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REMARKS, QUESTIONS AND CONJECTURES ON LANDAU–KOLMOGOROV–TYPE INEQUALITIES

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(*communicated by M. K. Kwong*)

Abstract. Results on Landau-Kolmogorov-type inequalities are surveyed with some new results given and some open questions raised. A multivariate analogue and analogues using different operators than the derivative are particularly emphasized. Equivalence between the periodic and nonperiodic case is shown. Failure of Kolmogorov's inequality for L_p spaces when $0 < p < 1$ is demonstrated.

1. The classic Landau-Kolmogorov inequality

The Landau-Kolmogorov inequality is given by

$$\|f^{(k)}\|_B \leqslant K(n,k,B)\|f^{(n)}\|_B^{k/n}\,\|f\|_B^{1-\frac{k}{n}}, \quad 0<k<n, \tag{1.1}$$

where B is a Banach space of functions on R (the reals) or a subset of R, and $K(n,k,B)$ are the best constants for the given n,k and B. The inequality (1.1) has been the subject of numerous articles. Without achieving the best constants (or a good estimate of them), the inequality follows by induction from the case $k=1$ and $n=2$. The best constants for $B=L_\infty(R)$ were given by Kolmogorov [Ko] who used the Euler splines (before that name was coined) for the extremal functions. The case $k=1$ and $n=2$ is attributed to Landau and to Hadamard.

It is known that, for $1 \leqslant p \leqslant \infty$,

$$\begin{aligned}1 = K\big(n,k,L_2(R)\big) &\leqslant K\big(n,k,L_p(R)\big)\\ &\leqslant K\big(n,k,L_\infty(R)\big) = K\big(n,k,L_1(R)\big).\end{aligned} \tag{1.2}$$

The first identity and second inequality of (1.2) (from left to right) are easy, the third inequality follows E. Stein [St], and the fourth equality can be found in [Di,I].

It should be noted that while many articles estimating $K\big(n,k,L_p(R)\big)$ were written, $K\big(n,k,L_p(R)\big)$ are known only for $p=1,2$ and ∞ (see [Ko], [St], [Di,I] and [Kw-Ze]).

The constants $K\big(n,k,L_p(R)\big)$ are closely related to $K\big(n,k,L_p(T)\big)$, and I enclose a statement and proof of that fact as I could not find a reference. (Recall $T=[-\pi,\pi]$ and $L_p(T)$ is the collection of 2π periodic functions satisfying $\|f\|_{L_p(T)} < \infty$.)

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THEOREM 1.1. *For* $1 \leqslant p \leqslant \infty$, *one has*

$$K\big(n,k,L_p(T)\big) = K\big(n,k,L_p(R)\big). \tag{1.3}$$

Proof. For $p = \infty$,

$$K\big(n,k,L_\infty(T)\big) \leqslant K\big(n,k,L_\infty(R)\big),$$

as that is valid for any norm for which translation is a weakly* continuous isometry [Di,II], and $L_\infty(T)$ certainly is such a norm. The construction by Kolmogorov of the extremal functions is periodic and hence will fit $L_\infty(T)$, and this yields the converse inequality. For $1 \leqslant p < \infty$, we assume that f $(f \not\equiv 0)$ is near extremal for $K\big(n,k,L_p(R)\big)$. Assuming f satisfies $f \not\equiv 0$, we have $f^{(\ell)} \not\equiv 0$, as the only polynomial in $L_p(R)$ $(p < \infty)$ is zero. We can now find a constant A, $A \geqslant 1$ such that, for $D_A \equiv \{x;\ |x| > A\}$,

$$\|f^{(\ell)}\|_{L_p(D_A)} \leqslant \varepsilon \quad \text{for} \quad \ell = 0,\dots,n.$$

We now define g_A such that $g_A = 1$ for $|x| \leqslant A$, $g_A(x) = 0$ for $|x| \geqslant A+1$ and $\|g_A^{(\ell)}\|_{L_p(D_A)} \leqslant M$ for $\ell = 0,1,\dots,n$. We note that the construction can be such that M does not depend on A. Clearly, for ε sufficiently small, $F = f \cdot g$ is also near extremal for $K\big(n,k,L_p(R)\big)$. A change of variable would not change the constant, and hence we may assume that $A+1 < \pi$. F being a function in L_p with support in $(-\pi,\pi)$, it can be extended periodically (with period 2π). Hence, we have

$$K\big(n,k,L_p(R)\big) \leqslant K\big(n,k,L_p(T)\big).$$

Assume now that $f \in L_p(T)$ is near extremal for $K\big(n,k,L_p(T)\big)$. We construct the function $g = 1$ for $|x| \leqslant m\pi$ and $g(x) = 0$ for $|x| \geqslant (m+1)\pi$ with $\|g^{(\ell)}(x)\|_{L_p(|x| \geqslant m\pi)} \leqslant M$.

Clearly, $F = g \cdot f$ is in $L_p(R)$ and

$$\|F^{(\ell)}\|_{L_p[-m\pi,m\pi]} = m\|f^{(\ell)}\|_{L_p(T)}.$$

Choosing m big enough, F satisfies

$$\|F^{(k)}\|_{L_p(R)} \leqslant \big(K\big(n,p,L_p(T)\big) + \varepsilon_1\big)\, \|F^{(n)}\|_{L_p(R)}^{k/n} \|F\|_{L_p(R)}^{1-\frac{k}{n}},$$

and hence

$$K\big(n,p,L_p(T)\big) \leqslant K\big(n,p,L_p(R)\big).$$

□

For $B = L_p[0,\infty) \equiv L_p(R_+)$, one has

$$K\big(n,k,L_p(R_+)\big) \leqslant K\big(n,k,L_\infty(R_+)\big). \tag{1.4}$$

A method of calculating $K\big(n,k,L_\infty(R_+)\big)$ was described by Schönberg and Cavaretta [Sc-Ca]. For these constants $\big(K\big(n,k,L_\infty(R_+)\big)$, which are computable, one does not have a formula as in the case of $K\big(n,k,L_\infty(R)\big)$. It was also shown that $K\big(n,k,L_1(R_+)\big) \neq K\big(n,k,L_\infty(R_+)\big)$.

In fact, for a strongly continuous semi-group of operators, $T(t)$, on a Banach space, B, and an infinitesimal generator A, one has (see remark in [Di,I, p. 150])

$$\|A^k f\|_B \leqslant K\big(n,k,T(t),B\big)\|A^n f\|_B^{k/n}\|f\|_B^{1-\frac{k}{n}} \tag{1.5}$$

and

$$K\big(n,k,T(t),B\big) \leqslant K\big(n,k,L_\infty(R_+)\big). \tag{1.6}$$

If, in addition, $T(t)$ is a group of isometries,

$$K\big(n,k,T(t),B\big) \leqslant K\big(n,k,L_\infty(R)\big). \tag{1.7}$$

While I believe that to calculate exactly $K\big(n,k,L_p(R)\big)$ or $K\big(n,k,L_p(R_+)\big)$ is almost hopeless except for $p=1,2,$ and ∞, one could perhaps hope to show that

$$K\big(n,k,L_p(R)\big) = K\big(n,k,L_q(R)\big), \quad \frac{1}{p}+\frac{1}{q}=1. \tag{A}$$

We cannot replace R by R_+ in (A), as in this case (A) would not be valid.

2. Multivariate Landau-Kolmogorov inequality

There are several generalizations of (1.1) to the multivariate situation, and for obvious reasons I am partial to the following:

$$\Big\| \frac{\partial^k f}{\partial\xi_1 \ldots \partial\xi_k} \Big\|_{L_p(R^d)} \leqslant C(n,k,p)\, \|\Delta^n f\|_{L_p(R^d)}^{k/2n}\, \|f\|_{L_p(R^d)}^{1-\frac{k}{2n}} \tag{2.1}$$

where $0<k<2n$, $1\leqslant p\leqslant\infty$, Δ^n is the n-th iterate of the Laplacian and ξ_i are directions in R^d [Di,III]. It was shown also that

$$C(n,k,p) \leqslant C(n,k,\infty) \tag{2.2}$$

and Timofeev (see [Ti]) proved that

$$C(1,1,\infty)=\sqrt{2}\,. \tag{2.3}$$

Using a result of Chen and Ditzian [Ch-Di,I], one can replace the left hand of (2.1) by $\big\|\frac{\partial^k f}{\partial\xi^k}\big\|_{L_p(R^d)}$.

I conjecture that

$$C(n,k,\infty) = K\big(2n,k,L_\infty(R)\big). \tag{B}$$

(A counter example if (B) is not valid would be of interest as well.)

For $1<p<\infty$, (2.1) is valid for $0\leqslant k\leqslant 2n$, and the proof follows by combining

$$\Big\| \frac{\partial^{2\ell} f}{\partial\xi_1,\ldots,\partial\xi_{2\ell}} \Big\|_{L_p} \leqslant A_p\|\Delta^\ell f\|_{L_p}, \quad 1<p<\infty, \tag{2.4}$$

which can be found in E. Stein's book [StII, p. 77];

$$\Big\| \frac{\partial}{\partial \xi} f \Big\|_{L_p} \leqslant \sqrt{2}\, \|f\|_{L_p}^{1/2}\, \|\Delta f\|_{L_p}^{1/2}, \quad 1 \leqslant p \leqslant \infty, \tag{2.5}$$

which followed from [Ti]; and

$$\|\Delta^m f\|_{L_p} \leqslant K(r,m,\infty)\|\Delta^r f\|_{L_p}^{m/r} \cdot \|f\|_{L_p}^{1-\frac{m}{r}}, \quad 1 \leqslant p \leqslant \infty, \quad m < r, \tag{2.6}$$

which follows as Δ is the infinitesimal generator of the Gauss-Weierstrass operator. However, this method will yield very big constants (as p tends to ∞ or 1), and I believe that, following (2.2), $C(n,k,p)$ for $1 \leqslant p \leqslant \infty$ and $k < 2n$ is bounded by a constant independent of k, n, d and p.

REMARK 2.1. For $k \leqslant n$, $1 \leqslant p \leqslant \infty$, (2.1) follows from (2.5) and (2.6) by induction.

The inequality (2.1) is with the norm $L_p(R^d)$, and it is shown below to be valid with the same constants for $L_p(T^d)$ where T^d is the d-dimensional torus.

THEOREM 2.2. *For f, a 2π periodic function in d (orthogonal) directions, one has for $k < 2n$*

$$\Big\| \frac{\partial^k f}{\partial \xi^k} \Big\|_{L_p(T^d)} \leqslant C_T(n,k,p)\|\Delta^n f\|_{L_p(T^d)}^{k/2n}\, \|f\|_{L_p(T^d)}^{1-\frac{k}{2n}}, \tag{2.7}$$

and the best possible constant $C_T(n,k,p)$ satisfies

$$C_T(n,k,p) = C(n,k,p), \quad 1 \leqslant p \leqslant \infty. \tag{2.8}$$

REMARK 2.3. We did not mention differentiability as a condition in this and other theorems. The situation is that as a distribution a function f in L_p has derivatives, and if $\Delta^n f$ (obtained as a distribution) is in L_p so is the k-th derivative, and it satisfies the appropriate inequality. Note also that for brevity we use $\frac{\partial^k f}{\partial \xi^k}$ rather than $\frac{\partial^k f}{\partial \xi_1 \ldots \partial \xi_k}$ (see also [Ch-Di,I]).

Proof. For $1 \leqslant p < \infty$, the proof is actually a repetition of the proof of Theorem 1.1 with minor modifications. The same can be said on the inequality

$$C_T(n,k,\infty) \leqslant C(n,k,\infty)$$

as in $L_\infty(T^d)$ translation is a weak* continuous isometry. For the other direction, in case $p = \infty$, we cannot use the construction of the extremal function for the given n and k as the problem of extremal functions for $C(n,k,\infty)$ is still open for general n and k. Assume that $f \in C^{2n}(R^d)$ such that for some ξ

$$\Big\| \Big(\frac{\partial}{\partial \xi}\Big)^k f \Big\|_{L_\infty(R^d)} \geqslant \big(C(n,k,\infty) - \varepsilon\big)\, \|\Delta^n f\|_{L_\infty(R^d)}^{k/2n}\, \|f\|_{L_\infty(R^d)}^{1-\frac{k}{2n}}.$$

(It is clear that, if (2.1) is satisfied for $p = \infty$, there exists $f \in C^{2n}(R^d)$ $f \not\equiv 0$ satisfying the above.) We can find a cube $S(A) = [-A;A] \times [-A,A] \times \cdots \times [-A,A]$

such that the norms of $\left\|\left(\frac{\partial}{\partial\xi}\right)^k f\right\|_{L_\infty(R^d)}$, $\|\Delta^n f\|_{L_\infty(R^d)}$ and $\|f\|_{L_\infty(R^d)}$ are almost achieved in $S(A)$.

We define a function g satisfying $g=1$ for $\max|x_i|<A$ and $g=0$ when $\max|x_i|\geqslant A+m$. We can construct g choosing m big enough to have $\left\|\frac{\partial^\ell}{\partial\xi^\ell}g\right|_{L_\infty(R^d)}\leqslant \varepsilon$ for $\ell=1,\dots,n$ and all ξ.

The function $F=f\cdot g$ is of compact support and satisfies

$$\left\|\frac{\partial^k}{\partial\xi^k}F\right\|_{L_\infty(S(A+m))}\geqslant\big(C(n,k,\infty)-\varepsilon_1\big)\,\|\Delta^n F\|^{k/2n}_{L_\infty(S(A+m))}\,\|f\|^{1-\frac{k}{n}}_{L_\infty(S(A+m))}.$$

We note that change of variable $y_i=x_i/(A+m+1)$ will not change the inequality, but then the new function is supported by a set in the interior of T^d, and its periodic extension will satisfy the above. This implies

$$C_T(n,k,\infty)\geqslant C(n,k,\infty)-\varepsilon_1$$

which concludes the proof.

□

REMARK 2.4. In the above, we showed that a near extremal periodic solution for (2.1) in $L_\infty(R^d)$ exists. I believe that an extremal solution for $L_\infty(R^d)$ exists and is periodic. For $L_p(R^d)$, $p<\infty$, only near extremal solutions exist. For $C_T(n,k,p)$, $1\leqslant p\leqslant\infty$, I believe that extremal solutions exist. (Not just near extremal.)

THEOREM 2.5. *For* $0\leqslant k\leqslant 2n$,

$$\left\|\frac{\partial^k f}{\partial\xi_1\dots\partial\xi_k}\right\|_{L_2(R^d)}\leqslant\|\Delta^n f\|^{k/2n}_{L_2(R^d)}\,\|f\|^{1-\frac{k}{2n}}_{L_2(R^d)}. \tag{2.9}$$

Proof. For $0<k\leqslant n$, (2.9) follows from

$$\left\|\frac{\partial}{\partial\xi}f\right\|^2_{L_2}\leqslant\|\mathrm{grad}\,f\|^2_{L_2}\leqslant|\langle\Delta f,f\rangle|\leqslant\|\Delta f\|_{L_2}\cdot\|f\|_{L_2}, \tag{2.10}$$

$$\|\Delta f\|^2_{L_2}=|\langle\Delta^2 f,f\rangle|\leqslant\|\Delta^2 f\|_{L_2}\cdot\|f\|_{L_2}, \tag{2.11}$$

and mathematical induction. For $n\leqslant k\leqslant 2n$, it is somewhat more convenient to show that $C_T(n,k,2)=1$ and obtain (2.9) via Theorem 2.2. We will show that

$$\left\|\frac{\partial^2}{\partial\xi\partial\eta}f\right\|_{L_2(T^d)}\leqslant\|\Delta f\|_{L_2(T^d)} \tag{2.12}$$

for any two directions ξ and η. Using (2.12), commutativity of $\frac{\partial}{\partial\xi}$ with $\frac{\partial}{\partial\eta}$ and with Δ, (2.11) for $L_2(T^d)$ and mathematical induction, we complete the proof for even k. For odd k, we need (2.10) for $L_2(T^d)$ as well as the above. To prove (2.12), we recall that the functions $(2\pi)^{-d/2}e^{ikx}$, $x\in T^d$, $k\in Z^d$, constitute a complete orthonormal system. For f, $\Delta f\in L_2(T^d)$,

$$f \sim (2\pi)^{-d/2} \sum_{k \in Z^d} a_k e^{ikx},$$

$$\|f\|^2_{L_2(T^d)} = \sum_{k \in Z^d} a_k^2,$$

$$\|\Delta f\|^2_{L_2(T^d)} = \sum_{k \in Z^d} |k|^4 a_k^2.$$

We also have

$$\frac{\partial}{\partial \xi} = \sum \alpha_i \frac{\partial}{\partial x_i}, \qquad \frac{\partial}{\partial \eta} = \sum \beta_i \frac{\partial}{\partial x_i}, \qquad k = (k_1, \dots, k_d),$$

$$\frac{\partial}{\partial \xi} \frac{\partial}{\partial \eta} f \sim (2\pi)^{-d/2} \sum_{k \in Z^d} (\alpha, k)(\beta, k) e^{ikx}$$

and, as $|\alpha| = |\beta| = 1$,

$$|(\alpha, k)| \leqslant |k| \quad \text{and} \quad |(\beta, k)| \leqslant |k|.$$

Hence we have $\frac{\partial}{\partial \xi} \frac{\partial}{\partial \eta} f \in L_2(T^d)$ and (2.12).

□

3. Kolmogorov-type inequality for different operators

We already mentioned that the Kolmogorov-type inequality is valid for operators that are infinitesimal generators of strongly continuous semi-groups of operators. Recently in [Di,IV], I showed that self-adjoint operators with a sequence of eigenvalues and eigenfunctions, for which a certain order of Cesáro summability of the eigenfunctions converges to f in B, also satisfy a Kolmogorov-type inequality with respect to that space B. That is, for such $P(D)$ and B, one has

$$\|P(D)^k f\|_{L_p[-\pi,\pi]} \leqslant C\big(n, k, P(D), p\big) \, \|P(D)^n f\|^{k/n}_{L_p[-\pi,\pi]} \, \|f\|^{1-\frac{k}{n}}_{L_p[-\pi,\pi]}. \tag{3.1}$$

I will not get into details here. I note that $P_1(D)f = \widetilde{f}\,'$ (the derivative of the harmonic conjugate) does satisfy the conditions, and hence (3.1) is valid with $P_1(D) = P(D)$.

I conjecture that, for $P_1(D)f = \widetilde{f}\,'$,

$$C\big(n, k, P_1(D), \infty\big) = K(n, k) \tag{C}$$

where $K(n, k)$ are Kolmogorov's constants. (For n and k even, (C) reduces to the classic Kolmogorov-type inequality.)

The inequality (3.1) is valid for operators like $\frac{d}{dx}\,(1 - x^2)\,\frac{d}{dx}$ in $L_p[-1, 1]$ and for Jacobi operators in appropriately weighted L_p spaces (see [Ch-Di,II] and [Di,IV]). The best constants are not known except for the simple case of $L_2[-1,1]$ or weighted L_2 for Jacobi weights where the best constant are all equal to 1. The identity

$$\langle P(D)f, P(D)f \rangle = \langle P(D)^2 f, f \rangle,$$

the Cauchy-Schwartz inequality and mathematical induction together will imply that the best constants in L_2 are all equal to 1 in the cases mentioned above.

4. Local supremum for the *k*-th derivative

When dealing with $L_\infty(R^d)$ or $L_\infty(T^d)$, (2.1) can clearly be replaced by

$$\Big\| \sup_\xi \Big| \frac{\partial^k}{\partial \xi^k} f(x) \Big| \Big\|_{L_\infty} \leqslant C(n,k,\infty) \, \|\Delta^n f\|_{L_\infty}^{k/2n} \, \|f\|_{L_\infty}^{1-\frac{k}{2n}}. \tag{4.1}$$

We will show below that (4.1) is valid also for $L_p(R^d)$ or $L_p(T^d)$, $1 \leqslant p < \infty$ (with a bigger constant). The statement and proof will pertain to $L_p(T^d)$, however the inequality is valid for $L_p(R^d)$ as well.

THEOREM 4.1. *Suppose* $f \in L_p(T^d)$, $1 \leqslant p \leqslant \infty$, *and* $\Delta^n f$ *defined in the distributional sense satisfies* $\Delta^n f \in L_p(T^d)$. *Then, for* $0 < k < 2n$,

$$\Big\| \sup_\xi \Big| \frac{\partial^k}{\partial \xi^k} f(x) \Big| \Big\|_{L_p(T^d)} \leqslant C \|\Delta^n f\|_{L_p(T^d)}^{k/2n} \, \|f\|_{L_p(T^d)}^{1-\frac{k}{2n}} \tag{4.2}$$

where C *is independent of* f.

REMARKS 4.2. (a) It was shown earlier that $(\frac{\partial}{\partial \xi})^k f(x)$ is in $L_p(T^d)$. The present result shows that $\sup_\xi |(\frac{\partial}{\partial \xi})^k f(x)|$ is finite a.e. and is in $L_p(T^d)$.

(b) For $p = 2$, $k = 1$ and $n = 1$, the constant C is equal to 1, using $|\mathrm{grad} f(x)|$ for $\sup_\xi |f(x)|$.

(c) The proof below yields constants that are bigger for $p < \infty$ and we do not automatically get $C(n,k,\infty)$ as an upper bound. It would be interesting to find out if they are, in fact, bigger than $C(n,k,\infty)$ for some $p < \infty$. (I suspect not.) It would be interesting to find out if, for $p = 2$ and other n and k (see (b)), the constants are still all equal to one. (For $k \leqslant n$ it is so.)

Proof. If $f \in C^{2n}(T^d)$, we can write, using [Di-Iv],

$$\max_\xi \Big| \frac{\partial^k}{\partial \xi^k} f(x) \Big| \leqslant A \max_{\xi_i \in S} \Big| \frac{\partial^k}{\partial \xi_i^k} f(x) \Big|, \tag{4.3}$$

where S is a finite set of ξ_i independent of x and f (consisting of $\binom{k+d-1}{k}$ vectors ξ_i which are (k,d) independent in the terminology of [Di-Iv]), and A can be chosen independently of f, ξ and x. As (2.1) is valid, we just increase the constant by the multiplicative factor A. To prove the general case, we have to take a little more care and it can be done in the following manner (for instance). Let T_m be the best approximant to f in L_p by a trigonometric polynomial in τ_m given by

$$\tau_m = \text{ span}\left\{e^{ik\cdot x} : |k| \equiv \Big(\sum_{i=1}^{d} |k_i|^2\Big)^{1/2} \leqslant m\right\}. \tag{4.4}$$

It was shown in [Ch-Di] that if $\Delta^n f \in L_p(T^d)$ (with Δ^n defined as a distribution), then

$$E_m(f)_p = \|f - T_m\|_{L_p(T^d)} \leqslant Cm^{-2n}\|\Delta^n f\|_{L_p(T^d)}. \tag{4.5}$$

We can now write

$$f = T_{2^\ell} + \sum_{j=1}^{\infty}\Big(T_{2^{\ell+j}} - T_{2^{\ell+j-1}}\Big) \tag{4.6}$$

which, using (4.5), converges nicely for any ℓ, and we will choose ℓ later.

We now have

$$\Big\| \sup_{\xi}\Big|\frac{\partial^k f}{\partial\xi^k}(x)\Big|\,\Big\|_p \leqslant \Big\|\max_{\xi}\Big|\frac{\partial^k}{\partial\xi^k}T_{2^\ell}(x)\Big|\,\Big\|_p + \sum_{j=1}^{\infty}\Big\|\max_{\xi}\Big|\frac{\partial^k}{\partial\xi^k}\Big(T_{2^{\ell+j}} - T_{2^{\ell+j-1}}\Big)(x)\Big|\,\Big\|_p$$

where $\|\cdot\|_{L_p(T^d)} \equiv \|\quad\|_p$.

We now use (4.3) to obtain

$$\Big\|\sup_{\xi}\Big|\frac{\partial^k}{\partial\xi^k}f(x)\Big|\,\Big\|_p \leqslant A\max_{\xi_i\in S}\Big[\Big\|\frac{\partial^k}{\partial\xi_i^k}T_{2^\ell}(x)\Big\|_p + \sum_{j=1}^{\infty}\Big\|\frac{\partial^k}{\partial\xi_i^k}\Big(T_{2^{\ell+j}} - T_{2^{\ell+j-1}}\Big)(x)\Big\|_p\Big]$$

(and using the Bernstein inequality)

$$\leqslant AB\Big[2^{\ell k}\Big\|T_{2^\ell}\Big\|_p + \sum_{j=1}^{\infty}2^{(\ell+j)k}\Big\|T_{2^{\ell+j}} - T_{2^{\ell+j-1}}\Big\|_p\Big]$$

(and using (4.5))

$$\leqslant AB\Big[2^{\ell k}\Big\|T_{2^\ell}\Big\|_p + 2C\sum_{j=1}^{\infty}2^{(\ell+j)k}2^{-(\ell+j-1)2n}\|\Delta^n f\|_p\Big]$$

(and using $\|T_{2^\ell}\|_p \leqslant 2\|f\|_p$ and geometric series)

$$\leqslant C_1\big[2^{\ell k}\|f\|_p + 2^{-\ell(2n-k)}\|\Delta^n f\|_p\big].$$

We now choose ℓ. If $\|\Delta^n f\|_p / \|f\|_p \geqslant 1$, we choose ℓ such that

$$2^\ell \leqslant \left(\frac{\|\Delta^n f\|_p}{\|f\|_p}\right)^{1/2n} \leqslant 2^{\ell+1},$$

and hence

$$\left\| \sup_\xi \left| \frac{\partial^k}{\partial \xi^k} f(x) \right| \right\|_p \leqslant C_1 \|\Delta^n f\|_p^{k/2n} \|f\|^{1-\frac{k}{n}} \left(1 + 2^{(2n-k)}\right).$$

For the case $\|\Delta^n f\|_p / \|f\|_p < 1$, we write $f = M + f - M$ where

$$\|f - M\|_p = \inf_C \|f - C\|_p.$$

Following [Di,IV] and the above,

$$\|f - M\|_p \leqslant C_2 \|\Delta^n f\|_p$$

but

$$\frac{\partial^k f}{\partial \xi^k} = \frac{\partial^k}{\partial \xi^k} (f - M),$$

and hence if $\|\Delta^n f\|_p \leqslant \|f\|_p$ we use the above (recall $k < 2n$) to obtain

$$\begin{aligned} \left\| \sup_\xi \left| \frac{\partial^k}{\partial \xi^k} f \right| \right\|_p &= \left\| \sup_\xi \left| \frac{\partial^k}{\partial \xi^k} (f - M) \right| \right\|_p \\ &\leqslant C_3 \|\Delta^n f\|_p \\ &\leqslant C_3 \|\Delta^n f\|_p^{k/2n} \|f\|_p^{1-\frac{k}{2n}}. \end{aligned}$$

□

We note that this proof does not use [Di,III] or earlier sections. It follows the technique (but not exactly the proof) in [Di,IV]. The present proof has the advantage that it facilitates generalizations with greater ease.

5. Kolmogorov-type inequality does not hold for L_p, $0 < p < 1$

I am sure that some were wondering if Kolmogorov inequality holds for $L_p(T)$ or $L_p(R)$, $0 < p < 1$. After all, the Bernstein inequality and the Jackson inequality hold for $L_p(T)$ when $0 < p < 1$. We will show here that, even for $k = 1$ and $n = 2$, there is no constant C_p, $0 < p < 1$, for which

$$\|f'\|_{L_p(T)} \leqslant C_p \|f''\|_{L_p(T)}^{1/2} \|f\|_{L_p(T)}^{1/2}.$$

Recall that the quasinorm $\|f\|_{L_p(T)}$ is given by

$$\|f\|_{L_p(T)} = \left(\int_{-\pi}^{\pi} |f(x)|^p dx\right)^{1/p}.$$

Define f by $f(0)=0$, $f'(x)=-f'(-x)$, $f\left(\frac{\pi}{2}+y\right)=f\left(\frac{\pi}{2}-y\right)$ and

$$f'(x)=\begin{cases} \frac{j-1}{N} & \text{for } \frac{\pi}{2}\frac{j-1}{N}\leqslant x\leqslant \frac{\pi j}{2N}-\frac{1}{N^2} \\ \frac{j-1}{N}+N\left(x-\frac{\pi j}{2N}+\frac{1}{N^2}\right) & \text{for } \frac{\pi j}{2N}-\frac{1}{N^2}\leqslant x\leqslant \frac{\pi j}{2N} \end{cases}$$

for $1\leqslant j\leqslant N$. Clearly, we have a periodic function which satisfies, for some $0<A$, $B<\infty$, $\|f'\|_{L_p}\sim A$ and $\|f\|_{L_p}\sim B$, as $N\gg 1$.

However,

$$\|f''\|_{L_p}=O\left(N^{(p-1)/p}\right)$$

which tends to zero as $N\to\infty$. Similar examples can be produced to show that a Kolmogorov-type inequality does not hold for $L_p(T)$, $0<p<1$, for any k and n $(0<k<n)$.

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