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SHARP INTEGRAL INEQUALITIES FOR C–MONOTONE FUNCTIONS OF SEVERAL VARIABLES

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(*communicated by C. E. M. Pearce*)

Abstract. Some sharp integral inequalities for C-monotone functions of several variables are proved. All cases of equality are found and some related results are pointed out and discussed.

1. Introduction

Let $-\infty < a < b \leqslant \infty$ and let f be a positive and integrable function on $[a,b]$. First we recall the inequality

$$\left(\int_a^b f^q(x)(x-a)^{q-1}dx\right)^{\frac{p}{q}} \leqslant pq^{\frac{-p}{q}} \int_a^b f^p(x)(x-a)^{p-1}dx,\ 0<p\leqslant q<\infty, \quad (1.1)$$

which holds for every decreasing function f. Here and in the sequel decreasing means non-increasing and increasing means non-decreasing.

The inequality (1.1) is sharp and equality occurs for every function of the type $f(x) = A\chi_{[a,t]}(x), a \leqslant t \leqslant b$ (χ denotes the characteristic function and A any positive constant) and (1.1) holds in the reversed direction if f is increasing. The inequality (1.1) was probably first discovered by Lorentz [9, p.39] c.f. also [7, p.100]. Various proofs and extensions can be found in [2], [3], [4], [9], [11], [13], [14], [15], and [17]. Moreover, also the analogous inequality

$$\left(\int_a^b f^q(x)(b-x)^{q-1}dx\right)^{\frac{p}{q}} \leqslant pq^{\frac{-p}{q}} \int_a^b f^p(x)(b-x)^{p-1}dx, 0<p\leqslant q<\infty, b<\infty, \quad (1.2)$$

holds for every increasing function f. Also (1.2) is sharp and equality occurs for $f(x) = C\chi_{[t,b]}(x), a \leqslant t \leqslant b$, and (1.2) holds in the reversed direction if f is decreasing. See e.g. [5], [6], [9], [13], [14], [15] and [18] for some different proofs and extensions.

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Of course, (1.1) and/or (1.2) can not hold in general (because they are a type of reversed Hölder inequalities). However, some variants of (1.1) and (1.2) do hold for more general functions. In particular, some generalizations of this type were derived in [15] for C-monotone functions and the sharpness of these inequalities were proved and analyzed in [16].

We recall from [16] that a function f is said to be C-decreasing [C-increasing], $C > 0$, if $f(t) \leqslant Cf(s)$ $[f(s) \leqslant Cf(t)]$ whenever $s \leqslant t,\ s,t \in (a,b)$. Moreover, f is said to be C-decreasing in mean relatively to g, where g is increasing, $g(a) = 0$ [g is decreasing and $g(b) = 0$] if, for all $x \in (a,b)$,

$$f(x)g(x) \leqslant C\int_a^x f(t)dg(t) \quad \left[Cf(x)g(x) \geqslant \int_x^b f(t)d\left[-g(t)\right]\right].$$

f is C-increasing in mean relatively to g, where g is increasing and $g(a) = 0$ [g is decreasing and $g(b) = 0$] if, for all $x \in (a,b)$,

$$Cf(x)g(x) \geqslant \int_a^x f(t)dg(t) \quad \left[f(x)g(x) \leqslant C\int_x^b f(t)d\left[-g(t)\right]\right].$$

Some examples and illustrations concerning these classes of functions were presented in [15]-[16]. In particular, we note that if f is C-increasing [C-decreasing], then f is C-increasing [C-decreasing] in mean with respect to any g of the type considered.

Moreover, in [1], and [14] some multidimensional versions of (1.1) and (1.2) have recently been proved and applied. In this paper we generalize and unify some of the results from [14]-[16] by proving some sharp multidimensional integral inequalities of the type (1.1)-(1.2) and the cases of equality are pointed out. The main results in this paper are stated and proved in Section 2. Some complementary results, examples and concluding remarks can be found in Section 3. In order not to disturb our discussions later on we finish this Section by stating some necessary notations and conventions:

Let $m \in \mathbb{N}$, $\mathbf{a} = (a_1, a_2, \ldots, a_m)$, $\mathbf{b} = (b_1, b_2, \ldots, b_m)$ and $\mathbf{x} = (x_1, x_2, \ldots, x_m) \in \mathbb{R}^m$. We assume that $-\infty < \mathbf{a} \leqslant \mathbf{b} \leqslant \infty$. The notation $\mathbf{a} < \mathbf{b}$, $(\mathbf{a} \leqslant \mathbf{b})$ means that $a_i < b_i$, $(a_i \leqslant b_i)$ for $i = 1, 2, \ldots, m$. We also use the (simplex) notation: if $\mathbf{a} = (a_1, a_2, \ldots, a_m)$, then $\mathbf{a}_i = (a_1, \ldots, a_{i-1}, a_{i+1}, \ldots, a_m)$, $i = 1, 2, \ldots, m-1$ and $\mathbf{a}_m = (a_1, a_2, \ldots, a_{m-1})$. Moreover $(\mathbf{a}, \mathbf{b}) = \{\mathbf{x} : a_i < x_i < b_i,\ i = 1, 2, \ldots, m\}$ and $d\mathbf{x} = dx_1 dx_2 \ldots\ldots dx_m$.

The functions f considered in this paper are assumed to be measurable and nonnegative. We also consider $\mathbf{g} = (g_1, g_2, \ldots, g_m)$, where $g_i = g_i(x_i)$ are nonnegative and differentiable for $i = 1, 2, \ldots, m$. Also put $d\mathbf{g}(\mathbf{x}) = dg_1(x_1) \ldots dg_m(x_m)$ and $d\left[\mathbf{g}^p(\mathbf{x})\right] = d\left[g_1^p(x_1)\right] \ldots d\left[g_m^p(x_m)\right]$, $p > 0$. Moreover, we say that $\mathbf{g}$ is increasing [decreasing] if $g_i, i = 1, 2, \ldots, m$, are increasing [decreasing].

DEFINITION 1.1. Let $\mathbf{C} = (C_1, C_2, \ldots, C_m), C_i > 0, i = 1, 2, \ldots, m.$ We say that $f(\mathbf{x})$ is $\mathbf{C}-$ decreasing in mean [$\mathbf{C}$ - increasing in mean] relatively to $\mathbf{g} = (g_1, g_2, \ldots, g_m)$, where $\mathbf{g}(\mathbf{a}) = 0$ and $\mathbf{g}$ is increasing, if, for all $\mathbf{x} \in (\mathbf{a}, \mathbf{b})$ and $i = 1, 2, \ldots, m$,

$$g_i(x_i) f(\mathbf{x}) \leqslant C_i \int_{a_i}^{x_i} f(x_1, \ldots, x_{i-1}, t_i, x_{i+1, \ldots,} x_m) dg_i(t_i)$$

$$\left[C_i g_i(x_i) f(\mathbf{x}) \geqslant \int_{a_i}^{x_i} f(x_1, \ldots, x_{i-1}, t_i, x_{i+1, \ldots,} x_m) dg_i(t_i) \right].$$

DEFINITION 1.2. Let $\mathbf{C} = (C_1, C_2, \ldots, C_m), C_i > 0, i = 1, 2, \ldots, m$. We say that $f(\mathbf{x})$ is $\mathbf{C}-$decreasing in mean [$\mathbf{C}-$ increasing in mean] relatively to $\mathbf{g} = (g_1, g_2, \ldots, g_m)$, where $g(\mathbf{b}) = 0$ and g_i is decreasing if, for all $\mathbf{x} \in (\mathbf{a}, \mathbf{b})$ and $i = 1, 2, \ldots, m$,

$$C_i g_i(x_i) f(\mathbf{x}) \geqslant \int_{x_i}^{b_i} f(x_1, \ldots, x_{i-1}, t_i, x_{i+1, \ldots,} x_m) d[-g_i(t_i)]$$

$$\left[g_i(x_i) f(\mathbf{x}) \leqslant C_i \int_{x_i}^{b_i} f(x_1, \ldots, x_{i-1}, t_i, x_{i+1, \ldots,} x_m) d[-g_i(t_i)] \right].$$

2. The main results

We first generalize (1.1) in the following way:

THEOREM 2.1. *Let* $m \in \mathbb{N}$ *and* $\mathbf{C} = (C_1, C_2, \ldots, C_m) \in \mathbb{R}_+^m$.

(a) *Suppose that* f *is* $\mathbf{C}$*-decreasing in mean relatively to* $\mathbf{g}$, *where* $\mathbf{g}$ *is increasing and* $\mathbf{g}(\mathbf{a}) = 0$. *Then, for any* $p \in (0, 1]$,

$$\int_{\mathbf{a}}^{\mathbf{b}} \mathbf{f}(\mathbf{x}) d\mathbf{g}(\mathbf{x}) \leqslant \left(\prod_{i=1}^{m} C_i^{(1-p)/p} \right) \left(\int_{\mathbf{a}}^{\mathbf{b}} \mathbf{f}^p(\mathbf{x}) d[\mathbf{g}^p(\mathbf{x})] \right)^{\frac{1}{p}}. \tag{2.1}$$

(b) *The inequality (2.1) is sharp and it reduces to an equality for every function of the type*

$$f(x_1, x_2, \ldots, x_m) = K \prod_{i=1}^{m} (g_i(x_i))^{C_i - 1}. \tag{2.2}$$

where $K \geqslant 0$. *If, in addition,* $f > 0$, *and* $\mathbf{g}$ *is strictly increasing, then equality holds in* (2.1) *if and only if* f *is of the type* (2.2).

REMARK 2.1. Our proof of Theorem 2.1 shows that if $p \geqslant 1$, then (2.1) holds in the reversed direction.

Proof. (2.1) is trivial for $p = 1$ so we assume that $p \in (0, 1)$.

(a) We prove (2.1) by using induction in the dimension m. Let $m = 1$, $x \in (a,b)$ and consider

$$h(x) = \left(\int_a^x f(t)dg(t)\right)^p - C_1^{1-p}\int_a^x f^p(t)d\left[g^p(t)\right].$$

Then

$$h'(x) = pf(x)g'(x)\left(\left(\int_a^x f(t)dg(t)\right)^{p-1} - C_1^{1-p}\left(f(x)g(x)\right)^{p-1}\right).$$

Moreover, since f is C_1- decreasing in mean relatively to g and $p \in (0,1)$ we have

$$\left(\int_a^x f(t)dg(t)\right)^{p-1} \leqslant C_1^{1-p}\left(f(x)g(x)\right)^{p-1}.$$

Thus $h'(x) \leqslant 0$, i.e., $h(x)$ is decreasing so that $h(b) \leqslant h(a) = 0$, which means that the inequality (2.1) is true for $m = 1$. We now assume that (2.1) is true for $n = m-1$ and prove that holds for $n = m$. Consider the function

$$F(\mathbf{x}) = \left(\int_{\mathbf{a}}^{\mathbf{x}} \mathbf{f}(\mathbf{t})d\mathbf{g}(\mathbf{t})\right)^p - \left(\prod_{i=1}^m C_i^{1-p}\right)\left(\int_{\mathbf{a}}^{\mathbf{x}} \mathbf{f}^p(\mathbf{t})d\left[\,\mathbf{g}^p(\mathbf{t})\right]\right).$$

We will prove that $\frac{\partial F}{\partial x_i} \leqslant 0$ for $i = 1,2,\ldots,m$. In view of symmetry we see that it is sufficient to prove that $\frac{\partial F}{\partial x_i} \leqslant 0$ for any i and we choose to prove that $\frac{\partial F(\mathbf{x}_m, x_m)}{\partial x_m} \leqslant 0$ when $a_m < x_m < b_m$. Since f is C_m-decreasing in mean relatively to g_m and $p \in (0,1)$ we have

$$\left(\int_{\mathbf{a}}^{\mathbf{x}} f(\mathbf{t})d\mathbf{g}(\mathbf{t})\right)^{p-1} \leqslant \left(\frac{1}{C_m}g_m(x_m)\int_{\mathbf{a}_m}^{\mathbf{x}_m} f(\mathbf{t}_m, x_m)d\mathbf{g}_m(\mathbf{t}_m)\right)^{p-1}. \tag{2.3}$$

By differentiating with respect to x_m we find that

$$\frac{\partial F(\mathbf{x}_m, x_m)}{\partial x_m} = p\left(\int_{\mathbf{a}}^{\mathbf{x}} f(\mathbf{t})d\mathbf{g}(\mathbf{t})\right)^{p-1}\int_{\mathbf{a}_m}^{\mathbf{x}_m} f(\mathbf{t}_m, x_m)d\mathbf{g}_{\mathbf{m}}(\mathbf{t}_{\mathbf{m}})g_m'(x_m)$$

$$-\left(\prod_{i=1}^m C_i^{1-p}\right)\int_{\mathbf{a}_m}^{\mathbf{x}_m} f^p(\mathbf{t}_{\mathbf{m}}, x_m)d\left[\mathbf{g}_m^p(\mathbf{t}_m)\right]p(g_m(x_m))^{p-1}g_m'(x_m).$$

Therefore, according to (2.3) and the induction assumption,

$$\frac{\partial F(\mathbf{x}_m, x_m)}{\partial x_m} \leqslant pC_m^{1-p}g_m'(x_m)(g_m(x_m))^{p-1}\left\{\left(\int_{\mathbf{a}_m}^{\mathbf{x}_m} f(\mathbf{t}_m, x_m)d\mathbf{g}_m(\mathbf{t}_m)\right)^p\right.$$

$$-\left(\prod_{i=1}^{m-1} C_i^{1-p}\right)\int_{\mathbf{a}_m}^{\mathbf{x}_m} f^p(\mathbf{t}_m, x_m)d\left[\mathbf{g}_m^p(\mathbf{t}_m)\right]\Bigg\}\leqslant 0.$$

We conclude that $F(\mathbf{b})\leqslant F(\mathbf{a})=0,$ which means that (2.1) holds for $n=m$ and, thus, in view of the induction axiom , that (2.1) holds for all $m\in\mathbb{N}$.

(b) First we note that by inserting

$$f(x_1,x_2,\ldots,x_m)=Kg_1(x_1)^{C_1-1}g_2(x_2)^{C_2-1}\ldots g_m(x_m)^{C_m-1}$$

into (2.1) an elementary calculation shows that (2.1) reduces to an equality (i.e. $F(\mathbf{x})\equiv 0$).We will now prove that this is the only possibility to have equality in (2.1). We consider the function $F(\mathbf{x})$ defined in (a). In (a) we proved that $\frac{\partial F}{\partial x_i}\leqslant 0$ for $i=1,2,\ldots,m,$ and we can have equality if and only if $\frac{\partial F}{\partial x_i}\equiv 0$ for $i=1,2,\ldots,m.$ Now we note that $\frac{\partial F}{\partial x_1}=0$ if and only if

$$g_1(x_1)^{p-1}\left(\prod_{i=1}^{m} C_i^{1-p}\right)\int_{\mathbf{a}_1}^{\mathbf{x}_1} f^p(x_1,\mathbf{t}_1)d\mathbf{g}_1^p(\mathbf{t}_1)=\left(\int_{\mathbf{a}}^{\mathbf{x}} f(\mathbf{t})d\mathbf{g}(\mathbf{t})\right)^{p-1}\int_{\mathbf{a}_1}^{\mathbf{x}_1} f(x_1,\mathbf{t}_1)d\mathbf{g}_1(\mathbf{t}_1). \tag{2.4}$$

Moreover, since f is C $_1-$ decreasing in mean relatively to $g_1(x_1)$ we have

$$g_1(x_1)\int_{\mathbf{a}_1}^{\mathbf{x}_1} f(x_1,\mathbf{t}_1)d\mathbf{g}_1(\mathbf{t}_1)\leqslant C_1\int_{\mathbf{a}}^{\mathbf{x}} f(\mathbf{t})d\mathbf{g}(\mathbf{t}) \tag{2.5}$$

and a $m-1-$ dimensional version of (2.1) reads

$$\left(\int_{\mathbf{a}_1}^{\mathbf{x}_1} f(x_1,\mathbf{t}_1)d\mathbf{g}_1(\mathbf{t}_1)\right)^p\leqslant\left(\prod_{i=2}^{m} C_i^{1-p}\right)\int_{\mathbf{a}_1}^{\mathbf{x}_1} f^p(x_1,\mathbf{t}_1)d\mathbf{g}_1^p(\mathbf{t}_1). \tag{2.6}$$

By comparing (2.4)-(2.6) we see that (2.4) can hold if and only if we have equality in both (2.5) and (2.6).Therefore we now assume that we have equality in (2.6) and differentiate (2.6) with respect to x_2 , i.e.,

$$g_2(x_2)^{p-1}\left(\prod_{i=2}^{m} C_i^{1-p}\right)\int_{a_3}^{x_3}\cdots\int_{a_m}^{x_m} f^p(x_1,x_2,t_3,\ldots,t_m)dg_3^p(t_3)\ldots dg_m^p(t_m)$$

$$=\left(\int_{\mathbf{a}_1}^{\mathbf{x}_1} f(x_1,\mathbf{t}_1)d\mathbf{g}_1(\mathbf{t}_1)\right)^{p-1}\int_{a_3}^{x_3}\cdots\int_{a_m}^{x_m} f(x_1,x_2,t_3,\ldots,t_m)dg_3(t_3)\ldots dg_m(t_m). \tag{2.7}$$

Furthermore, since f is C_2- decreasing in mean relatively $g_2(x_2)$,

$$g_2(x_2)\int_{a_3}^{x_3}\cdots\cdot\int_{a_m}^{x_m} f(x_1,x_2,t_3,\ldots,t_m)dg_3(t_3)\ldots dg_m(t_m)\leqslant C_2\int_{\mathbf{a_1}}^{\mathbf{x_1}} f(x_1,\mathbf{t}_1)d\mathbf{g_1}(\mathbf{t_1}) \tag{2.8}$$

and the following $(m-2)-$ dimensional equality version of (2.1) holds:

$$\left(\int_{a_3}^{x_3} \dots \int_{a_m}^{x_m} f(x_1,x_2,t_3,\dots,t_m)dg_3(t_3)\dots dg_m(t_m)\right)^p$$

$$\leqslant \left(\prod_{i=3}^{m} C_i^{1-p}\right) \int_{a_3}^{x_3} \dots \int_{a_m}^{x_m} f^p(x_1,x_2,t_3,\dots,t_m)dg_3^p(t_3)\dots dg_m^p(t_m). \qquad (2.9)$$

We proceed in the same way and by induction we finally find that

$$\left(\int_{a_m}^{x_m} f(x_1,x_2,\dots,x_{m-1},t_m)dg_m(t_m)\right)^p = C_m^{1-p}\int_{a_m}^{x_m} f^p(x_1,x_2,\dots,x_{m-1},t_m)dg_m^p(t_m). \qquad (2.10)$$

By differentiating (2.10) with respect to x_m we obtain that

$$\left(\int_{a_m}^{x_m} f(x_1,x_2,\dots,x_{m-1},t_m)dg_m(t_m)\right)^{p-1} f(x_1,x_2,\dots,x_{m-1},x_m)$$

$$= c_m^{1-p} f^p(x_1,x_2,\dots,x_{m-1},x_m)g_m(x_m)^{p-1}$$

so that

$$\int_{a_m}^{x_m} f(x_1,x_2,\dots,x_{m-1},t_m)dg_m(t_m) = \frac{1}{C_m} f(x_1,x_2,\dots,x_{m-1},x_m)g_m(x_m).$$

By differentiating again with respect to x_m we end up with the following differential equation:

$$(C_m-1)f(x_1,x_2,\dots,x_{m-1},x_m)g_m'(x_m) = f_m'(x_1,x_2,\dots,x_{m-1},x_m)g_m(x_m). \qquad (2.11)$$

The solution of (2.11) is $f(x_1,x_2,\dots,x_{m-1},x_m) = K_o g_m(x_m)^{c_m-1}$, where K_o may depend on $x_1,x_2,\dots,x_{m-1,}$ i.e., K_o is of the type $K_o = h_{m-1}(x_1,x_2,\dots,x_{m-1).}$ By plugging this extremal function into the equality version of (2.1) we find that

$$\left(\prod_{i=1}^{m-1} C_i^{1-p}\right)\int_{\mathbf{a}_m}^{\mathbf{x}_m} h_{m-1}^p(t_1,\dots,t_{m-1})d\mathbf{g}_m^p(\mathbf{t}_m)\frac{g_m(x_m)^{pC_m}}{C_m^p}$$

$$= \left(\int_{\mathbf{a}_m}^{\mathbf{x}_m} h_{m-1}(t_1,\dots,t_{m-1})d\mathbf{g}_m(\mathbf{t}_m)\right)^p \frac{g_m(x_m)^{pC_m}}{C_m^p}.$$

By now repeating our procedure we successively obtain that

$$h_{m-1}(x_1,\dots,x_{m-1}) = h_{m-2}(x_1,\dots,x_{m-2})g_{m-1}(x_{m-1})^{C_{m-1}-1},$$

$$\vdots$$

$$h_1(x_1) = Kg_1(x_1)^{C_1-1}.$$

We conclude that

$$\begin{aligned} f(x_1,\ldots,x_m) &= h_{m-1}(x_1,\ldots,x_{m-1})g_m(x_m)^{C_m-1} \\ &= h_{m-2}(x_1,\ldots,x_{m-2})g_{m-1}(x_{m-1})^{C_{m-1}-1}g_m(x_m)^{C_m-1} \\ &= Kg_1(x_1)^{C_1-1}\ldots g_{m-1}(x_{m-1})^{C_{m-1}-1}g_m(x_m)^{C_m-1}. \end{aligned}$$

The proof is complete. □

We also state the corresponding generalization of (1.2):

THEOREM 2.2. *Let* $m \in \mathbb{N}$ *and* $\mathbf{C} = (C_1, C_2, \ldots, C_m) \in \mathbb{R}_+^m$.

(a) *Assume that* f *is* $\mathbf{C}-$ *increasing in mean relatively to* $\mathbf{g}$ *where* $\mathbf{g}$ *is decreasing and* $\mathbf{g}(\mathbf{b}) = 0$. *Then, for any* $p \in (0,1]$,

$$\int_{\mathbf{a}}^{\mathbf{b}} f(\mathbf{x})d\left[-\mathbf{g}(\mathbf{x})\right] \leqslant \left(\prod_{i=1}^{m} C_i^{(1-p)/p}\right)\left(\int_{\mathbf{a}}^{\mathbf{b}} f^p(\mathbf{x})d\left[-\mathbf{g}^p(\mathbf{x})\right]\right)^{\frac{1}{p}}. \tag{2.12}$$

(b) *The inequality* (2.12) *is sharp and it reduces to an equality for every function* f *of the type*

$$f(x_1,x_2,\ldots,x_m) = K\prod_{i=1}^{m}(g_i(x_i))^{C_i-1}.$$

where $K \geqslant 0$. *If, in addition,* $f > 0$ *and* $\mathbf{g}$ *is strictly decreasing, then equality holds in* (2.12) *if and only if* f *is of this type.*

Proof. The proof of Theorem 2.2 is similar to that of Theorem 2.1 so we leave out the details. Moreover, we note that for the case when all $b_i < \infty$ Theorems 2.1 and 2.2 can obviously be directly deduced from each other by making an obvious transformation $(x_i \to a_i + b_i - x_i,\ i = 1,2,\ldots,m)$. □

REMARK 2.2. If $p \geqslant 1$, then (2.12) holds in the reversed direction.

3. Further results and remarks

The concept of C- monotone in mean can be generalized to the multidimensional case in various ways. Instead of making the (”local”) version we have done in Definition 1.1 the following (”global”) one can be a reasonable alternative:

DEFINITION 3.1. Let $C > 0$. We say that $f(\mathbf{x})$ is C-decreasing in mean [C-increasing in mean] relatively to $\mathbf{g}$, where $\mathbf{g}(\mathbf{a}) = 0$ and $\mathbf{g}$ is differentiable and increasing, if, for all $\mathbf{x} \in (\mathbf{a},\mathbf{b})$,

$$\left(\prod_{i=1}^{m} g_i(x_i)\right) f(\mathbf{x}) \leqslant C\int_{\mathbf{a}}^{\mathbf{x}} f(\mathbf{t})d\mathbf{g}(\mathbf{t}) \quad \left[C\left(\prod_{i=1}^{m} g_i(x_i)\right) f(\mathbf{x}) \geqslant \int_{\mathbf{a}}^{\mathbf{x}} f(\mathbf{t})d\mathbf{g}(\mathbf{t})\right].$$

REMARK 3.1. We note that $f(\mathbf{x})$ is C-decreasing or (C,C, ... ,C)-decreasing in mean [C-increasing or (C,C, ... ,C)-increasing in mean] relatively to any $\mathbf{g}$ e.g. if $f(\mathbf{x})$ is C-decreasing [C-increasing] in the usual sense, i.e. , if

$$f(\mathbf{x}) \leqslant Cf(\mathbf{t}) \quad [f(\mathbf{t}) \leqslant Cf(\mathbf{x})]$$

for all $\mathbf{t} \leqslant \mathbf{x}$. (Of course 1-decreasing or $\mathbf{1}$ -decreasing [1-increasing or $\mathbf{1}$-increasing] means decreasing [increasing] in the usual sense.)

REMARK 3.2. We see that if f is $\mathbf{C}$-decreasing in mean relatively to $\mathbf{g}$ ($\mathbf{C} = (C_1, C_2, \dots, C_m)$), then f is C-decreasing in mean relatively to $\mathbf{g}$, where $C = \prod_{i=1}^{m} C_i$.

We also complement Remarks 3.1 and 3.2 with the following examples:

EXAMPLE 3.1. Let $f_1(x_1,x_2) = K_1 x_1^{C-1} + K_2 x_2^{C-1}, g_1(x_1) = x_1$ and $g_2(x_2) = x_2, 0 \leqslant x_1, x_2 \leqslant 1$ and $C \geqslant 1$. An elementary calculation shows that $f_1(x_1,x_2)$ is C-decreasing (even C-constant) with respect to (g_1,g_2). Moreover $f_1(x_1,x_2)$ is also (C,C)-decreasing with respect to (g_1,g_2).

EXAMPLE 3.2. Let $f_2(x_1,x_2) = x_1^{C_1-1} x_2^{C_2-1}, g_1(x_1) = x_1$ and $g_2(x_2) = x_2, 0 \leqslant x_1, x_2 \leqslant 1$ and $C_1, C_2 \geqslant 1$. Then obviously $f_2(x_1,x_2)$ is both C_1C_2- decreasing (even C_1C_2- constant) and (C_1,C_2)-decreasing (even (C_1,C_2)-constant) with respect to (g_1,g_2).

It seems not to be easy to prove an inequality similar to that in Theorem 2.1 for C-decreasing functions. However, if we add the condition that f is increasing this result follows even from our Theorem 2.1.

PROPOSITION 3.1. *Let $m \in \mathbb{N}$, $C > 1$, and $\mathbf{g}$ is differentiable and increasing and $\mathbf{g}(\mathbf{a}) = 0$. If f is increasing and C- decreasing in mean relatively to $\mathbf{g}$, then, for any $p \in (0,1]$,*

$$\left(\int_{\mathbf{a}}^{\mathbf{b}} f(\mathbf{x}) d\mathbf{g}(\mathbf{x}) \right)^p \leqslant C^{m(1-p)} \int_{\mathbf{a}}^{\mathbf{b}} f^p(\mathbf{x}) d\left[\mathbf{g}^p(\mathbf{x})\right]. \tag{3.1}$$

REMARK 3.3. We have assumed that f is both increasing and C-decreasing in mean. For the one-dimensional case similar classes of functions have previously been considered in other contexts e.g. by Muckenhoupt ([12], p. 213) and Gehring ([7], p.266).

Proof. The proposition is a consequence of Theorem 2.1 (a), if we prove that the function f is $C-$ decreasing in mean in each variable. Because of symmetry it is enough to prove this for the first variable and we shall give two proofs of this.

I. By using Definition 3.1, Fubini's theorem and integration by parts we have

$$
\begin{aligned}
f(x_1,\dots,x_m)\prod_{I=1}^{m} g_i(x_i) &\leqslant C\int_{\mathbf{a}}^{\mathbf{x}} f(t_1,\dots,t_m)dg_1(t_1)\dots dg_m(t_m)\\
&= C\int_{\mathbf{a}_m}^{\mathbf{x}_m}\left[\int_{a_m}^{x_m} f(t_1,\dots,t_m)dg_m(t_m)\right]dg_1(t_1)\dots dg_{m-1}(t_{m-1})\\
&= C\int_{\mathbf{a}_m}^{\mathbf{x}_m}\Big[f(t_1,\dots,t_{m-1},x_m)g_m(x_m)\\
&\quad -\int_{a_m}^{x_m} g_m(t_m)d_m f(t_1,\dots,t_m)\Big]\,dg_1(t_1)\dots dg_{m-1}(t_{m-1})\\
&\leqslant Cg_m(x_m)\int_{\mathbf{a}_m}^{\mathbf{x}_m} f(t_1,\dots,t_{m-1},x_m)g_m(x_m)dg_1(t_1)\dots dg_{m-1}(t_{m-1}).
\end{aligned}
$$

Here $d_m f(t_1,..,t_m)$ is the Borel measure generated with the function $t_m \longmapsto f(t_1,..,t_m)$. Proceeding in this way will give us the claim.

II. We have from Definition 3.1 and Fubini's theorem that

$$
\begin{aligned}
&f(x_1,..,x_m)g_1(x_1)\\
&\leqslant C\int_{a_1}^{x_1}\left[\frac{1}{g_2(x_2),..,g_m(x_m)}\int_{a_2}^{x_2}\cdots\int_{a_m}^{x_m} f(t_1,..,t_m)dg_2(t_2)\dots dg_m(t_m)\right]dg_1(t_1)\\
&\leqslant C\int_{\mathbf{a}_1}^{\mathbf{x}_1} f(t_1,x_2,..,x_m)dg_1(t_1).
\end{aligned}
$$

where the last inequality is a consequence of increasing property of the function f. □

REMARK 3.4. It is obvious from the second proof that the condition that f is increasing can be weakened to the following (mean -increasing) condition:

$$
\int_{\mathbf{a}_i}^{\mathbf{x}_i} f(t_1,..t_{i-1},x_i,t_{i+1},\dots,t_m)d\mathbf{g}_i(\mathbf{t}_i)\leqslant f(x_1,\dots,x_m)\prod_{j\neq i} g_j(x_j).
$$

Finally we'll discuss Theorem 2.1 (b), just supposing that $f \geqslant 0$. It is easy to see that functions of the type $f(x_1,x_2,\dots x_m) = f_1(x_1)\dots f_m(x_m)$ are solutions of the equality problem in (2.1), where $f_i, \ i = 1,\dots,m,$ are the solutions of the one-dimensional equality problems

$$\begin{aligned} C_i \int_{a_i}^{x_i} f_i(t_i)dg_i(t_i) &= f_i(x_i)g_i(x_i), x_i \in U_i \subseteq (a_i,b_i) \\ and\ f_i(x_i) &= 0,\ x_i \in U_i^c \end{aligned}$$

(see [18] for details). Note that f is positive on $\prod_{i=1}^m U_i$, and 0 on $\left(\prod_{i=1}^m U_i\right)^c$.

We shall also sketch the proof of the reversed implication. Of course if we have equality in (2.1), then the function f, has to be locally of the form (2.2). What we want to prove is that the support of f is of product type. Notice first that now we have to be more careful in proceeding from equalities as in (2.4) to (2.5) or (2.7) to (2.9), because for example if

$$\int_{\mathbf{a}_1}^{\mathbf{x}_1} f(x_1,\mathbf{t}_1)d\mathbf{g}_1(\mathbf{t}_1) = 0$$

we cannot conclude equality in (2.6). Take $(x_1,x_2,\dots x_m) \in \operatorname{Int}\mathcal{N}$ (topological interior) and assume that

$$\int_{a_m}^{x_m} f(\mathbf{x}_m,t_m)dg_m(t_m) > 0.$$

Then (since $f^{-1}(0,+\infty)$ is an open set) it is obvious that all the integrals of the type

$$\int_{a_i}^{x_i}\cdots\int_{a_m}^{x_m} f(x_1,x_2,\dots,x_{i-1},t_i,\dots t_m)dg_i(t_i)\dots dg_m(t_m)$$

are strictly positive, so we can proceed from equality in (2.5) to equality in (2.6) etc., so with equality in (2.6), (2.10) and lower dimensional analogues we obtain

$$g_1(x_1)\dots g_{m-1}(x_{m-1})\int_{a_m}^{x_m} f(x_1,\dots,x_{m-1},t_m)dg_m(t_m) = C_1\dots C_{m-1}\int_{\mathbf{a}}^{\mathbf{x}} f(\mathbf{t})d\mathbf{g}(\mathbf{t}).$$

By differentiating this equality with respect to the variable x_m (notice that the function on the left side doesn't change with small changes in the variable x_m since $(x_1,\dots,x_m) \in \operatorname{Int}\mathcal{N}$), we obtain

$$\int_{a_1}^{x_1}\int_{a_2}^{x_2}\cdots\int_{a_{m-1}}^{x_{m-1}} f(t_1,t_2,\dots t_{m-1},x_m)dg_1(t_1)dg_2(t_2)\dots dg_{m-1}(t_{m-1}) = 0$$

which obviously gives the product structure of $(\operatorname{Int}\mathcal{N})^c$.

In particular, this investigation shows the following:

REMARK 3.6. A version of Theorem 2.1 (b) can be formulated also for the case $f \geqslant 0$, but the formulation of this result will be more complicated and all cases of equality can still be described.

The same can be done for Theorem 2.2.

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