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THE HAUSDORFF AND THE QUASI HAUSDORFF OPERATORS ON THE SPACES L^p, $1 \leqslant p < \infty$

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Abstract. The operators indicated in the title are defined by means of Lebesgue-Stieltjes integrals of real- (or complex-) valued funtions with respect to σ-finite positive (or signed, or complex) measures μ defined on the Borel measurable subsets of $\mathbf{R}_+$ (or $\mathbf{R}$). We give simple sufficient conditions in terms of μ in order that these operators be bounded on the Lebesgue space $L^p(\mathbf{R}_+)$ (or $L^p(\mathbf{R})$) for some $1 \leqslant p < \infty$. These sufficient conditions are exact even in the wellknown special cases of the Cesàro and Copson operators. We also prove that the Hausdorff and the quasi Hausdorff operators are adjoint of one another, under an appropriate condition in terms of μ. On closing, we reveal an interrelation among these operators and the Fourier transform of a function in $L^1(\mathbf{R})$.

1. Definitions and auxiliary results on $\mathbf{R}_+$

We shall consider Lebesgue-Stieltjes integrals with respect to a σ-finite signed measure μ defined on the Borel measurable subsets of $\mathbf{R}_+ := [0, \infty)$. As it is well known, a signed measure is an extended real-valued, countably additive set function μ on the class of all measurable sets of a measure space, such that $\mu(\emptyset) = 0$, and such that μ assumes at most one of the values $+\infty$ and $-\infty$. For more details, we refer the reader to [2, Chap. VI].

Following Hardy [4, Chap. XI], we define the Hausdorff operator $\mathscr{H} = \mathscr{H}_\mu$ with respect to a signed measure μ as follows. First, we define $\mathscr{H}f$ for continuous functions $f : \mathbf{R}_+ \to \mathbf{R}$ with compact support by

$$\mathscr{H}_\mu f(x) := \int_0^\infty f(xt)\,d\mu(t), \quad x > 0, \tag{1.1}$$

provided the right-hand side exists as a Lebesgue-Stieltjes integral. In case μ is a finite signed measure, this integral can be equally considered as a Riemann-Stieltjes integral. Clearly, the operator $\mathscr{H}$ is linear.

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Let $1 \leqslant p < \infty$ be fixed. We shall prove in Theorem 1 of Section 2 below that, under an appropriate condition in terms of μ, the operator $\mathscr{H} : L^p(\mathbf{R}_+) \to L^p(\mathbf{R}_+)$ is bounded when it is applied to continuous functions with compact support. The subclass of such functions is dense in $L^p(\mathbf{R}_+)$ whenever $1 \leqslant p < \infty$. Thus, the operator $\mathscr{H}$ can be extended for the whole space $L^p(\mathbf{R}_+)$ with the same operator norm.

Next, we define the quasi Hausdorff operator $\mathscr{H}^* = \mathscr{H}^*_\mu$ with respect to the same signed measure μ. First, we define $\mathscr{H}^* f$ for continuous functions $f : \mathbf{R}_+ \to \mathbf{R}$ with compact support as follows:

$$\mathscr{H}^*_\mu f(x) := \int_0^\infty \frac{1}{t} f\left(\frac{x}{t}\right) d\mu(t), \quad x > 0, \tag{1.2}$$

provided again that the right-hand side exists as a Lebesgue-Stieltjes integral. If μ is a finite signed measure and the compact support of f does not contain the zero (origin), then the integral in (1.2) exists even as a Riemann-Stieltjes integral. Clearly, the operator $\mathscr{H}^*$ is linear.

Again, let $1 \leqslant p < \infty$ be fixed. We shall also prove in Theorem 1 of Section 2 below that, under an appropriate condition in terms of μ, the operator $\mathscr{H}^* : L^p(\mathbf{R}_+) \to L^p(\mathbf{R}_+)$ is bounded when it is applied to continuous functions with compact support. So, the operator $\mathscr{H}^*$ can also be extended for the whole space $L^p(\mathbf{R}_+)$ with the same operator norm.

A few remarks are appropriate here.

(i) In fact, Hardy [4, Chap. XI] defined $\mathscr{H}_\mu f$ in the case when f is a continuous function on $\mathbf{R}_+$ and μ is a finite signed measure supported on the unit interval $[0,1]$. The adjective "quasi" in the name of $\mathscr{H}^*$ was used also by Hardy [4, Chap. XI], who defined it for infinite sequences of real numbers containing only a finite number of nonzero terms.

(ii) However, considering a noncontinuous function f in $L^p(\mathbf{R}_+)$ for some $1 \leqslant p < \infty$, we face with the following problem: f should be measurable with respect to the Lebesgue measure and at the same time with respect to the signed measure μ. It is well known that if $f \in L^p(\mathbf{R}_+)$ for some $1 \leqslant p < \infty$, then there exists a Borel measurable function $f_1 \in L^p(\mathbf{R}_+)$ such that $f(x) = f_1(x)$ almost everywhere with respect to the Lebesgue measure. Therefore, it seems to be reasonable to consider only Borel measurable functions in $L^p(\mathbf{R}_+)$. This is in accordance with the common agreement that in any Lebesgue space $L^p(\mathbf{R}_+)$ two functions are called equivalent if they differ only on a set of Lebesgue measure zero. In other words, the elements of $L^p(\mathbf{R}_+)$ are actually function classes, each class consisting of equivalent functions, and according what we have said above, each class can be represented by a Borel measurable function. This suggests that in definitions (1.1) and (1.2), the function $f \in L^p(\mathbf{R}_+)$ should be exchanged by a Borel measurable function f_1 equivalent to f. The next problem is that f_1 is not uniquely determined by f. Since μ is not assumed to be absolutely continuous with respect to the Lebesgue measure, $\mathscr{H}_\mu f_1$ and $\mathscr{H}_\mu f_2$ may be essentally different functions (with respect to the measure μ) in the case of equivalent (with respect to the Lebesgue measure) functions f_1 and f_2, both being Borel measurable on $\mathbf{R}_+$. In other words, $\mathscr{H}_\mu f_1$ and $\mathscr{H}_\mu f_2$ may differ on a set of positive measure with respect to μ, in spite of the fact that f_1 and f_2 differ only on a set of Lebesgue measure zero. This is the reason why we define the operators $\mathscr{H}$ and $\mathscr{H}^*$ in two steps, while making use of a "density" approach familiar in functional analysis.

(iii) This approach does not apply in the case of the nonseparable space $L^\infty(\mathbf{R}_+)$. Instead, we may consider either the subspace $C_b(\mathbf{R}_+)$ consisting of the continuous and bounded functions $f : \mathbf{R}_+ \to \mathbf{R}$, endowed with the norm

$$\|f\|_C := \sup_{0 \leqslant x < \infty} |f(x)|; \tag{1.3}$$

or we may consider the even narrower subspace $C_0(\mathbf{R}_+)$ consisting of the continuous functions $f : \mathbf{R}_+ \to \mathbf{R}$ that tend to zero as $x \to \infty$. In the latter case, (1.3) is of the form

$$\|f\|_C := \max_{0 \leqslant x < \infty} |f(x)|.$$

(iv) The notations $\mathscr{H}$ and $\mathscr{H}^*$ tacitly indicate that these operators are adjoint to one another in a certain sense. Indeed, this will be made precise in Theorem 2 below.

Next, we present two auxiliary results, which will play important roles in our proofs. The first of them is the Jordan decomposition of a signed measure.

LEMMA 1. *(see, e.g. [2, p. 123]). Any signed measure μ can be represented as the difference of two positive measures μ^+ and μ^-:*

$$\mu(E) = \mu^+(E) - \mu^-(E) \tag{1.4}$$

for every measurable set E. If μ is finite or σ-finite, then so also are μ^+ and μ^-; at least one of the measures μ^+ and μ^- is always finite.

This gives rise to the notion of the total variation $|\mu|$ of μ, defined by

$$|\mu|(E) := \mu^+(E) + \mu^-(E) \tag{1.5}$$

for every measurable set E. The following statements are obvious: $|\mu|$ is a positive measure; and if μ is finite or σ-finite, then so is $|\mu|$.

The next equivalent definition of $|\mu|$ is also well known (which explains the term "total variation"):

$$|\mu|(E) := \sup_{\mathscr{P}} \sum |\mu(E_k)|$$

for every measurable set E, where the supremum is extended over any (finite or countable) partition of E into disjoint measurable subsets $E_k : E = \bigcup E_k$.

In this context, μ^+ and μ^- are called respectively the upper variation and the lower variation of μ.

The second auxiliary result we need is the Minkowski inequality for integrals (see, e.g. [1, p. 14]). For the reader's convenience, we present it in the following

LEMMA 2. *If μ and ν are σ-finite positive measures, $F(x,t)$ is a function measurable with respect to the product measure $\nu \times \mu$, and $1 \leqslant p < \infty$, then*

$$\left\{\int \left| \int F(x,t)\, d\mu(t) \right|^p d\nu(x) \right\}^{1/p} \leqslant \int \left\{ \int \left| F(x,t) \right|^p d\nu(x) \right\}^{1/p} d\mu(t).$$

2. Boundedness of $\mathscr{H}$ and $\mathscr{H}^*$ on $L^p(\mathbf{R}_+)$

Let $1 \leqslant p < \infty$ and denote by p^* the exponent conjugate to p, that is, let $1/p + 1/p^* = 1$ with the agreement that $1/\infty := 0$.

Our first main result is the following

THEOREM 1. *Assume that μ is a σ-finite signed measure on $\mathbf{R}_+$.*

(i) If

$$\mathscr{K}_\mu(p) := \int_0^\infty t^{-1/p} d|\mu|(t) < \infty \tag{2.1}$$

for some $1 \leqslant p < \infty$, then the Hausdorff operator $\mathscr{H}_\mu$ is bounded on $L^p(\mathbf{R}_+)$:

$$\|\mathscr{H}_\mu\|_p := \sup_{\|f\|_p \leqslant 1} \|\mathscr{H}_\mu f\|_p \leqslant \mathscr{K}_\mu(p). \tag{2.2}$$

(ii) If $\mathscr{K}_\mu(p^*) < \infty$ for some $1 \leqslant p < \infty$, then the quasi Hausdorff operator $\mathscr{H}_\mu^*$ is bounded on $L^p(\mathbf{R}_+)$:

$$\|\mathscr{H}_\mu^*\|_p \leqslant \mathscr{K}_\mu(p^*). \tag{2.3}$$

We recall that the the norm $\|.\|_p$ in $L^p(\mathbf{R}_+)$ is defined by

$$\|f\|_p := \left\{\int_0^\infty |f(x)|^p dx\right\}^{1/p}, \quad 1 \leqslant p < \infty.$$

REMARK. It is not difficult to check that if μ is a finite signed measure on $\mathbf{R}_+$, that is, if condition (2.1) is satisfied for $p = \infty$, then both $\mathscr{H}_\mu : C_b(\mathbf{R}_+) \to C_b(\mathbf{R}_+)$ and $\mathscr{H}_\mu : C_0(\mathbf{R}_+) \to C_0(\mathbf{R}_+)$ are bounded operators, provided that $\mu(\{0\}) = 0$ in the latter case.

Analogously, it is easy to show that if the condition $\mathscr{K}_\mu(\infty^*) = \mathscr{K}_\mu(1) < \infty$ is satisfied, then both $\mathscr{H}_\mu^* : C_b(\mathbf{R}_+) \to L^\infty(\mathbf{R}_+)$ and $\mathscr{H}_\mu^* : C_0(\mathbf{R}_+) \to C_b(\mathbf{R}_+)$ are bounded operators. We recall that the space $L^\infty(\mathbf{R}_+)$ is defined by the norm

$$\|f\|_\infty := \operatorname*{ess\,sup}_{0<x<\infty} |f(x)| < \infty.$$

Proof of Theorem 1.

(i) First, we treat the special case when μ is a σ-finite positive measure on $\mathbf{R}_+$ and $f : \mathbf{R}_+ \to \mathbf{R}$ is continuous with compact support. (However, our estimates below are valid in the more general setting when $f \in L^p(\mathbf{R}_+)$ is Borel measurable.) . We make use of Lemma 2 (this time $d\nu(x) := dx$, the Lebesgue measure on $\mathbf{R}_+$) as follows:

$$\begin{aligned}\|\mathscr{H}_\mu f\|_p &:= \left\{\int_0^\infty \left|\int_0^\infty f(xt) d\mu(t)\right|^p dx\right\}^{1/p} \\ &\leqslant \int_0^\infty \left\{\int_0^\infty |f(xt)|^p dx\right\}^{1/p} d\mu(t) \\ &= \int_0^\infty \left\{\int_0^\infty t^{-1}|f(s)|^p ds\right\}^{1/p} d\mu(t) \\ &= \|f\|_p \int_0^\infty t^{-1/p} d\mu(t) =: \|f\|_p \mathscr{K}_\mu(p).\end{aligned}$$

Second, we treat the case when μ is a σ-finite signed measure on $\mathbf{R}_+$ and $f:\mathbf{R}_+\to\mathbf{R}$ is continuous with compact support. By Lemma 1, the ordinary Minkowski inequality, and by what has been proved just above, we obtain that

$$\begin{aligned}\|\mathscr{H}_\mu f\|_p &\leqslant \|\mathscr{H}_{\mu^+}f\|_p+\|\mathscr{H}_{\mu^-}f\|_p\\ &\leqslant \Big\{\mathscr{K}_{\mu^+}(p)+\mathscr{K}_{\mu^-}(p)\Big\}\|f\|_p=\mathscr{K}_\mu(p)\|f\|_p.\end{aligned}$$

This proves (2.2) when f is a continuous function with compact support.

Third, we complete the proof of (2.2) in the general case, that is, when μ is a σ-finite signed measure on $\mathbf{R}_+$ and $f\in L^p(\mathbf{R}_+)$ for some $1\leqslant p<\infty$. Since the subclass of continuous functions with compact support on $\mathbf{R}_+$ is dense in $L^p(\mathbf{R}_+)$ when $1\leqslant p<\infty$, there exists a sequence $\{f_n:n=1,2,\ldots\}$ of such functions such that

$$\lim_{n\to\infty}\|f_n-f\|_p=0.$$

By the above reasoning, the sequence $\{\mathscr{H}_\mu f_n:n=1,2,\ldots\}$ is Cauchy in $L^p(\mathbf{R}_+)$:

$$\lim_{m,n\to\infty}\|\mathscr{H}_\mu f_m-\mathscr{H}_\mu f_n\|_p\leqslant\lim_{m,n\to\infty}\mathscr{K}_\mu(p)\|f_m-f_n\|_p=0.$$

By the completeness of $L^p(\mathbf{R}_+)$, there exists an element in $L^p(\mathbf{R}_+)$, say $\mathscr{H}_\mu f$, such that

$$\lim_{n\to\infty}\|\mathscr{H}_\mu f_n-\mathscr{H}_\mu f\|_p=0.$$

This $\mathscr{H}_\mu f$ is uniquely determined, up to equivalence with respect to the Lebesgue measure. It is plain that this extension of $\mathscr{H}_\mu$ to the whole space $L^p(\mathbf{R}_+)$ remains linear and bounded:

$$\|\mathscr{H}_\mu f\|_p=\lim_{n\to\infty}\|\mathscr{H}_\mu f_n\|_p\leqslant\lim_{n\to\infty}\mathscr{K}_\mu(p)\|f_n\|_p=\mathscr{K}_\mu(p)\|f\|_p.$$

This proves (2.2) in the general case.

(ii) The proof of (2.3) runs along the same lines. Again, first we treat the special case when μ is a σ-finite positive measure on $\mathbf{R}_+$ and $f:\mathbf{R}_+\to\mathbf{R}$ is continuous with compact support. By Lemma 2,

$$\begin{aligned}\|\mathscr{H}^*_\mu f\|_p &:= \Big\{\int_0^\infty\Big|\int_0^\infty\frac{1}{t}f\big(\frac{x}{t}\big)d\mu(t)\Big|^p dx\Big\}^{1/p}\\ &\leqslant\int_0^\infty\Big\{\int_0^\infty\Big|\frac{1}{t}f\big(\frac{x}{t}\big)\Big|^p dx\Big\}^{1/p}d\mu(t)\\ &=\int_0^\infty\Big\{\int_0^\infty t^{1-p}|f(s)|^p ds\Big\}^{1/p}d\mu(t)\\ &=\|f\|_p\int_0^\infty t^{(1-p)/p}d\mu(t)=:\|f\|_p\mathscr{K}_\mu(p^*),\end{aligned}$$

since $p^*=(p-1)/p$, provided $1<p<\infty$. In case $p=1$, via Fubini's theorem, (2.3) follows immediately.

The general case when μ is a σ-finite signed measure on $\mathbf{R}_+$ and $f \in L^p(\mathbf{R}_+)$ for some $1 \leqslant p < \infty$, follows from the first part just proved exactly in the same way as in the case (i). □

REMARK. It is instructive to consider the particular case of Theorem 1 when μ is the ordinary Lebesgue measure supported on the unit interval $(0,1)$:

$$d\mu(t) := \chi_{(0,1)}(t)dt. \tag{2.4}$$

Then definition (1.1) is of the form

$$\mathscr{H}f(x) := \int_0^1 f(xt)dt = \frac{1}{x}\int_0^x f(s)ds, \quad x > 0,$$

which is the familiar Cesàro operator applied to f, in symbol: $\mathscr{C}f(x)$; while definition (1.2) is of the form

$$\mathscr{H}^*f(x) := \int_0^1 \frac{1}{t}f(\frac{x}{t})dt = \int_x^\infty \frac{f(s)}{s}ds, \quad x > 0,$$

which is the so-called Copson operator applied to f, in symbol: $\mathscr{C}^*f(x)$.

Due to the exact inequalities of Hardy (see, for example, [3, Theorems 327 and 328]), the operator norms of $\mathscr{C}$ and $\mathscr{C}^*$ are the following:

$$\|\mathscr{C}\|_p = p^* \quad \text{for} \quad 1 < p \leqslant \infty,$$

while

$$\|\mathscr{C}^*\|_p = p \quad \text{for} \quad 1 \leqslant p < \infty.$$

The strength of Theorem 1 is illustrated by the fact that from (2.1) - (2.3) it follows that

$$\|\mathscr{C}\|_p \leqslant p^* \quad \text{and} \quad \|\mathscr{C}^*\|_p \leqslant p$$

for the corresponding ranges of p. Simple examples show that $\mathscr{C}$ is not bounded on $L^1(\mathbf{R}_+)$, while $\mathscr{C}^*$ is even not defined for all $f \in C_b(\mathbf{R}_+)$. For instance, in the capacity of a counterexample, see $f = \chi_{(0,1)}$ in the former case, while $f \equiv 1$ in the latter case.

3. $\mathscr{H}$ and $\mathscr{H}^*$ as adjoint operators

As we have mentioned in Section 1, the operators $\mathscr{H}$ and $\mathscr{H}^*$ are adjoint of one another in a certain sense. This is formulated in our second main result as follows.

THEOREM 2. *Assume that μ is a σ-finite signed measure on $\mathbf{R}_+$ such that condition (2.1) is satisfied for some $1 < p < \infty$. If $f \in L^p(\mathbf{R}_+)$ and $g \in L^{p^*}(\mathbf{R}_+)$, then*

$$\int_0^\infty [\mathscr{H}_\mu f(x)]g(x)dx = \int_0^\infty f(x)[\mathscr{H}_\mu^* g(x)]dx. \tag{3.1}$$

Accordingly, the quasi Hausdorff operator $\mathscr{H}^*$ may be called the adjoint (to the) Hausdorff operator $\mathscr{H}$.

Proof. By the boundedness of the operators $\mathscr{H}$ on $L^p(\mathbf{R}_+)$ and $\mathscr{H}^*$ on $L^{p^*}(\mathbf{R}_+)$, respectively, we may assume again that both f and g are continuous functions with compact support on $\mathbf{R}_+$. From Theorem 1 it also follows that both integrals in (3.2) exist as Lebesgue integrals. Applying Fubini's theorem yields (3.1) as follows:

$$\begin{aligned}\int_0^\infty [\mathscr{H}_\mu f(x)]g(x)dx &:= \int_0^\infty \Big\{\int_0^\infty f(xt)d\mu(t)\Big\}g(x)dx \\ &= \int_0^\infty \Big\{\int_0^\infty f(xt)g(x)dx\Big\}d\mu(t) \\ &= \int_0^\infty \Big\{\int_0^\infty f(s)g(\frac{s}{t})\frac{ds}{t}\Big\}d\mu(t) \\ &= \int_0^\infty f(s)\Big\{\int_0^\infty \frac{1}{t}g(\frac{s}{t})d\mu(t)\Big\}ds \\ &=: \int_0^\infty f(s)[\mathscr{H}_\mu^* g(s)]ds.\end{aligned}$$

□

Now, we claim that each of the statements (i) and (ii) in Theorem 1 can be deduced from the other by means of a duality argument. More exactly, the following Corollary 1 is a simple consequence of Theorem 2.

COROLLARY 1. *Assume that* μ *is a* σ*-finite signed measure on* $\mathbf{R}_+$ *such that condition (2.1) is satisfied for some* $1 < p < \infty$. *Then*

$$\|\mathscr{H}_\mu\|_p = \|\mathscr{H}_\mu^*\|_{p^*}. \tag{3.2}$$

Proof. Applying the reverse Hölder inequality, then (3.1), and finally the usual Hölder inequality, results in turn into the following:

$$\begin{aligned}\|\mathscr{H}\|_p &:= \sup_{\|f\|_p\leqslant 1} \|\mathscr{H}f\|_p \\ &= \sup_{\|f\|_p\leqslant 1}\{\sup_{\|g\|_{p^*}\leqslant 1}(\mathscr{H}f,g)\} \\ &= \sup_{\|g\|_{p^*}\leqslant 1}\{\sup_{\|f\|_p\leqslant 1}(f,\mathscr{H}^*g)\} \\ &\leqslant \sup_{\|g\|_{p^*}\leqslant 1}\|\mathscr{H}^*g\|_{p^*} =: \|\mathscr{H}^*\|_{p^*}.\end{aligned} \tag{3.3}$$

For the sake of brevity in writing, above we denoted by $(\mathscr{H}f,g)$ the left-hand side in (3.1), while by $(f,\mathscr{H}^*g)$ its right-hand side.

The inequality converse to (3.3):

$$\|\mathscr{H}^*\|_{p^*} \leqslant \|\mathscr{H}\|_p$$

can be proved analogously. This completes the proof of (3.2). □

4. Extension to functions $f : \mathbf{R} \to \mathbf{C}$

We shall extend the results of Sections 2 and 3 in two directions at the same time by considering

(i) functions and measures defined on the whole real line $\mathbf{R} := (-\infty, \infty)$, and

(ii) complex-valued functions instead of real-valued ones.

Accordingly, let μ be a σ-finite complex measure defined on the Borel measurable subsets of $\mathbf{R}$. We remind the reader that a complex measure on the class of all measurable sets of a measurable space is a set function μ such that

$$\mu(E) = \mu_1(E) + i\mu_2(E) \tag{4.1}$$

for every measurable set E, where $i = \sqrt{-1}$, the imaginary unit, and where μ_1 and μ_2 are signed measures. Furthermore, μ is called finite or σ-finite, if so are μ_1 and μ_2. (See [2, Chap. VI].)

In the sequel, let μ be a σ-finite complex measure on $\mathbf{R}$. We shall define the Hausdorff operator $\mathscr{H} = \mathscr{H}_\mu$ and the quasi Hausdorff operator $\mathscr{H}^* = H^*_\mu$ with respect to μ also in two steps. First, given a continuous function $f : \mathbf{R} \to \mathbf{C}$ with compact support, we set

$$\mathscr{H}_\mu f(x) := \int_{-\infty}^{\infty} f(xt) d\mu(t) \tag{4.2}$$

and

$$\mathscr{H}^*_\mu f(x) := \int_{-\infty}^{\infty} \frac{1}{|t|} f(\frac{x}{t}) d\mu(t), \quad x \in \mathbf{R}, \tag{4.3}$$

provided that the right-hand side exists as a Lebesgue-Stieltjes integral in both cases (cf. (1.1) and (1.2), respectively). Second, by means of Theorem 1^* below we extend the bounded operators $\mathscr{H}$ and $\mathscr{H}^*$ from the subclass of continuous functions with compact support to the whole Lebesgue space $L^p(\mathbf{R})$ for any given p, $1 \leqslant p < \infty$, under an appropriate condition in terms of μ.

THEOREM 1^*. *Assume that μ is a σ-finite complex measure on $\mathbf{R}$.*

(i) *If $\mathscr{K}_{\mu_j}(p) < \infty$ for some $1 \leqslant p < \infty$ and $j = 1, 2$, where μ_1 and μ_2 are from* (4.1), *then the Hausdorff operator $\mathscr{H}_\mu$ is bounded on $L^p(\mathbf{R})$.*

(ii) *If $\mathscr{K}_{\mu_j}(p^*) < \infty$ for some $1 \leqslant p < \infty$ and $j = 1, 2$, then the quasi Hausdorff operator $\mathscr{H}^*_\mu$ is bounded on $L^p(\mathbf{R}_+)$.*

This time $\|.\|_p$ is defined by

$$\|f\|_p := \Big\{ \int_{-\infty}^{\infty} |f(x)|^p dx \Big\}^{1/p}, \quad 1 \leqslant p < \infty.$$

REMARK. In case $p = \infty$, instead of $L^p(\mathbf{R})$ we may consider either its subspace $C_b(\mathbf{R})$ consisting of the continuous and bounded funcions $f : \mathbf{R} \to \mathbf{C}$, endowed with the norm

$$\|f\|_C := \sup_{-\infty < x < \infty} |f(x)|$$

(cf. (1.3)); or we may consider the even narrower subspace $C_0(\mathbf{R})$ consisting of the continuous functions $f:\mathbf{R}\to\mathbf{C}$ that tend to zero as $|x|\to\infty$.

We point out that $\mathscr{C}_0(\mathbf{R})$ coincides with the closure (in the norm $\|.\|_C$ defined just above) of the space $\mathscr{S}$ of tempered distributions; that is, of the space of infinitely differentiable, complex-valued, rapidly decreasing functions endowed with the topology defined by the countable system of seminorms

$$p_{m,\alpha}(f) := \sup_{-\infty<x<\infty}(1+|x|^m)|f^{(\alpha)}(x)|, \quad m,\alpha=0,1,\ldots,$$

each seminorm being finite for $f\in\mathscr{S}$.

Now, it is easy to check that if μ_1 and μ_2 in (4.1) are finite signed measures on $\mathbf{R}$, then both $\mathscr{H}_\mu : C_b(\mathbf{R})\to C_b(\mathbf{R})$ and $\mathscr{H}_\mu : C_0(\mathbf{R})\to C_0(\mathbf{R})$ are bounded operators, provided that $\mu(\{0\})=0$ in the latter case.

Analogously, one can also show that if the condition $\mathscr{K}_{\mu_j}(\infty^*)=\mathscr{K}_{\mu_j}(1)<\infty$ is satisfied for $j=1,2$, where μ_1 and μ_2 are from (4.1), then both $\mathscr{H}^*_\mu : C_b(\mathbf{R})\to L^\infty(\mathbf{R})$ and $\mathscr{H}^*_\mu : C_0(\mathbf{R})\to C_b(\mathbf{R})$ are bounded operators.

Again, the operator $\mathscr{H}^*$ can be considered to be the adjoint to $\mathscr{H}$ in the following sense.

THEOREM 2^*. *Assume that μ is a σ-finite complex measure on $\mathbf{R}$ such that $\mathscr{K}_{\mu_j}(p)<\infty$ for some $1<p<\infty$ and $j=1,2$, where μ_1 and μ_2 are from* (4.1). *If $f\in L^p(\mathbf{R})$ and $g\in L^{p^*}(\mathbf{R})$, then*

$$\int_{-\infty}^{\infty}[\mathscr{H}_\mu f(x)]g(x)dx = \int_{-\infty}^{\infty} f(x)[\mathscr{H}^*_\mu g(x)]dx.$$

Furthermore, Corollary 1 remains valid for σ-finite complex measures on $\mathbf{R}$, too.

We recall that the Fourier transform $\hat{f}$ of a function $f\in L^1(\mathbf{R})$ is defined by

$$\hat{f}(u) := (2\pi)^{-1/2}\int_{-\infty}^{\infty} f(x)e^{-iux}dx, \quad u\in\mathbf{R}. \tag{4.4}$$

The following corollary of Theorem 1^* reveals another interesting interplay between the Hausdorff operator $\mathscr{H}$ and its adjoint $\mathscr{H}^*$.

COROLLARY 2. *Assume that μ is a σ-finite complex measure on $\mathbf{R}$, and $f\in L^1(\mathbf{R})$.*

(i) If $\mathscr{K}_{\mu_j}(1)<\infty$ for $j=1,2$, where μ_1 and μ_2 are from (4.1), *then*

$$(\mathscr{H}_\mu f)^\wedge(u) = \mathscr{H}^*_\mu\hat{f}(u), \quad u\in\mathbf{R}. \tag{4.5}$$

(ii) If $\mathscr{K}_{\mu_j}(\infty)<\infty$ for $j=1,2$, then

$$(\mathscr{H}^*_\mu f)^\wedge(u) = \mathscr{H}_\mu\hat{f}(u), \quad u\in\mathbf{R}. \tag{4.6}$$

Identity (4.6) explains why $\mathscr{H}^*$ may be viewed as the harmonic Hausdorff operator. In this context, $\mathscr{H}$ may be called the harmonic quasi Hausdorff operator.

Proof. (i) Due to the condition $\mathscr{K}_{\mu_j}(1) < \infty$ for $j = 1, 2$, Theorem 1 * guarantees that $\mathscr{H}f \in L^1(\mathbf{R})$, so (4.4) makes sense. By definition (4.2), Fubini's theorem, and integration by the substitution $s := xt$, for any $u \in \mathbf{R}$, we conclude the following:

$$
\begin{aligned}
(2\pi)^{1/2}(\mathscr{H}f)^\wedge(u) &:= \int_{-\infty}^{\infty} (\mathscr{H}f)(x)e^{-iux}dx \\
&= \int_{-\infty}^{\infty} e^{-iux}\Big\{\int_{-\infty}^{\infty} f(xt)d\mu(t)\Big\}dx \\
&= \int_{-\infty}^{\infty} \Big\{\int_{-\infty}^{\infty} f(xt)e^{-iux}dx\Big\}d\mu(t) \\
&= \int_{0}^{\infty} \Big\{\int_{-\infty}^{\infty} f(s)e^{-ius/t}\frac{ds}{t}\Big\}d\mu(t) \\
&\qquad + \int_{-\infty}^{0} \Big\{\int_{\infty}^{-\infty} f(s)e^{-ius/t}\frac{ds}{t}\Big\}d\mu(t) \\
&= (2\pi)^{1/2}\Big\{\int_{0}^{\infty} \frac{1}{t}\hat{f}(\frac{u}{t})d\mu(t) - \int_{-\infty}^{0} \frac{1}{t}\hat{f}(\frac{u}{t})d\mu(t)\Big\} \\
&= (2\pi)^{1/2}\int_{-\infty}^{\infty} \frac{1}{|t|}\hat{f}(\frac{u}{t})d\mu(t) =: (2\pi)^{1/2}\mathscr{H}^*\hat{f}(u).
\end{aligned}
$$

This proves (4.5).

(ii) The proof of (4.6) can be done analogously. We do not enter into details. □

REMARK. On closing, we reformulate definitions (4.2) and (4.3) in the particular case when μ is absolutely continuous with respect to the Lebesgue measure on $\mathbf{R}$:

$$d\mu(t) = \phi(t)dt, \quad \text{where} \quad \phi \in L^1(\mathbf{R}). \tag{4.7}$$

Then (4.2) and (4.3) are respectively of the following form:

$$\mathscr{H}_\mu f(x) := \int_{-\infty}^{\infty} f(xt)\phi(t)dt = \frac{1}{|x|}\int_{-\infty}^{\infty} f(s)\phi(\frac{s}{x})ds, \quad 0 \neq x \in \mathbf{R},$$

and

$$\mathscr{H}_\mu^* f(x) := \int_{-\infty}^{\infty} \frac{1}{|t|}f(\frac{x}{t})\phi(t)dt = \int_{-\infty}^{\infty} f(s)\phi(\frac{x}{s})\frac{ds}{|s|}, \quad x \in \mathbf{R}.$$

These definitions immediately make sense for any function f in $L^p(\mathbf{R})$ for any $1 \leqslant p \leqslant \infty$. The reason is that in the case of (4.7), the subsets of $\mathbf{R}$ of measure zero with respect to the Lebesgue measure and with respect to μ coincide. Consequently, if two functions are equivalent with respect to the Lebesgue measure, then they are equivalent with respect to μ, as well.

In the further particular case (2.4), that is, when $\phi := \chi_{(0,1)}$, the indicator function of the unit interval $(0,1)$ in (4.7), we get respectively the definitions of the Cesàro and Copson operators on $\mathbf{R}$:

$$\mathscr{C}f(x) := \int_{0}^{1} f(xt)dt = \frac{1}{x}\int_{0}^{x} f(s)ds, \quad 0 \neq x \in \mathbf{R},$$

and

$$\mathscr{C}^* f(x) := \int_0^1 \frac{1}{t} f\left(\frac{x}{t}\right) dt = \begin{cases} \int_x^\infty \frac{f(s)}{s} ds, & x > 0, \\ -\int_{-\infty}^x \frac{f(s)}{s} ds, & x < 0. \end{cases}$$

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