Mathematical
Inequalities
& Applications
Volume 3, Number 1 (2000), 89–96

AN EXTENSION OF SPECHT'S THEOREM VIA KANTOROVICH INEQUALITY AND RELATED RESULTS

TAKEAKI YAMAZAKI

(*communicated by T. Furuta*)

Abstract. In this paper, we shall show the following result.
"If $MI \geqslant A \geqslant mI > 0$ with $M > m > 0$, then

$$K_+(m^r, M^r, \tfrac{p}{r})^{\frac{1}{p}} (A^r x, x)^{\frac{1}{r}} \geqslant (A^p x, x)^{\frac{1}{p}}$$

for $p > r > 0$, where

$$K_+(m, M, p) = \frac{(p-1)^{p-1}}{p^p} \frac{(M^p - m^p)^p}{(M-m)(mM^p - Mm^p)^{p-1}}."$$

This result is an extension of Specht's theorem [6] as a converse of the arithmetic-geometric mean inequality.
"If $x_1, x_2, \cdots, x_n \in [m, M]$ with $M > m > 0$, then

$$M_h \sqrt[n]{x_1 x_2 \cdots x_n} \geqslant \frac{x_1 + x_2 + \cdots + x_n}{n}$$

where $h = \frac{M}{m} > 1$ and $M_h = \dfrac{h^{\frac{1}{h-1}}}{e \log h^{\frac{1}{h-1}}}$."

Secondly, we shall show an application for operator inequalities, that is, *"if $A \geqslant B \geqslant 0$ satisfying $MI \geqslant B \geqslant mI > 0$ with $M > m > 0$, then*

$$A^p - B^p \geqslant \frac{-(mM^p - Mm^p)}{M-m} \left\{ K_+(m, M, p)^{\frac{1}{p-1}} - 1 \right\}$$

for $p > 1$."

1. Introduction

We shall consider bounded linear operators on a complex Hilbert space H. An operator T is said to be positive (denoted by $T \geqslant 0$) if $(Tx, x) \geqslant 0$ for all $x \in H$. Also, an operator T is strictly positive (denoted by $T > 0$) if T is positive and invertible.

Very recently, J. I. Fujii and Y. Seo [3] defined determinant for positive invertible operators as follows:

Mathematics subject classification (1991): 47A30, 47A63.
Key words and phrases: Kantorovich inequality, positive operator, Specht's ratio.

DEFINITION 1. (Determinant for positive operators [3]) *Let A be a positive invertible operator. Then the determinant $\Delta_x(A)$ for A at a unit vector $x \in H$ is defined as follows:*

$$\Delta_x(A) \overset{\text{def}}{=} \exp\langle (\log A)x, x\rangle .$$

On the determinant $\Delta_x(A)$, the following theorem was shown in [1].

THEOREM A. ([1]) *Let A be a positive invertible operator. Then for every unit vector $x \in H$, $f(p) = (A^p x, x)^{\frac{1}{p}}$ is an increasing function for $p > 0$, and $\lim_{p\to 0}(A^p x, x)^{\frac{1}{p}} = \Delta_x(A)$. Especially $(Ax, x) \geqslant \Delta_x(A)$ holds for every unit vector $x \in H$.*

As a converse of $(Ax, x) \geqslant \Delta_x(A)$ in Theorem A, J. I. Fujii, S. Izumino and Y. Seo [2] showed the following theorem.

THEOREM B. ([2]) *Let A be a positive invertible operator satisfying $MI \geqslant A \geqslant mI > 0$ with $M > m > 0$. Then*

$$M_h(p)\Delta_x(A^p) \geqslant (A^p x, x) \tag{1.1}$$

holds for $p > 0$ and every unit vector $x \in H$, where $h = \frac{M}{m} > 1$ and

$$M_h(p) = \frac{h^{\frac{p}{h^p-1}}}{e\log\left(h^{\frac{p}{h^p-1}}\right)}. \tag{1.2}$$

REMARK 1. By putting $p = 1$,

$$A = \begin{pmatrix} x_1 & 0 & \cdots & 0 \\ 0 & x_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & x_n \end{pmatrix} \quad \text{and} \quad x = \frac{1}{\sqrt{n}}\begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}$$

in Theorem B, then we have the following Specht's theorem [6] between arithmetic mean and geometric mean as follows:

Let $x_1, x_2, \cdots, x_n \in [m, M]$ with $M > m > 0$. Then

$$M_h \sqrt[n]{x_1 x_2 \cdots x_n} \geqslant \frac{x_1 + x_2 + \cdots + x_n}{n} \tag{1.3}$$

holds, where $h = \frac{M}{m} > 1$, $M_h = M_h(1)$ and $M_h(p)$ is defined in (1.2).

It is well known that

$$\frac{x_1 + x_2 + \cdots + x_n}{n} \geqslant \sqrt[n]{x_1 x_2 \cdots x_n} \tag{1.4}$$

holds for positive numbers $x_1, x_2, \cdots, x_n$. (1.3) means a converse of arithmetic-geometric mean inequality (1.4). We remark that M_h is called *Specht's ratio* and it is shown in [2] and [6] that $M_h(p)$ and M_h are optimal in Theorem B and (1.3), respectively.

Related to Theorem A, the following Hölder-McCarthy inequality is well-known result: *Let A be a positive operator, then $(A^p x,x) \geqslant (Ax,x)^p$ holds for $p \geqslant 1$ and every unit vector $x \in H$.* As a converse of Hölder-McCarthy inequality, the following Theorem C is slight modification of [5, Corollary 15]

THEOREM C. ([5, Corollary 15]) *Let A be a positive invertible operator satisfying $MI \geqslant A \geqslant mI > 0$ with $M > m > 0$. Then*

$$\frac{mM^p - Mm^p}{M-m}\left\{K_+(m,M,p)^{\frac{1}{p-1}} - 1\right\} \geqslant (A^p x,x) - (Ax,x)^p \tag{1.5}$$

holds for $p > 1$ and every unit vector $x \in H$, where

$$K_+(m,M,p) = \frac{(p-1)^{p-1}}{p^p}\frac{(M^p - m^p)^p}{(M-m)(mM^p - Mm^p)^{p-1}}. \tag{1.6}$$

We remark that $(A^p x,x) - (Ax,x)^p \geqslant 0$ holds for $p > 1$ by Hölder-McCarthy inequality. Related to Theorem C, T. Furuta [4] showed the following Theorem D as an extension of Kantorovich inequality: *Let A be a positive invertible operator satisfying $MI \geqslant A \geqslant mI > 0$. Then $\frac{(m+M)^2}{4mM}(Ax,x)^2 \geqslant (A^2x,x)$ holds for every unit vector $x \in H$.*

THEOREM D. ([4]) *Let A be a positive invertible operator satisfying $MI \geqslant A \geqslant mI > 0$ with $M > m > 0$. Then*

$$K_+(m,M,p)(Ax,x)^p \geqslant (A^p x,x) \geqslant (Ax,x)^p \tag{1.7}$$

hold for all $p > 1$ and every unit vector $x \in H$, where $K_+(m,M,p)$ is defined in (1.6).

The last inequality of (1.7) also holds by Hölder-McCarthy inequality. We put $p = 2$ in Theorem D, then we have Kantorovich inequality.

For positive operators A and B, $A \geqslant B \geqslant 0$ ensures $A^p \geqslant B^p$ for any $p \in [0,1]$ by well-known Löwner-Heinz theorem. However, it is also well known that $A \geqslant B \geqslant 0$ does not always ensure $A^p \geqslant B^p$ for any $p > 1$. Related to this result, the following result was shown in [4] as an application of Theorem D.

THEOREM E. ([4]) *Let $A \geqslant B \geqslant 0$ satisfying $MI \geqslant B \geqslant mI > 0$ with $M > m > 0$. Then $K_+(m,M,p)A^p \geqslant B^p$ holds for all $p > 1$, where $K_+(m,M,p)$ is defined in* (1.6).

Related to Theorem E, we showed a parallel result to Theorem E as follows:

THEOREM F. ([7]) *Let A and B be positive and invertible operators on a Hilbert space H satisfying $MI \geqslant B \geqslant mI > 0$ with $M > m > 0$. Then the following assertions are mutually equivalent:*

(i) $\log A \geqslant \log B$.

(ii) *$M_h(p)A^p \geqslant B^p$ holds for all $p > 0$, where $h = \frac{M}{m} > 1$ and $M_h(p)$ is defined in* (1.2).

We remark that a simplified proof of Theorem F was shown in [1].

In this paper, firstly we shall show an extension of Theorem B. Secondly, we shall show an application of Theorem C as a parallel result to Theorem E.

2. An extension of Theorem B

THEOREM 1. *Let A be a positive invertible operator satisfying $MI \geqslant A \geqslant mI > 0$ with $M > m > 0$. Then*

$$K_+\left(m^r, M^r, \frac{p}{r}\right)^{\frac{1}{p}} (A^r x, x)^{\frac{1}{r}} \geqslant (A^p x, x)^{\frac{1}{p}} \tag{2.1}$$

holds for $p > r > 0$ and every unit vector $x \in H$, where $K_+(m, M, p)$ is defined in (1.6).

Very recently, we showed the properties of $K_+(m, M, p)$ in [7] as follows:

THEOREM G. ([7]) *Let $K_+(m, M, p)$ be defined in (1.6). Then*

$$F(p, r, m, M) = K_+\left(m^r, M^r, \frac{p+r}{r}\right)$$

is an incerasing function of p, r and M, and also a decreasing function of m for $p > 0$, $r > 0$ and $M > m > 0$. And the following inequality holds:

$$\left(\frac{M}{m}\right)^p \geqslant K_+\left(m^r, M^r, \frac{p+r}{r}\right) \geqslant 1 \tag{2.2}$$

for any $p > 0$, $r > 0$ and $M > m > 0$.

THEOREM H. ([7]) *Let $K_+(m, M, p)$ be defined in (1.6). Then for $p > 0$ and $M > m > 0$,*

$$\lim_{r \to +0} K_+\left(m^r, M^r, \frac{p}{r}\right) = M_h(p),$$

where $h = \frac{M}{m} > 1$, and $M_h(p)$ is defined in (1.2).

Let $p = 1$ and $r \to +0$ in Theorem 1, then we have the following Theorem B' by using Theorem A and Theorem H. And the following Theorem B' is equivalent to Theorem B.

THEOREM B'. *Let A be a positive invertible operator satisfying $MI \geqslant A \geqslant mI > 0$ with $M > m > 0$. Then*

$$M_h(1)\Delta_x(A) \geqslant (Ax, x) \tag{2.3}$$

holds for every unit vector $x \in H$, where $h = \frac{M}{m} > 1$ and $M_h(p)$ is defined in (1.2).

Proof of the equivalence relation between Theorem B and Theorem B'. Put $p = 1$ in (1.1), then we have Theorem B'.

Conversely, for each $p > 0$, $MI \geqslant A \geqslant mI > 0$ is equivalent to $M^p I \geqslant A^p \geqslant m^p I > 0$. Then we have

$$M_{h^p}(1)\Delta_x(A^p) \geqslant (A^p x, x)$$

holds for every unit vector $x \in H$ by Theorem B'. Hence we obtain Theorem B since $M_{h^p}(1) = M_h(p)$. $\square$

Briefly speaking, let $f(p)$ be defined as $f(p) = (A^p x, x)^{\frac{1}{p}}$ for unit vector $x \in H$, then Theorem B' asserts $\lim_{r\to+0} \left\{K_+\left(m^r, M^r, \frac{1}{r}\right)\right\} f(0) \geqslant f(1)$, and also Theorem 1 asserts $K_+(m^r, M^r, \frac{p}{r})^{\frac{1}{p}} f(r) \geqslant f(p)$ for $p > r > 0$. Hence Theorem 1 is an extension of Theorem B' which is equivalent to Theorem B.

REMARK 2. For positive numbers $x_1, x_2, \cdots, x_n$,

$$\left(\frac{x_1^p + x_2^p + \cdots + x_n^p}{n}\right)^{\frac{1}{p}} \geqslant \left(\frac{x_1^r + x_2^r + \cdots + x_n^r}{n}\right)^{\frac{1}{r}} \tag{2.4}$$

holds for $p > r > 0$, that is, $g(p) = (\frac{x_1^p + x_2^p + \cdots + x_n^p}{n})^{\frac{1}{p}}$ is an increasing function for $p > 0$. By putting

$$A = \begin{pmatrix} x_1 & 0 & \cdots & 0 \\ 0 & x_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & x_n \end{pmatrix} \quad \text{and} \quad x = \frac{1}{\sqrt{n}} \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix}$$

in Theorem 1, then we have a converse inequality of (2.4) as follows:

Let $x_1, x_2, \cdots, x_n \in [m, M]$ with $M > m > 0$. Then

$$K_+\left(m^r, M^r, \frac{p}{r}\right)^{\frac{1}{p}} \left(\frac{x_1^r + \cdots + x_n^r}{n}\right)^{\frac{1}{r}} \geqslant \left(\frac{x_1^p + \cdots + x_n^p}{n}\right)^{\frac{1}{p}} \tag{2.5}$$

holds for $p > r > 0$, wehre $K_+(m, M, p)$ is defined in (1.6).

Briefly speaking, for positive numbers $x_1, x_2, \cdots, x_n \in [m, M]$ with $M > m > 0$, let $g(p)$ be defined as $g(p) = (\frac{x_1^p + x_2^p + \cdots + x_n^p}{n})^{\frac{1}{p}}$, then (1.3) asserts $\lim_{r\to+0} \left\{K_+\left(m^r, M^r, \frac{1}{r}\right)\right\} g(0) \geqslant g(1)$ by using Theorem H and considering $g(p) \to \sqrt[n]{x_1 x_2 \cdots x_n}$ as $p \to +0$, and also (2.5) asserts $K_+(m^r, M^r, \frac{p}{r})^{\frac{1}{p}} g(r) \geqslant g(p)$ for $p > r > 0$. Hence (2.5) is an extension of (1.3).

Proof of Theorem 1. $MI \geqslant A \geqslant mI > 0$ is equivalent to $M^r I \geqslant A^r \geqslant m^r I > 0$ for $r > 0$. By using Theorem D to $M^r I \geqslant A^r \geqslant m^r I > 0$, we have

$$K_+(m^r, M^r, p_1)(A^r x, x)^{p_1} \geqslant (A^{rp_1} x, x) \tag{2.6}$$

holds for $p_1 > 1$ and every unit vector $x \in H$. Put $p_1 = \frac{p}{r} > 1$ in (2.6), then we have

$$K_+\left(m^r, M^r, \frac{p}{r}\right) (A^r x, x)^{\frac{p}{r}} \geqslant (A^{r\frac{p}{r}} x, x). \tag{2.7}$$

(2.7) is equivalent to

$$K_+\left(m^r, M^r, \frac{p}{r}\right)^{\frac{1}{p}} (A^r x, x)^{\frac{1}{r}} \geqslant (A^p x, x)^{\frac{1}{p}} \tag{2.1}$$

for $p > r > 0$ and every unit vector $x \in H$.

Hence the proof of Theorem 1 is complete. □

3. An application of Theorem C to operator inequality

THEOREM 2. *Let A and B be positive invertible operators satisfying $A \geqslant B \geqslant 0$ and $MI \geqslant B \geqslant mI > 0$ with $M > m > 0$. Then*

$$A^p - B^p \geqslant \frac{-(mM^p - Mm^p)}{M-m}\left\{K_+(m,M,p)^{\frac{1}{p-1}} - 1\right\} \tag{3.1}$$

holds for $p > 1$, where $K_+(m,M,p)$ is defined in (1.6).

We remark that

$$0 \geqslant \frac{-(mM^p - Mm^p)}{M-m}\{K_+(m,M,p)^{\frac{1}{p-1}} - 1\} \geqslant -m(M^{p-1} - m^{p-1})$$

holds for $p > 1$ and $M > m > 0$ by using (2.2) in Theorem G. Let $p \to 1$ in (3.1), then we have $A - B \geqslant 0$.

Proof of Theorem 2. By using Theorem C to $MI \geqslant B \geqslant mI > 0$, then we have for $p > 1$ and every unit vector $x \in H$,

$$\begin{aligned}\frac{mM^p - Mm^p}{M-m}\left\{K_+(m,M,p)^{\frac{1}{p-1}} - 1\right\} &\geqslant (B^p x,x) - (Bx,x)^p \\ &\geqslant (B^p x,x) - (Ax,x)^p \quad \text{by } A \geqslant B \geqslant 0 \\ &\geqslant (B^p x,x) - (A^p x,x)\end{aligned}$$

and the last inequality holds by Hölder-McCarthy inequality since $p > 1$. Then we have

$$A^p - B^p \geqslant \frac{-(mM^p - Mm^p)}{M-m}\left\{K_+(m,M,p)^{\frac{1}{p-1}} - 1\right\} \tag{3.1}$$

holds for $p > 1$ and every unit vector $x \in H$. □

4. Simplified proofs of Theorem C and Theorem D

Theorem C and Theorem D were shown by using elaborate differential calculus separately. In this section, we give short proofs of Theorem C and Theorem D by only using the following relation between generalized arithmetic mean and generalized geometric mean.

THEOREM I. *Let a and b be positive numbers, then*

$$\lambda a + (1-\lambda)b \geqslant a^\lambda b^{1-\lambda}$$

holds for any $\lambda \in [0,1]$.

We cite the following lemma to give proofs of Theorem C and Theorem D.

LEMMA 3. *Let A be a positive invertible operator satisfying $MI \geqslant A \geqslant mI > 0$ with $M > m > 0$. Then*

$$\frac{M^p - m^p}{M-m}(Ax,x) \geqslant (A^p x,x) + \frac{mM^p - Mm^p}{M-m} \tag{4.1}$$

holds for $p > 1$ and every unit vector $x \in H$.

Proof. For $p > 1$, $f(t) = t^p$ is a real valued continuous convex function on $[m, M]$, so that we have

$$\frac{M^p - m^p}{M - m}(t - m) + m^p \geqslant t^p \qquad \text{for any } t \in [m, M]. \tag{4.2}$$

By applying the standard operational calculus of positive operator A to (4.2) since $MI \geqslant A \geqslant mI > 0$, we have

$$\frac{M^p - m^p}{M - m}(A - m) + m^p \geqslant A^p. \tag{4.3}$$

By (4.3), the following (4.4) holds for every unit vector $x \in H$:

$$\frac{M^p - m^p}{M - m}\left((Ax, x) - m\right) + m^p \geqslant (A^p x, x). \tag{4.4}$$

(4.4) coincide with (4.1), i.e.,

$$\frac{M^p - m^p}{M - m}(Ax, x) \geqslant (A^p x, x) + \frac{mM^p - Mm^p}{M - m}. \tag{4.1}$$

□

Short proof of Theorem C. By using Lemma 3, we have

$$\begin{aligned}
(A^p x, x) + \frac{mM^p - Mm^p}{M - m} &\leqslant \frac{M^p - m^p}{M - m}(Ax, x) \\
&= \frac{M^p - m^p}{M - m}\left(\frac{1}{p}\right)^{\frac{1}{p}} p^{\frac{1}{p}}(Ax, x) \\
&= \left[\left\{\left(\frac{1}{p}\right)^{\frac{1}{p}} \frac{M^p - m^p}{M - m}\right\}^{\frac{p}{p-1}}\right]^{\frac{p-1}{p}} \{p(Ax, x)^p\}^{\frac{1}{p}} \\
&\leqslant \frac{p-1}{p}\left\{\left(\frac{1}{p}\right)^{\frac{1}{p}} \frac{M^p - m^p}{M - m}\right\}^{\frac{p}{p-1}} + \frac{1}{p}\{p(Ax, x)^p\} \\
&= \frac{p-1}{p^{\frac{p}{p-1}}}\left(\frac{M^p - m^p}{M - m}\right)^{\frac{p}{p-1}} + (Ax, x)^p,
\end{aligned}$$

and the last inequality holds by using Theorem I since $p > 1$. Then we obtain

$$\frac{p-1}{p^{\frac{p}{p-1}}}\left(\frac{M^p - m^p}{M - m}\right)^{\frac{p}{p-1}} - \frac{mM^p - Mm^p}{M - m} \geqslant (A^p x, x) - (Ax, x)^p. \tag{4.5}$$

(4.5) is equivalent to

$$\frac{mM^p - Mm^p}{M - m}\left\{K_+(m, M, p)^{\frac{1}{p-1}} - 1\right\} \geqslant (A^p x, x) - (Ax, x)^p \tag{1.5}$$

for $p > 1$ and every unit vector $x \in H$. □

Short proof of Theorem D. By using Lemma 3, we have

$$\begin{aligned}
\frac{M^p - m^p}{M-m}(Ax,x) &\geqslant (A^px,x) + \frac{mM^p - Mm^p}{M-m} \\
&= \frac{1}{p}\{p(A^px,x)\} + \frac{p-1}{p}\left\{\frac{p}{p-1}\frac{mM^p - Mm^p}{M-m}\right\} \\
&\geqslant \{p(A^px,x)\}^{\frac{1}{p}}\left\{\frac{p}{p-1}\frac{mM^p - Mm^p}{M-m}\right\}^{\frac{p-1}{p}} \\
&= \frac{p}{(p-1)^{\frac{p-1}{p}}}\left(\frac{mM^p - Mm^p}{M-m}\right)^{\frac{p-1}{p}}(A^px,x)^{\frac{1}{p}},
\end{aligned}$$

and the last inequality holds by using Theorem I since $p > 1$. Then we obtain

$$\frac{(p-1)^{\frac{p-1}{p}}}{p}\left(\frac{M-m}{mM^p - Mm^p}\right)^{\frac{p-1}{p}}\frac{M^p - m^p}{M-m}(Ax,x) \geqslant (A^px,x)^{\frac{1}{p}}. \tag{4.6}$$

(4.6) is equivalent to

$$K_+(m,M,p)(Ax,x)^p \geqslant (A^px,x) \tag{1.7}$$

for $p > 1$ and every unit vector x. □

Acknowledgement. I would like to express my cordial thanks to Professor Takayuki Furuta for his guidance and encouragement.

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(Received August 31, 1999)

Takeaki Yamazaki
Department of Applied Mathematics
Faculty of Science
Science University of Tokyo
1-3 Kagurazaka, Shinjuku
Tokyo 162-8601, Japan
e-mail: tyamaz@am.kagu.sut.ac.jp