X)$ bounded. Let $S : X \to 2^X$ be continuous such that each $S(x)$ is compact convex and $T : X \to 2^{E^*}$ be an h-demi operator and be upper hemi-continuous along line segments in X to the weak*-topology on E^* such that each $T(x)$ is weak*-compact convex. Suppose that for each $A \in \mathcal{F}(X)$ and each $y \in co(A)$ there exist $\bar{x} \in A$ and $\bar{u} \in T(\bar{x})$ such that $\beta_0(y)[Re\langle \bar{u}, y-\bar{x}\rangle + h(y) - h(\bar{x})] + \Sigma_{p \in E^*} \beta_p(y) Re\langle p, y-\bar{x}\rangle \leqslant 0$ for every family $\{\beta_0, \beta_p : p \in E^*\}$ of non-negative real-valued functions from X into [0,1]. Suppose further that*

(a) for each $y \in \Sigma = \{y \in X : \sup_{x \in S(y)} [\inf_{u \in T(x)} Re\langle u, y-x\rangle + h(y) - h(x)] > 0\}$, T is upper semicontinuous at some point x in $S(y)$ with $\inf_{u \in T(x)} Re\langle u, y-x\rangle + h(y) - h(x)] > 0$ and

(b) there exists a non-empty compact subset K of X and a point $x_0 \in X$ such that $x_0 \in K \cap S(y)$ and $\min_{u \in T(x_0)} Re\langle u, y-x_0\rangle + h(y) - h(x_0) > 0$ for all $y \in X \setminus K$.

Then there exists $\hat{y} \in K$ such that

(i) $\hat{y} \in S(\hat{y})$ and

(ii) there exists a point $\hat{w} \in T(\hat{y})$ with $Re\langle \hat{w}, \hat{y}-x\rangle \leqslant h(x) - h(\hat{y})$ for all $x \in S(\hat{y})$.

Proof. By virtue of Theorem 3.1, it suffices to show that the set

$$\Sigma = \{y \in X : \sup_{x \in S(y)} [\inf_{u \in T(x)} Re\langle u, y-x\rangle + h(y) - h(x)] > 0\}$$

is open in X. Indeed, let $y_0 \in \Sigma$; then by hypothesis, T is upper semicontinuous at some point x_0 in $S(y_0)$ with $\inf_{u \in T(x_0)} Re\langle u, y_0 - x_0\rangle + h(y_0) - h(x_0) > 0$. Let

$$\alpha := \inf_{u \in T(x_0)} Re\langle u, y_0 - x_0\rangle + h(y_0) - h(x_0).$$

Then $\alpha > 0$. Also let

$$W := \{w \in E^* : \sup_{z_1, z_2 \in X} |\langle w, z_1 - z_2\rangle| < \alpha/6\}.$$

Then W is a strongly open neighborhood of 0 in E^* so that $U_1 := T(x_0) + W$ is an open neighborhood of $T(x_0)$ in E^*. Since T is upper semicontinuous at x_0, there exists an open neighborhood V_1 of x_0 in X such that $T(x) \subset U_1$ for all $x \in V_1$.

As the map $x \longmapsto \inf_{u \in T(x_0)} Re\langle u, x_0 - x\rangle + h(x_0) - h(x)$ is continuous at x_0, there exists an open neighborhood V_2 of x_0 in X such that

$$|\inf_{u \in T(x_0)} Re\langle u, x_0 - x\rangle + h(x_0) - h(x)| < \alpha/6 \quad \text{for all} \quad x \in V_2.$$

Let $V_0 := V_1 \cap V_2$; then V_0 is an open neighborhood of x_0 in X. Since $x_0 \in V_0 \cap S(y_0) \neq \emptyset$ and S is lower semicontinuous at y_0, there exists an open neighborhood N_1 of y_0 in X such that $S(y) \cap V_0 \neq \emptyset$ for all $y \in N_1$.

Since the map $y \longmapsto \inf_{u \in T(x_0)} Re\langle u, y - y_0\rangle + h(y) - h(y_0)$ is continuous at y_0, there exists an open neighborhood N_2 of y_0 in X such that

$$|\inf_{u \in T(x_0)} Re\langle u, y - y_0\rangle + h(y) - h(y_0)| < \alpha/6 \quad \text{for all} \quad y \in N_2.$$

Let $N_0 := N_1 \cap N_2$. Then N_0 is an open neighborhood of y_0 in X such that for each $y_1 \in N_0$, we have

(*i*) $S(y_1) \cap V_0 \neq \emptyset$ as $y_1 \in N_1$; so we can choose any $x_1 \in S(y_1) \cap V_0$;
(*ii*) $|\inf_{u \in T(x_0)} Re\langle u, y_1 - y_0\rangle + h(y_1) - h(y_0)| < \alpha/6$ as $y_1 \in N_2$;
(*iii*) $T(x_1) \subset U_1 = T(x_0) + W$ as $x_1 \in V_1$;
(*iv*) $|\inf_{u \in T(x_0)} Re\langle u, x_0 - x_1\rangle + h(x_0) - h(x_1)| < \alpha/6$ as $x_1 \in V_2$.

It follows that

$$\begin{aligned}
&\inf_{u \in T(x_1)} Re\langle u, y_1 - x_1\rangle + h(y_1) - h(x_1) \\
&\geqslant \inf_{[u \in T(x_0)+W]} Re\langle u, y_1 - x_1\rangle + h(y_1) - h(x_1) \qquad (\text{by } (iii)), \\
&\geqslant \inf_{u \in T(x_0)} Re\langle u, y_1 - x_1\rangle + h(y_1) - h(x_1) + \inf_{u \in W} Re\langle u, y_1 - x_1\rangle \\
&\geqslant \inf_{u \in T(x_0)} Re\langle u, y_1 - y_0\rangle + h(y_1) - h(y_0) \\
&+ \inf_{u \in T(x_0)} Re\langle u, y_0 - x_0\rangle + h(y_0) - h(x_0) \\
&+ \inf_{u \in T(x_0)} Re\langle u, x_0 - x_1\rangle + h(x_0) - h(x_1) + \inf_{u \in W} Re\langle u, y_1 - x_1\rangle \\
&\geqslant -\frac{\alpha}{6} + \alpha - \frac{\alpha}{6} - \frac{\alpha}{6} = \frac{\alpha}{2} > 0 \qquad (\text{by } (ii) \text{ and } (iv));
\end{aligned}$$

therefore

$$\sup_{x\in S(y_1)}\Big[\inf_{u\in T(x)} Re\langle u, y_1 - x\rangle + h(y_1) - h(x)\Big] > 0$$

as $x_1 \in S(y_1)$. This shows that $y_1 \in \Sigma$ for all $y_1 \in N_0$, so that Σ is open in X. This completes the proof. □

The compact version of this result follows immediately from Theorem 3.2 when X is also compact.

4. Generalized quasi-variational inequalities for only upper hemi-continuous operators on non compact sets

In this section we shall obtain some general theorems on solutions of the generalized quasi-variational inequalities for only upper hemi-continuous operators on non compact sets.

In Theorem 3.1 if h is, in addition, continuous on X, then h can be defined only on X and T need not be an h-demi operator. Thus we have the following result for only upper hemi-continuous operators T:

THEOREM 4.1. *Let E be a locally convex Hausdorff topological vector space, X be a non-empty paracompact convex subset of E and $h : X \to \mathbb{R}$ be convex and continuous. Let $S : X \to 2^X$ be upper semicontinuous such that each $S(x)$ is compact convex and $T : X \to 2^{E^*}$ be upper hemi-continuous along line segments in X to the weak*-topology on E^* such that each $T(x)$ is weak*-compact convex. Suppose that the set*

$$\Sigma = \{y \in X : \sup_{x\in S(y)}\Big[\inf_{u\in T(x)} Re\langle u, y - x\rangle + h(y) - h(x)\Big] > 0\}$$

is open in X and for each $A \in \mathcal{F}(X)$ and each $y \in co(A)$ there exist $\bar{x} \in A$ and $\bar{u} \in T(\bar{x})$ such that $\beta_0(y)[Re\langle \bar{u}, y - \bar{x}\rangle + h(y) - h(\bar{x})] + \Sigma_{p\in E^*}\beta_p(y)Re\langle p, y - \bar{x}\rangle \leqslant 0$ for every family $\{\beta_0, \beta_p : p \in E^*\}$ of non-negative real-valued functions from X into [0,1]. Suppose further that there exists a non-empty compact subset K of X and a point $x_0 \in X$ such that $x_0 \in K \cap S(y)$ and $\min_{u\in T(x_0)} Re\langle u, y - x_0\rangle + h(y) - h(x_0) > 0$ for all $y \in X \setminus K$. Then there exists a point $\hat{y} \in K$ such that

(i) $\hat{y} \in S(\hat{y})$ and

(ii) there exists a point $\hat{w} \in T(\hat{y})$ with $Re\langle \hat{w}, \hat{y} - x\rangle \leqslant h(x) - h(\hat{y})$ for all $x \in S(\hat{y})$.

Proof. The proof is exactly same until (c) in the proof of Theorem 3.1. Then, we shall prove (c) as follows:

(c) Suppose that $A \in \mathcal{F}(X)$, $x, y \in co(A)$ and $\{y_\alpha\}_{\alpha\in\Gamma}$ is a net in X converging to y with $\phi(tx + (1 - t)y, y_\alpha) \leqslant 0$ for all $\alpha \in \Gamma$ and all $t \in [0, 1]$.

For $t = 1$ we have $\phi(x, y_\alpha) \leqslant 0$ for all $\alpha \in \Gamma$, i.e.,

$$\beta_0(y_\alpha)\Big[\min_{u\in T(x)} Re\langle u, y_\alpha - x\rangle + h(y_\alpha) - h(x)\Big] + \Sigma_{p\in E^*}\beta_p(y_\alpha)Re\langle p, y_\alpha - x\rangle \leqslant 0$$

for all $\alpha \in \Gamma$. It follows that

$$\limsup_{\alpha}[\beta_0(y_\alpha)(\min_{u\in T(x)} Re\langle u, y_\alpha - x\rangle + h(y_\alpha) - h(x))] + \Sigma_{p\in E^*}\beta_p(y)Re\langle p, y-x\rangle \leqslant 0.$$

Thus by continuity, we have

$$\begin{aligned}&\beta_0(y)[\min_{u\in T(x)} Re\langle u, y-x\rangle + h(y) - h(x)] + \Sigma_{p\in E^*}\beta_p(y)Re\langle p, y-x\rangle \\ &= \beta_0(y)[\limsup_{\alpha}(\min_{u\in T(x)} Re\langle u, y_\alpha - x\rangle + h(y_\alpha) - h(x))] + \Sigma_{p\in E^*}\beta_p(y)Re\langle p, y-x\rangle \\ &\leqslant 0.\end{aligned}$$

Hence $\phi(x,y) \leqslant 0$.

The rest of the proof is exactly same as in the proof of Theorem 3.1. Consequently Theorem 4.1 is proved.

When $h \equiv 0$ in Theorem 4.1, we can have the standard form of this result. The compact versions of Theorem 4.1 and Theorem 4.1 with $h \equiv 0$ follow immediately when X is also compact. Since these are trivial, we omit the statements of these results.

Again, if the map $S : X \to 2^X$ is, in addition, lower semicontinuous and for each $y \in \Sigma$, T is upper semi-continuous at some point x in $S(y)$ with $\inf_{u\in T(x)} Re\langle u, y-x\rangle + h(y) - h(x)] > 0$, then the set Σ in Theorem 4.1 is always open in X and we have the following:

THEOREM 4.2. *Let E be a locally convex Hausdorff topological vector space, X be a non-empty paracompact convex subset of E and $h : X \to \mathbb{R}$ be convex and continuous. Let $S : X \to 2^X$ be continuous such that each $S(x)$ is compact convex and $T : X \to 2^{E^*}$ be upper hemi-continuous along line segments in X to the weak*-topology on E^* such that each $T(x)$ is weak*-compact convex. Suppose that for each $A \in \mathcal{F}(X)$ and each $y \in co(A)$ there exist $\bar{x} \in A$ and $\bar{u} \in T(\bar{x})$ such that $\beta_0(y)[Re\langle \bar{u}, y-\bar{x}\rangle + h(y) - h(\bar{x})] + \Sigma_{p\in E^*}\beta_p(y)Re\langle p, y-\bar{x}\rangle \leqslant 0$ for every family $\{\beta_0, \beta_p : p \in E^*\}$ of non-negative real-valued functions from X into [0,1]. Suppose further that*

(a) for each $y \in \Sigma = \{y \in X : \sup_{x\in S(y)}[\inf_{u\in T(x)} Re\langle u, y-x\rangle + h(y) - h(x)] > 0\}$, T is upper semicontinuous at some point x in $S(y)$ with $\inf_{u\in T(x)} Re\langle u, y-x\rangle + h(y) - h(x)] > 0$ and

(b) there exists a non-empty compact subset K of X and a point $x_0 \in X$ such that $x_0 \in K \cap S(y)$ and $\min_{u\in T(x_0)} Re\langle u, y-x_0\rangle + h(y) - h(x_0) > 0$ for all $y \in X\setminus K$.

Then there exists a point $\hat{y} \in K$ such that

(i) $\hat{y} \in S(\hat{y})$ and

(ii) there exists a point $\hat{w} \in T(\hat{y})$ with $Re\langle \hat{w}, \hat{y} - x\rangle \leqslant h(x) - h(\hat{y})$ for all $x \in S(\hat{y})$.

When $h \equiv 0$ in Theorem 4.2, we can have the standard form of this result. The compact versions of Theorem 4.2 and Theorem 4.2 with $h \equiv 0$ follow immediately when X is also compact. Since these are trivial results, we omit their statements.

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