ZYGMUND TYPE INEQUALITIES FOR DOUBLE
SINGULAR CAUCHY–STIELTJES INTEGRAL

A. GAZIEV

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Abstract. For the double singular Cauchy-Stiltjes integral over spanning set of a bicylindric domain, a Zygmund type inequality connecting partial and mixed moduli of continuity of the singular integral and its density, is obtained. On this basis, some spaces are constructed, invariant with respect to the double singular integral.

1. Let γ^k be a closed Jordan rectifiable curve (c.j.r.c.) on the complex plane z_k $(k = 1, 2)$, which divides the complex plane into two parts, the interior D_k^+ and the exterior D_k^-. The curves γ^1 and γ^2 define four bicylindric domains $D^\pm = D_1^\pm \times D_2^\pm$ with the boundaries having the common part $\triangle = \gamma^1 * \gamma^2$ known as spanning set. Let

$$\Phi_\psi(z) = \frac{1}{(2\pi i)^2} \int\limits_\triangle \frac{f(\tau)}{\prod\limits_{k=1}^2 (\tau_k - z_k)} d\psi(\tau) \tag{1}$$

be the double Cauchy-Stiltjes type integral, where $z = (z_1, z_2), \tau = (\tau_1, \tau_2), d\psi(\tau) = d\psi_1(\tau)d\psi_2(\tau)$, $f(\tau) \in C_\triangle$, where $C_\triangle$ is the space of continuous functions on $\triangle, \psi_k(\tau)$ being functions of bounded variation on $\gamma^k(k = 1, 2)$. Under the investigation of limiting values of the function $\Phi_\psi(z)$ there appear the following singular integrals:

$$g_\psi^{1,2}(t) = \int\limits_\triangle \frac{\Delta f(\tau; t)}{\prod\limits_{k=1}^2 (\tau_k - z_k)} d\psi(\tau),$$

$$g_\psi^{1,0}(t)) = \int\limits_{\gamma^1} \frac{\Delta f(\tau_{t_2}; t)}{\tau_1 - t_1} d\psi_1(\tau_1), \tag{2}$$

$$g_\psi^{0,2}(t) = \int\limits_{\gamma^2} \frac{\Delta f(\tau_{t_1}; t)}{\tau_2 - t_2} d\psi(\tau_2),$$

where

$$\Delta f(\tau;t) \overset{def}{=} \Delta f(\tau_1, \tau_2; t_1, t_2) = f(\tau_1, \tau_2) - f(t_1, \tau_2) - f(\tau_1, t_2) + f(t_1, t_2),$$

$$\Delta f(\tau_{t_2};t) \overset{def}{=} \Delta f(\tau_1, t_2; t_1, t_2) = f(\tau_1, t_2) - f(t_1, t_2),$$

$$\Delta f(\tau_{t_1};t) \overset{def}{=} \Delta f(t_1, \tau_2; t_1, t_2) = f(t_1, \tau_2) - f(t_1, t_2).$$

We denote

$$\tilde{f}_\psi(t) = g_\psi^{1,2}(t) + g_{\psi_1}^{1,0}(t) + g_{\psi_2}^{0,2}(t) . \tag{3}$$

In the case when $\psi_i(t) = t \ (i = 1, 2)$, we write $\tilde{f}(t) = g^{1,2}(t) + g^{1,0}(t) + g^{0,2}(t)$.

In the case when $\gamma^i (i = 1, 2)$ is the unit circle, the singular integral $g^{1,2}(t)$ is reduced to

$$h(x, y) = \int\limits_{-\pi}^{\pi} \int\limits_{-\pi}^{\pi} f(x + t, y + \tau) \operatorname{ctg} \frac{t}{2} \operatorname{ctg} \frac{\tau}{2} dt d\tau .$$

It is known that the space $H^2(\varphi_1, \varphi_2)$ (see [4]) is not invariant for the singular integral h, see [1]-[3] in the case $\varphi_1(\delta) = \varphi_2(\delta) = \delta^\alpha, 0 < \alpha < 1$) and [4] for the general case. At the same time in [5] it was proved that the function spaces

$$L^{\alpha,\beta} = \{f \in C_{[-\pi,\pi]^2} : \omega_f(\delta_1, \delta_2) = O(\delta_1^\alpha \cdot \delta_2^\beta), 0 < \alpha, \beta < 1\}$$

are invariant for the singular integral h.

In [6] for the function h there was obtained an inequality of Zygmund type. Based on that inequality, in [7] there was constructed the class $\Im_{\infty,\infty}$ invariant with respect to the singular integral h. In the one-dimensional case this class was introduced in [8]. In [4] for the function h there was proved some analogue of the Plemeli-Privalov theorem.

In [9] there was proved an analogue of Zygmund inequality for the function $\tilde{f}(t)$, in terms of the characteristic $\theta(\delta)$ introduced in [10].

In the one-dimensional case in [11] there was obtained some estimate of Zygmund type for the singular integral $\tilde{f}_\psi$ in terms of the characteristic $\theta_\psi(\delta)$ introduced in the same paper, and an analogue of Plemeli-Privalov and Magnaradze ([8]) theorems were obtained.

An analogue of Zygmund inequality in terms of the characteristic $\theta(\delta)$ was obtained in [12] in the n-dimensional case for the function $\tilde{f}$.

In this paper we give a Zygmund type inequality connecting partial and mixed continuity moduli of the functions f and $\tilde{f}_\psi$. With the help of these estimates a Banach space invariant with respect to the singular integral $\tilde{f}_\psi$ is constructed.

2. As in [11], we denote

$$\theta_k^{\psi_k} = \int\limits_{\gamma_\delta^k(t_k)} |d\psi_k(\tau_k)|, \quad \theta_k^{\psi_k}(\delta) = \sup_{t_k \in \gamma^k} \theta_k^{\psi_k}(t_k, \delta),$$

where $\gamma_\delta^k(t_k) = \{\tau \in \gamma^k : |\tau - t_k| \leqslant \delta, \delta \in (0, d_k]\}$, $d_k = sup|\tau_k - t_k|, k = 1, 2$. Functions $\varphi_1(\delta)$ and $\varphi_2(\delta)$ non-increasing in the interval $[0, \int\limits_\gamma |d\psi(\tau)|]$ are said to be

equal to each other $(\varphi_1 \overset{ess}{=} \varphi_2)$, if they are equal on some dense set which contains the point $\int\limits_\gamma |d\psi(\tau)|$ (see [11]).

The monotonous increasing function $\theta^\psi(\delta)$, defined by

$$\breve{\theta}^\psi(\delta) \overset{ess}{=} sup\{t : \theta^\psi \leqslant \delta\}, \quad \delta \in \left(0, \int\limits_\gamma |d\psi(\tau)|\right),$$

is called the generalized inverse to the function $\theta(\delta)$, see [10,14] .

Let Q be a domain in the complex plane $\mathbf{C}$ and $H(Q)$ the class of functions golomorphic in Q and continuous in $\bar{Q}$. Let also D be a bounded region in $\mathbf{C}$ with the boundary $\partial D = \gamma$ which is (c.j.r.c.). For $F \in H(C\bar{D})$ we usually assume that $\lim_{z\to\infty} F(z) = 0$.

In the case when $d\psi(t) = F(t)dt$, where $F(t)$ is limiting value of a function analytic in $D^\pm$ and continuous up to the boundary, we denote

$$\theta^\psi \overset{def}{=} \theta^F(\delta) = \sup_{t\in\gamma} \theta^F(t,\delta), \quad \theta^F(t,\delta) = \int\limits_{\gamma_\delta(t)} |F(\tau)||d\tau|, \ \delta \in (0,d] \ ,$$

so that

$$\theta^\psi \overset{def}{=} \theta(\delta) = \sup_{t\in\gamma} \int\limits_{\gamma_\delta(t)} |dt|$$

in the case $F(t) \equiv 1$.

From the definition it follows that $\theta^F(\delta)$ is a non-negative and non-decreasing function on $(0,d]$ and $\theta^F(t,\delta) \leqslant \theta^F(\delta) \leqslant C\theta(\delta)$ with the constant C depending on F.

We denote

$$\Im_0(\gamma) = \{f \in C(\gamma) : \int_0^d \frac{\omega_f(\xi)}{\xi}d\theta(\xi) < \infty\} \tag{A}$$

and

$$\Im_0^F(\gamma) = \{f \in C(\gamma) : \int_0^d \frac{\omega_f(\xi)}{\xi}d\theta^F(\xi) < \infty\} \ , \tag{B}$$

where $F \in H(D)$ (or $F \in H(C\bar{D})$).

Let $F \in H(D)$ (or $F \in H(C\bar{D})$). If $F(t) \neq 0 \forall t \in \gamma$, then $\theta^F(\delta) \sim \theta(\delta)$ (here $f \sim g \equiv \exists C_1, C_2$ such that $C_1 f(t) \leqslant g(t) \leqslant C_2 f(t) \forall t \in (0,d]$) and in this case $\Im_0(\gamma) = \Im_0^F(\gamma)$, that is, $(A) \Leftrightarrow (B)$. In the general case the conditions (A) and (B) are not equivalent. Since $\theta^F(\delta) \leqslant C_F\theta(\delta)$, we have $\int_0^d \frac{\omega_f(\xi)}{\xi}d\theta^F(\xi) \leqslant C_F \int_0^d \frac{\omega_f(\xi)}{\xi}d\theta(\xi)$, whence $\Im_0(\gamma) \subset \Im_0^F(\gamma)$. To study the properties of the integral (2), we arrive at the necessity to choose the following basic characteristics of functions $f \in C_\triangle$:

1) mixed modulus of continuity ($\delta = (\delta_1, \delta_2), \delta_1 > 0, \delta_2 > 0, \xi = (\xi_1, \xi_2)$):

$$\overset{1,2}{\omega_f}(\delta) \overset{def}{=} \delta_1 \cdot \delta_2 \sup_{\xi_1 \geqslant \delta_1, \xi_2 \geqslant \delta_2} \frac{\overset{1,2}{\omega}(f;\xi_1,\xi_2)}{\xi_1\xi_2} = \delta \sup_{\xi\geqslant\delta} \frac{\overset{1,2}{\omega}(f,\xi)}{\xi},$$

where $\overset{1,2}{\omega}(f,\delta) \overset{def}{=} \sup\limits_{|\tau_1-t_1|\leqslant\delta_1,|\tau_2-t_2|\leqslant\delta_2} |\Delta f(\tau;t)|$, and

2) partial continuity moduli

$$\overset{1,0}{\omega}_f(\delta_1) = \delta_1 \sup\limits_{\xi_1\geqslant\delta_1} \frac{\overset{1,0}{\omega}(f,\xi_1)}{\xi_1}, \quad \overset{1,0}{\omega}(f,\delta_1) = \sup\limits_{t_2\in\gamma^2} \sup\limits_{|t_1-\tau_1|\leqslant\delta_1} |\Delta f(\tau_{t_2};t)|$$

and

$$\overset{0,2}{\omega}_f(\delta_2) = \delta_2 \sup\limits_{\xi_2\geqslant\delta_2} \frac{\overset{0,2}{\omega}(f,\xi_2)}{\xi_2}, \quad \overset{0,2}{\omega}(f,\delta_2) = \sup\limits_{t_1\in\gamma^1} \sup\limits_{|\tau_2-t_2|\leqslant\delta_2} |\Delta f(\tau_{t_1};t)|.$$

By $\Phi^1_{(0,d]}$ we denote the set of those non-negative increasing functions $\varphi(\delta)$ on $(0,d]$, for which $\lim_{\delta\to 0}\varphi(\delta) = 0$ and $\frac{\varphi(\delta)}{\delta}$ decreases.

Let $\Phi_{(0,d_1]\cdot(0,d_2]} = \Phi_{T^2}$ denote the set of functions $\omega(\delta_1,\delta_2) = \omega(\delta)$ defined on $T^2 = [0,d_1]\times[0,d_2]$ and belonging to Φ^1 in each argument, i.e.

1) $\omega(\delta) \in \Phi^1_{0,d_2]}$ in δ_2 for any fixed δ_1,

2) $\omega(\delta) \in \Phi^1_{0,d_1]}$ in δ_1 for any fixed δ_2.

In [15] it was shown that the properties 1) and 2) are characteristic for continuity moduli in the sense that for every $\omega\in\Phi_{T^2}$ there exist such a function $f\in C_\triangle$, that

$$\omega_f(\delta_1,\delta_2) \sim \omega(\delta_1,\delta_2), \quad \overset{1,0}{\omega}_f(\delta_1) \sim \omega(\delta_1,d_2), \quad \overset{0,2}{\omega}_f(\delta_2) \sim \omega(d_1,\delta_2),$$

By V_γ (see [11]) we denote the set of all functions with bounded variation on γ, for which the integral $\left| \int\limits_{\gamma\setminus\gamma_\varepsilon(t)} \frac{d\psi(\tau)}{\prod_{k=1}^2(\tau_k-t_k)} \right|$ is uniformly bounded.

Let a function $\omega(\delta_1,\delta_2)$ be defined on T^2, non-negative and satisfying the condition

$$\int_0^{d_1}\int_0^{d_2} \frac{\omega(\xi_1,\xi_2)}{\xi_1\xi_2} d\theta_1^{\psi_1}(\xi_1)d\theta_2^{\psi_2}(\xi_2) < \infty.$$

We introduce the Zygmund type operator

$$Z(\omega;\delta,\theta^\psi,\check\theta^\psi) \overset{def}{=} Z(\omega;\delta_1,\delta_2,\theta_1^{\psi_1},\theta_2^{\psi_2},\check\theta_1^{\psi_1},\check\theta_2^{\psi_2})$$

$$= \int_0^{\theta_1^{\psi_1}(\delta_1)} \int_0^{\theta_2^{\psi_2}(\delta_2)} \frac{\omega(\check\theta_1^{\psi_1}(\xi_1),\check\theta_2^{\psi_2}(\xi_2))}{\check\theta_1^{\psi_1}(\xi_1)\cdot\check\theta_2^{\psi_2}(\xi_2)}d\xi_1 d\xi_2$$

$$+ \delta_1 \int_{\theta_1^{\psi_1}(\delta_1)}^{d_1} \int_0^{\theta_2^{\psi_2}(\delta_2)} \frac{\omega(\check\theta_1^{\psi_1}(\xi_1),\check\theta_2^{\psi_2}(\xi_2))}{(\check\theta_1^{\psi_1}(\xi_1))^2\cdot\check\theta_2^{\psi_2}(\xi_2)}d\xi_1 d\xi_2$$

$$+ \delta_2 \int_0^{\theta_1^{\psi_1}(\delta_1)} \int_{\theta_2^{\psi_2}(\delta_2)}^{d_2} \frac{\omega(\check\theta_1^{\psi_1}(\xi_1),\check\theta_2^{\psi_2}(\xi_2))}{\check\theta_1^{\psi_1}(\xi_1)\cdot(\check\theta_2^{\psi_2}(\xi_2))^2}d\xi_1 d\xi_2$$

$$+ \delta_1\delta_2 \int_{\theta_1^{\psi_1}(\delta_1)}^{d_1} \int_{\theta_1^{\psi_1}(\delta_1)}^{d_2} \frac{\omega(\check\theta_1^{\psi_1}(\xi_1),\check\theta_2^{\psi_2}(\xi_2))}{(\check\theta_1^{\psi_1}(\xi_1))^2\cdot(\check\theta_2^{\psi_2}(\xi_2))^2}d\xi_1 d\xi_2.$$

It is not hard to show that $Z(\omega; \delta, \theta^\psi, \breve{\theta}^\psi) \in \Phi_{T^2}$ and

$$Z(\omega; \delta, \theta^\psi, \breve{\theta}^\psi) = Z_1\left(Z_2(\omega(\xi_1, \cdot), \delta_2, \theta_2^{\psi_2}, \breve{\theta}_2^{\psi_2}), \delta_1, \theta_1^{\psi_1}, \breve{\theta}_1^{\psi_1}\right)$$

$$= \int_0^{\theta_1^{\psi_1}(\delta_1)} \frac{Z_2(\omega(\breve{\theta}_1^{\xi_1}, \cdot), \delta_2, \theta_2^{\psi_2}, \breve{\theta}_2^{\psi_2})}{\breve{\theta}_1^{\psi_1}(\xi_1)} d\xi_1 + \delta_1 \int_{\theta_1^{\psi_1}(\delta_1)}^{d_1} \frac{Z_2(\omega(\breve{\theta}_1^{\xi_1}, \cdot), \delta_2, \theta_2^{\psi_2}, \breve{\theta}_2^{\psi_2})}{(\breve{\theta}_1^{\psi_1}(\xi_1))^2} d\xi_1$$

and

$$Z_2(\omega(\breve{\theta}_1^{\psi_1}(\xi_1), \cdot), \delta_2, \theta_2^{\psi_2}, \breve{\theta}_2^{\psi_2})$$

$$= \int_0^{\theta_2^{\psi_2}(\delta_2)} \frac{\omega(\breve{\theta}_1^{\psi_1}(\xi_1), \breve{\theta}_2^{\psi_2}(\xi_2))}{\breve{\theta}_2^{\psi_2}(\xi_2)} d\xi_2 + \delta_2 \int_{\theta_2^{\psi_2}(\delta_2)}^{d_2} \frac{\omega(\breve{\theta}_1^{\psi_1}(\xi_1), \breve{\theta}_2^{\psi_2}(\xi_2))}{(\breve{\theta}_2^{\psi_2}(\xi_2))^2} d\xi_2.$$

For further goals we need the following technical lemma (see [11]).

LEMMA 1. *Let $g(t)$ be a function non-decreasing on $(0, d]$ and $\psi(t)$ a function of bounded variation on γ, then $\forall \varepsilon', \varepsilon'' \in (0, d], \varepsilon' < \varepsilon''$*

$$\int_{\gamma_{\varepsilon''}(t) \backslash \gamma_{\varepsilon'}(t)} g(|\tau - t|)|d\psi(\tau)| = \int_{\varepsilon'}^{\varepsilon''} g(\tau) d\theta^\psi(t, \tau).$$

Let us denote

$$g_{\psi, \varepsilon}^{1,2}(t) = \int_{\triangle \backslash \triangle_\varepsilon} \frac{\Delta f(\tau; t)}{\tau - t} d\psi(\tau),$$

where $\varepsilon = (\varepsilon_1, \varepsilon_2)$, so that $\triangle \backslash \triangle_\varepsilon = \gamma^1 \backslash \gamma_{\varepsilon_1}^1(t_1) \times \gamma^2 \backslash \gamma_{\varepsilon_2}^2(t_2)$.

As usual, we say that the integral $g_\psi^{1,2}$ exists in the principal value sense, if there exists the finite limit

$$\lim_{\varepsilon_1 \to 0, \varepsilon_2 \to 0} g_{\psi, \varepsilon_1, \varepsilon_2}^{1 \cdot 2}.$$

THEOREM 1. *Let $f \in C_\triangle, \psi_k \in V_\gamma$ $(k = 1, 2)$. If*

$$\int_0^d \frac{\omega_f^{1,2}(\eta)}{\eta} d\theta^\psi(\eta) \stackrel{def}{=} \int_0^{d_1} \int_0^{d_2} \frac{\omega_f^{1,2}(\eta_1, \eta_2)}{\eta_1 \eta_2} d\theta_1^{\psi_1}(\eta_1) d\theta_2^{\psi_2}(\eta_2) < \infty,$$

then the limits $\lim_{\varepsilon_1 \to 0} g_{\psi, \varepsilon}^{1 \cdot 2}(t_1, t_2)$, $\lim_{\varepsilon_2 \to 0} g_{\psi, \varepsilon}^{1 \cdot 2}(t_1, t_2)$, $\lim_{\varepsilon_1 \to 0, \varepsilon_2 \to 0} g_{\psi, \varepsilon}^{1 \cdot 2}(t_1, t_2)$ with any fixed $\varepsilon_2 \in (0, d_2]$ in the first limit, and any fixed $\varepsilon_1 \in (0, d_1]$ in the second one, exist uniformly in t_1, t_2.

Proof. It can be seen directly that

$$
g^{1,2}_{\varepsilon_1,\varepsilon_2}(t_1,t_2) - g^{1,2}_{\eta_1,\eta_2}(t_1,t_2) = \int\limits_{\gamma^1_{\eta_1}(t_1)\setminus\gamma^1_{\varepsilon_1}(t_1)} \int\limits_{\gamma^2_{\eta_2}(t_1)\setminus\gamma^2_{\varepsilon_2}(t_2)} \frac{\Delta f(\tau;t)}{\tau-t}d\psi(\tau)
$$

$$
+ \int\limits_{\gamma^1\setminus\gamma^1_{\eta_1}(t_1)} \int\limits_{\gamma^2_{\eta_2}(t_1)\setminus\gamma^2_{\varepsilon_2}(t_2)} \frac{\Delta f(\tau;t)}{\tau-t}d\psi(\tau) + \int\limits_{\gamma^1_{\eta_1}(t_1)\setminus\gamma^1_{\varepsilon_1}(t_1)} \int\limits_{\gamma^2\setminus\gamma^2_{\varepsilon_2}(t_2)} \frac{\Delta f(\tau;t)}{\tau-t}d\psi(\tau)
$$

$$
\stackrel{def}{=} \Im_1 + \Im_2 + \Im_3,
$$

where $0 < \varepsilon_1 < \eta_1 \leqslant d_1$ and $0 < \varepsilon_2 < \eta_2 \leqslant d_2$.

We estimate every term separately. Obviously,

$$
|\Im_1| \leqslant \int\limits_{\gamma^1_{\eta_1}(t_1)\setminus\gamma^1_{\varepsilon_1}(t_1)} \int\limits_{\gamma^2_{\eta_2}(t_2)\setminus\gamma^2_{\varepsilon_2}(t_2)} \frac{\omega^{1,2}_f(|\tau_1 - t_1|;|\tau_2 - t_2|)}{|\tau_1 - t_2||\tau_2 - t_2|}|d\psi_1(\tau_1)||d\psi_2(t_1)|.
$$

The case 1): $\eta_1 \leqslant \varepsilon_2$. Applying subsequently Lemma 1 and Theorem from [14], p. 241, and taking into account that $\theta^{\psi_k}_k(t_k,\delta) \leqslant \theta^{\psi_k}_k(\delta)$ and $\frac{\omega^{1,2}_f(\delta_1,\delta_2)}{\delta_k}\downarrow$ $(k=1,2)$, we obtain

$$
|\Im_1| \leqslant \int\limits_0^{\eta_1}\int\limits_0^{\eta_2} \frac{\omega^{1,2}_f(\xi_1,\xi_2)}{\xi_1\cdot\xi_2}d\theta^{\psi_1}_1(\xi_1)d\theta^{\psi_2}_2(\xi_2).
$$

The case 2): $\varepsilon_1 \leqslant \varepsilon_2 \leqslant \eta_1 \leqslant \eta_2$. In this case we have

$$
|\Im_1| \leqslant \left(\int\limits_{\gamma^1_{\varepsilon_1}(t_1)\setminus\gamma^1_{\eta_1}(t_1)} + \int\limits_{\gamma^1_{\eta_1}(t_1)\setminus\gamma^2_{\varepsilon_2}(t_1)}\right)\left(\int\limits_{\gamma^2_{\eta_1}(t_2)\setminus\gamma^2_{\varepsilon_2}(t_2)} + \int\limits_{\gamma^2_{\eta_2}(t_2)\setminus\gamma^2_{\eta^2_1}(t_1)}\right)
$$

$$
\times\frac{\omega^{1,2}_f(|\xi_1-t_1|,|\xi_2-t_2|)}{|\xi_1-t_1|\cdot|\xi_2-t_2|}|d\psi_1(\xi_1)||d\psi_2(\xi_2)|
$$

Taking into account the case 1), we obtain

$$
|\Im_1| \leqslant \left(\int\limits_0^{\varepsilon_1}\int\limits_0^{\eta_1} + \int\limits_0^{\varepsilon_2}\int\limits_0^{\eta_2} + \int\limits_0^{\eta_1}\int\limits_0^{\eta_1} + \int\limits_0^{\eta_1}\int\limits_0^{\eta_2}\right)\frac{\omega^{1,2}_f(\xi_1),\xi_2)}{\xi_1\cdot\xi_2}d\theta^{\psi_1}_1(\xi_1)d\theta^{\psi_2}_2(\xi_2).
$$

The integral $\Im_1$ in the cases $\varepsilon_1 \leqslant \varepsilon_2 \leqslant \eta_2 \leqslant \eta_1$, $\varepsilon_2 \leqslant \varepsilon_1 \leqslant \eta_2 \leqslant \eta_1$, $\varepsilon_2 \leqslant \varepsilon_1 \leqslant \eta_1 \leqslant \eta_2$ is estimated in a similar way.

Passing to the integral $\Im_3$, we have

$$
|\Im_3| \leqslant \int\limits_0^{\eta_1}\int\limits_0^{d_2} \frac{\omega^{1,2}_f(\xi_1,\xi_2)}{\xi_1\xi_2}d\theta^{\psi_1}_1(\xi_1)d\theta^{\psi_2}_2(\xi_2),
$$

if $\eta_1 \leqslant \eta_2$, and

$$|\Im_3| \leqslant \left(\int\limits_0^{\eta_2}\int\limits_0^{d_2} + \int\limits_0^{\eta_1}\int\limits_0^{d_2} \right) \frac{\overset{1,2}{\omega}_f(\xi_1,\xi_2)}{\xi_1\xi_2} d\theta_1^{\psi_1}(\xi_1) d\theta_2^{\psi_2}(\xi_2).$$

if $\varepsilon_1 \leqslant \eta_2 \leqslant \eta_1$. The integral $\Im_3$ is estimated similarly in the case $\eta_2 \leqslant \varepsilon_1 \leqslant \eta_1$. In the symmetrical way the integral $\Im_2$ is estimated. The above estimates prove the theorem.

In the following theorem we use the notation

$$\Im_0^{\psi}(\triangle) = \Big\{ f \in C_{\triangle} : \int\limits_0^d \frac{\overset{1,2}{\omega}_f(\xi)}{\xi} d\theta^{\psi}(\xi) < \infty,$$

$$\int\limits_0^{d_1} \frac{\overset{1,0}{\omega}_f(\xi_1)}{\xi_1} d\theta_1^{\psi_1}(\xi_1) < \infty, \int\limits_0^{d_2} \frac{\overset{0,2}{\omega}_f(\xi_2)}{\xi_2} d\theta_2^{\psi_2}(\xi_2) < \infty \Big\}.$$

THEOREM 2. *Let* $f \in \Im_0^{\psi}(\triangle)$ *with* $\psi = (\psi_1, \psi_2)$, $\psi_k \in V_k$ $(k = 1, 2)$. *Then the following inequalities are valid*

$$\overset{1,2}{\omega}_{\tilde{f}}(\delta) \leqslant CZ(\overset{1,2}{\omega}_f; \delta; \theta^{\psi}, \check{\theta}^{\psi}), \quad 0 < \delta_k \leqslant d_k, k = 1, 2,$$

$$\overset{1,0}{\omega}_{\tilde{f}}(\delta_1) \leqslant C \left(Z(\overset{1,2}{\omega}_f; \delta_1, d_2, \theta^{\psi}, \check{\theta}^{\psi} + Z(\overset{1,0}{\omega}_f; \delta_1, \theta_1^{\psi_2}, \check{\theta}_1^{\psi_1})) \right), \quad 0 < \delta_1 \leqslant d_1,$$

$$\overset{0,2}{\omega}_{\tilde{f}}(\delta_2) \leqslant C \left(Z(\overset{1,2}{\omega}_f; d_1, \delta_2, \theta^{\psi}, \check{\theta}^{\psi}) + (Z(\overset{0,2}{\omega}_f; \delta_2, \theta_2^{\psi_2}, \check{\theta}_2^{\psi_2})) \right), \quad 0 < \delta_2 \leqslant d_2. \quad (4)$$

Proof. Let $t_k, t_k' \in \gamma^k, |t_k - t_k'| = \varepsilon_k \leqslant d_k, k = 1, 2$. For the function $\tilde{f}_{\psi}(t)$ we estimate mixed and partial increments

$$\Delta \tilde{f}_{\psi}(t; t') \overset{def}{=} \tilde{f}_{\psi}(t_1, t_2) - \tilde{f}_{\psi}(t_1', t_2) - \tilde{f}_{\psi}(t_1, t_2') + \tilde{f}_{\psi}(t_1', t_2')$$

$$= \Delta \overset{1,2}{g}_{\psi}(t; t') + \Delta \overset{1,0}{g}_{\psi_1}(t; t') + \Delta \overset{0,2}{g}_{\psi_2}(t; t'), \quad (5)$$

$$\Delta \tilde{f}_{\psi}(t_{t_1'}; t') \overset{def}{=} \tilde{f}_{\psi}(t_1', t_2) - \tilde{f}_{\psi}(t_1', t_2) = \Delta \overset{1,2}{g}_{\psi}(t_{t_1'}; t') + \Delta \overset{0,2}{g}_{\psi_2}(t_{t_1'}; t') + \Delta \overset{1,0}{g}_{\psi_1}(t_{t_1'}; t') \quad (6)$$

and

$$\Delta \tilde{f}_{\psi}(t_{t_2'}; t') \overset{def}{=} \tilde{f}_{\psi}(t_1, t_2') - \tilde{f}_{\psi}(t_1', t_2') = \Delta \overset{1,2}{g}_{\psi}(t_{t_2'}; t') + +\Delta \overset{0,2}{g}_{\psi_2}(t_{t_2'}; t') + \Delta \overset{1,0}{g}_{\psi_1}(t_{t_2'}; t'). \quad (7)$$

To estimate $\Delta \overset{1,2}{g}_{\psi}(t; t')$, we observe that

$$\Delta \overset{1,2}{g}_{\psi}(t; t') = \int\limits_{\gamma^1} \frac{g_2(\Delta f; \tau_{t_2}, t_{t_2'})}{\tau_1 - t_1} d\psi_1(\tau_1) - \int\limits_{\gamma^1} \frac{g_2(\Delta f; \tau_{t_2}, t')}{\tau_1 - t_1'} d\psi_1(\tau_1) \quad (8)$$

where

$$g_2(\triangle f; \tau_{t_2}, t_{t_2'}) = \int_{\gamma^2} \frac{\triangle f(\tau; t)}{\tau_2 - t_2} d\psi_2(\tau_2) - \int_{\gamma^2} \frac{\triangle f(\tau, t_{t_2'})}{\tau_2 - t_2'} d\psi_2(\tau_2) \qquad (9)$$

and similarly for $g_2(\Delta f; \tau_{t_2}, t')$.

It is easy to see that $\Delta f(\tau; t) - \triangle f(\tau; t_{t_k'}) = -\Delta f(\tau_{t_k}; t_{t'_k}), k = 1, 2$, so that

$$g_2(\Delta f; \tau_{t_2}, t_{t_2'}) = \int_{\gamma_{\varepsilon_2}^2(t_2)} \frac{\Delta f(\tau; t)}{\tau_2 - t_2} d\psi_2(\tau_2) - \int_{\gamma_{\varepsilon_2}^2(t_2')} \frac{\Delta f(\tau, t_{t_2'})}{\tau_2 - t_2'} d\psi_2(\tau_2)$$

$$+ (t_2 - t_2') \int_{\gamma^2 \setminus \gamma_{\varepsilon_2}^2} \frac{\Delta f(\tau; t)}{(\tau_2 - t_2)(\tau_2 - t_2')} d\psi_2(\tau_2) - \int_{\gamma^2 \setminus \gamma_{\varepsilon_2}^2} \frac{\Delta f(\tau_{t_2}, t_{t_2'})}{\tau_2 - t_2'} d\psi_2(\tau_2)$$

$$+ \int_{\gamma_{\varepsilon_2}^2(t_2') \setminus \gamma_{\varepsilon_2}^2(t_2)} \frac{\Delta f(\tau; t)}{\tau_2 - t_2} d\psi(\tau_2) + \int_{\gamma_{\varepsilon_2}^2(t_2) \setminus \gamma_{\varepsilon_2}^2(t_2')} \frac{\Delta f(\tau; t_{t_2'})}{\tau_2 - t_2'} d\psi_2(\tau_2) \stackrel{def}{=} \sum_{k=0}^6 \Im_k, \qquad (10)$$

where $\gamma_{\varepsilon_k}^k = \gamma_{\varepsilon_k}^k(t_k) \bigcup \gamma_{\varepsilon_k}^k(t_k'), k = 1, 2$. Basing on the ideas of the proof of Theorem 1 from [11] and Lemma 1, it is easy to obtain the estimates:

$$|\Im_i| \leqslant \int_0^{\varepsilon_2} \frac{\overset{1,2}{\omega_f}(|\tau_1 - t_1|, \xi_2)}{\xi_2} d\theta_2^{\psi_2}(\xi_2), \ i = 1, 2, \ |\Im_3| \leqslant \varepsilon_2 \int_{\varepsilon_2}^{d_2} \frac{\overset{1,2}{\omega_f}(|\tau_1 - t_1|, \xi_2)}{\xi_2^2} d\theta_2^{\psi_2}(\xi_2),$$

$$|\Im_4| \leqslant 3\varepsilon_2 \int_{\varepsilon_2}^{d_2} \frac{\overset{1,2}{\omega_f}(|\tau_1 - t_1|)}{\xi_2^2} d\theta_2^{\psi_2}(\xi_2) + M \overset{1,2}{\omega_f}(|\tau_1 - t_1|, \varepsilon_2),$$

where

$$M = \sup \left| \int_{\gamma^2 \setminus \gamma_{\varepsilon_2}^2} \frac{d\psi_2(\tau_2)}{\tau_2 - t_2'} \right|$$

and

$$|\Im_i| \leqslant 2\varepsilon_2 \int_{\varepsilon_2}^{d_2} \frac{\overset{1,2}{\omega_f}(|\tau_1 - t_1|, \xi_2)}{\xi_2^2} d\theta_{t_2}^{\psi_2}(\xi_2), \ i = 5, 6.$$

Gathering the estimates for the integrals $\Im_i(i = 1, 2, \ldots, 6)$ and taking into account the theorem from [14], p.241, we obtain:

$$|g_2(\Delta f; \tau_{t_2}; t_{t_2'})| \leqslant C \Big[\int_0^{\theta_{t_2}^{\psi_2}(\varepsilon_2)} \frac{\overset{1,2}{\omega_f}(|\tau_1 - t_1|, \breve{\theta}_{t_2}^{\psi_2}(\xi_2))}{\breve{\theta}_{t_2}^{\psi_2}(\xi_2)} d\xi_2$$

$$+\varepsilon_2 \int\limits_{\theta_{t_2}^{\psi_2}(\varepsilon_2)}^{d_2} \frac{\overset{1,2}{\omega_f}\left(|\tau_1-t_1|,\check{\theta}_{t_2}^{\psi_2}(\xi_2)\right)}{(\check{\theta}_{t_2}^{\psi_2}(\xi_2))^2}d\xi_2 + \overset{1,2}{\omega_f}\left(|\tau_1-t_1|,\xi_2\right)\Bigg]. \tag{11}$$

Since the function

$$F(\delta,\eta) = \int\limits_0^\eta \frac{\overset{1,2}{\omega_f}\left(\delta,\check{\theta}_{t_2}(\xi_2)\right)}{\check{\theta}_{t_2}^{\psi_2}(\xi_2)}d\xi_2 + \eta\int\limits_\eta^{d_2} \frac{\overset{1,2}{\omega_f}\left(\delta,\check{\theta}_{t_2}^{\psi_2}(\xi_2)\right)}{(\check{\theta}_{t_2}^{\psi_2}(\xi_2))^2}d\xi_2$$

is increasing in η for any $\delta \in (0,d_1]$ and $\check{\theta}_2^{\psi_2}(\xi_2) \leqslant \check{\theta}_{t_2}^{\psi_2}(\xi_2)$, we obtain

$$\overset{1,2}{\omega_f}\left(|\tau_1-t_1|,\varepsilon_2)\right| \leqslant \int\limits_0^{\theta_2^{\psi_2}(\varepsilon_2)} \frac{\overset{1,2}{\omega_f}\left(|\tau_1-t_1|,\check{\theta}_2^{\psi_2}(\xi_2)\right)}{\check{\theta}_2^{\psi_2}(\xi_2)}d\xi_2 .$$

From the inequality (11), taking $\varepsilon_2 \leqslant \delta_2$, we have

$$|g_2(\Delta f;\tau_{t_2};t_{t_2'})| \leqslant C\left(\int\limits_0^{\theta_2^{\psi_2}(\delta_2)} \frac{\overset{1,2}{\omega_f}\left(|\tau_1-t_1|,\check{\theta}_2^{\psi_2}(\xi_2)\right)}{\check{\theta}_2^{\psi_2}(\xi_2)}d\xi_2\right.$$

$$+\delta_2 \int\limits_{\theta_2^{\psi_2}(\delta_2)}^{d_2} \frac{\overset{1,2}{\omega_f}\left(|\tau_1-t_1|,\check{\theta}_2^{\psi_2}(\xi_2)\right)}{(\check{\theta}_2^{\psi_2}(\xi_2))^2}d\xi_2\right) \overset{def}{=} Z_2\left(\overset{1,2}{\omega_f}\left(|\tau_1-t_1|,\delta_2,\theta_2^{\psi_2},\check{\theta}_2^{\psi_2}\right)\right). \tag{12}$$

Here and below C denote absolute constants, which are different at different in different inequalities.

The equality (8) differs from (9) by the fact that in (8) the integration is taken over through γ', not γ^2 and the density Δf is replaced by g_2. Therefore, to derive the estimate for the difference (8), we may use the partition of this difference similar to that in (10). We have

$$\Delta g_\psi^{1,2}(t,t') = \int\limits_{\gamma_{\varepsilon_1}^1(t_1)} \frac{g_2(\Delta f;\tau_{t_2},t_{t_2'})}{\tau_1-t_1}d\psi_1(\tau_1) -$$

$$-\int\limits_{\gamma_{\varepsilon_1}^1(t_1')} \frac{g_2(\Delta f;\tau_{t_2},t')}{\tau_1-t_1'}d\psi_1(\tau_1) + (t_1-t_1')\int\limits_{\gamma^1|\gamma_{\varepsilon_1}^1} \frac{g_2(\Delta f;\tau_{t_2},t_{t_2'})}{(\tau_1-t_1)(\tau_1-t_1')}d\psi_1(\tau_1$$

$$+\int\limits_{\gamma_{\varepsilon_1}^1(t_1')\backslash\gamma_{\varepsilon_1}^1(t_1)} \frac{g_2(\Delta f;\tau_{t_2},t_{t_2'})}{\tau_1-t_1}d\psi(\tau_1) - \int\limits_{\gamma_{\varepsilon_1}^1(t_1)\backslash\gamma_{\varepsilon_1}^1(t_1')} \frac{g_2(\Delta f;\tau_{t_2},t')}{\tau_1-t_1'}d\psi_1(\tau_1)$$

$$+\int\limits_{\gamma\backslash\gamma_{\varepsilon_1}^1} \frac{g_2(\Delta f;\tau_{t_2},t_{t_2'})-g_2(\Delta f;\tau_{t_2},t')}{\tau_1-t_1'}d\psi_1(\tau_1), \tag{13}$$

with

$$g_2(\Delta f\,;\tau_{t_2},t_{t_2'}) - g_2(\Delta f\,;\tau_{t_2},t')$$
$$= \int\limits_{\gamma^2} \frac{\Delta f\,(\tau_{t_1'};t)}{\tau_2 - t_2} d\psi_2(\tau_2) - \int\limits_{\gamma^2} \frac{\Delta f\,(\tau_{t_1'};t_{t_2'})}{\tau_2 - t_2'} d\psi_2(\tau_2). \tag{14}$$

We denote the terms in the right-hand side of (13) as $i_k, k = 1, 2, \ldots, 6$.

Taking into account the estimate for the difference (9), we can obtain the following estimate for (14):

$$|g_2(\Delta f\,;\tau_{t_2},t_{t_2'}) - g_2(\Delta f\,;\tau_{t_2},t')| \leqslant C \left(\int\limits_0^{\theta^{\psi_2}(\varepsilon_2)} \frac{\overset{1,2}{\omega_f}\,(|\tau_1 - t_1|, \check{\theta}^{\psi_2}(\varepsilon_2))}{\check{\theta}^{\psi_2}} \check{\theta}^{\psi_2} d\xi_2^{\psi_2}(\xi_2) \right.$$

$$\left. + \varepsilon_2 \int\limits_{\theta^{\psi_2}(\varepsilon_2)}^{d_2} \frac{\overset{1,2}{\omega_f}\,(|\tau_1 - t_1|, \check{\theta}^{\psi_2}(\xi_2))\check{\theta}^{\psi_2}(\xi_2)^2}{d} \xi_2 \right) \overset{def}{=} Z_2 \left(\overset{1,2}{\omega_f}\,|\tau_1 - t_1|, \varepsilon_2, \theta_2^{\psi_2}, \check{\theta}_2^{\psi_2} \right). \tag{15}$$

Keeping the bounds (12) and (15) in mind, we may estimate the integrals $i_1, \ldots, i_6$ in the same way as it was done for the integrals $\Im_1, \ldots, \Im_6$ and then for $\varepsilon_1 \leqslant \delta_1$ we get

$$| \overset{1,2}{\Delta} g_\psi(t;t')| \leqslant C \left(\int\limits_0^{\theta_1^{\psi_1}(\delta_1)} \frac{Z_2(\overset{1,2}{\omega_f}\,(\check{\theta}_1(\xi_1), \cdot), \delta_2, \theta_2^{\psi_2}, \check{\theta}_2^{\psi_2})}{\check{\theta}_1^{\psi_1}(\xi_1)} d\xi_1 \right.$$

$$\left. + \delta_1 \int\limits_{\theta_1^{\psi_1}(\delta_1)}^{l_1} \frac{Z_2(\overset{1,2}{\omega_f}\,(\check{\theta}_1(\xi_1), \cdot), \delta_2, \theta_2^{\psi_2}, \check{\theta}_2^{\psi_2})}{(\check{\theta}_1^{\psi_1}(\xi_1))^2} d\xi_1 \right)$$

$$\overset{def}{=} Z_1(Z_2(\overset{1,2}{\omega_f}\,(\xi_1, \cdot), \delta_2, \theta_2^{\psi_2}, \check{\theta}_2^{\psi_2}), \delta_1, \theta_1^{\psi_1}, \check{\theta}_1^{\psi_1}) \overset{def}{=} Z(\overset{1,2}{\omega_f}\,;\delta, \theta^\psi, \check{\theta}^\psi). \tag{16}$$

Now we have to estimate $\Delta \overset{1,0}{g_{\psi_1}} (t;t')$ and $\Delta \overset{0,2}{g_2} (t;t')$. Estimating $\Delta \overset{1,0}{g_{\psi_1}} (t;t')$ and $\Delta \overset{0,2}{g_2} (t;t')$ similarly to actions in (9), we obtain

$$|\Delta \overset{1,0}{g_{\psi_1}} (t;t')| \leqslant C \left(\int\limits_0^{\theta_2^{\psi_2}(\delta_2)} \int\limits_0^{\theta_1^{\psi_1}(\delta_1)} \frac{\overset{1,2}{\omega_f}\,(\check{\theta}_1^{\psi_1}(\xi_1), \check{\theta}_2^{\psi_2}(\xi_2))}{\check{\theta}_1^{\psi_1}(\xi_1) \cdot \check{\theta}_2^{\psi_2}(\xi_2)} d\xi_1 d\xi_2 \right.$$

$$\left. + \delta_1 \int\limits_0^{\theta_2^{\psi_2}(\delta_2)} \int\limits_{\theta_1^{\psi_1}(\delta_1)}^{l_1} \frac{\overset{1,2}{\omega_f}\,(\check{\theta}_1^{\psi_1}(\xi_1), \check{\theta}_2^{\psi_2}(\xi_2))}{(\check{\theta}_1^{\psi_1}(\xi_1))^2 \cdot \check{\theta}_2^{\psi_2}(\xi_2)} d\xi_1 d\xi_2 \right)$$

$$= \int\limits_0^{\theta_2^{\psi_2}(\delta_2)} \frac{Z_1(\overset{1,2}{\omega_f}\,(\cdot, \check{\theta}_2^{\psi_2}(\xi_2)), \delta_1, \theta_1^{\psi_1}, \check{\theta}_1^{\psi_1})}{\check{\theta}_2^{\psi_2}(\xi_2)} d\xi_2. \tag{17}$$

In the same way we get:

$$|\Delta \overset{0,2}{g}_{\psi_2}(t;t')| \leqslant \int\limits_0^{\theta_1^{\psi_1}(\delta_1)} \frac{Z_2(\overset{1,2}{\check{\omega}_f}(\check{\theta}_1^{\psi_1}(\xi_1),\cdot),\delta_2,\theta_2^{\psi_2},\check{\theta}_2^{\psi_2})}{\check{\theta}_1^{\psi_1}(\xi_1)}d\xi_1. \tag{18}$$

Gathering the estimates (16), (17), (18) we finally obtain:

$$\overset{1,2}{\omega_f}(\delta_1,\delta_2) = O(Z_1(Z_2(\overset{1,2}{\omega_f}(\xi_1,\cdot),\delta_2,\theta_2^{\psi_2},\check{\theta}_2^{\psi_2}),\delta_1,\theta_1^{\psi_1},\check{\theta}_1^{\psi_1})$$
$$= O(Z(\overset{1,2}{\omega_f};\delta,\theta^\psi,\check{\theta}^\psi)). \tag{19}$$

Let us estimate $\Delta \tilde{f}_\psi(t_{t_2'};t')\tilde{f}_\psi(t_1,t_2') - \tilde{f}_\psi(t_1',t_2')$. We have

$$\Delta\tilde{f}_\psi(t_{t_2'};t') == \int\limits_{\gamma^2}\frac{d\psi_2(\tau_2)}{\tau_2-t_2^1}\left(\int\limits_{\gamma^1}\frac{\Delta f(\tau;t_{t_2'})}{\tau_1-t_1}d\psi_1(\tau_1) - \int\limits_{\gamma^1}\frac{\Delta f(\tau;t')}{\tau_1-t_1'}d\psi_1(\tau_1)\right)$$
$$+ \int\limits_{\gamma^2}\frac{\Delta f(\tau_{t_1};t_{t_2'})}{\tau_2-t_2'}d\psi_2(\tau_2) - \int\limits_{\gamma^2}\frac{\Delta f(\tau_{t_1'};t')}{\tau_2-t_2'}d\psi_2(\tau_2)$$
$$+ \int\limits_{\gamma^1}\frac{\Delta f(\tau_{t_2};t_{t_2'})}{\tau_1-t_1}d\psi_1(\tau_1) - \int\limits_{\gamma^1}\frac{\Delta f(\tau_{t_2'};t')}{\tau_1-t_1'}d\psi_1(\tau_1)$$
$$= \Delta\overset{1,2}{g}_\psi(t_{t_2'};t') + \Delta\overset{0,2}{g}_{\psi_2}(t_{t_2'};t') + \Delta\overset{1,0}{g}_{\psi_1}(t_{t_2'};t').$$

Now we have to estimate $|\Delta\overset{1,2}{g}_\psi(t_{t_2'};t')|$. We have

$$|\Delta\overset{1,2}{g}_\psi(t_{t_2'};t')| \leqslant c\left(\int\limits_0^{\theta_1^{\psi_1}(\delta_1)}\int\limits_0^{l_2}\frac{\overset{1,2}{\omega_f}(\check{\theta}_1^{\psi_1}(\xi_1),\check{\theta}_2^{\psi_2}(\xi_2))}{\check{\theta}_1^{\psi_1}(\xi_1)\cdot\check{\theta}_2^{\psi_2}(\xi_2)}d\xi_1 d\xi_2\right.$$
$$\left. + \varepsilon_1\int\limits_{\theta_1^{\psi_1}(\varepsilon_1)}^{l_1}\int\limits_0^{l_2}\frac{\overset{1,2}{\omega_f}(\check{\theta}_1^{\psi_1}(\xi_1),\check{\theta}_2^{\psi_2}(\xi_2))}{(\check{\theta}_1^{\psi_1}(\xi_1))^2\cdot(\check{\theta}_2^{\psi_2}(\xi_2))}d\xi_1 d\xi_2\right).$$

To have bounds for $\Delta\overset{0,2}{g}_{\psi_2}(t_{t_2'};t')$ and $\Delta\overset{1,0}{g}_{\psi_1}(t_{t_2'};t')$, we use Lemma 1 and apply the theorem from [14]. Taking into account the inequality

$$\overset{1,2}{\omega_f}(|\varepsilon_1-t_1|,\xi_2) \leqslant \int\limits_0^{\theta_2^{\psi_2}(\varepsilon_2)}\frac{\overset{1,2}{\omega_f}(\check{\theta}_1^{\psi_1}(\xi_1),\xi_2)}{\check{\theta}_1^{\psi_1}(\xi_1)}d\xi_1,$$

we arrive at the estimate

$$|\Delta\overset{0,2}{g}_{\psi_2}(t_{t_2'};t')| \leqslant \int\limits_{\gamma^2}\frac{|\Delta f(\tau_{t_1};t_{t_2'}) - \Delta f(\tau_{t_1'};t')|}{|\tau_2-t_2'|}|d\psi(\tau_2)| = \int\limits_{\gamma^2}\frac{|\Delta f(\tau_{t_1};t')|}{|\tau_2-t_2'|}|d\psi(\tau_2)|$$

$$\leqslant \int_0^{l_2} \frac{\overset{1,2}{\breve{\omega}_f}(|t_1-t_1'|,\breve{\theta}_2^{\psi_2}(\xi_2))}{\breve{\theta}_2^{\psi_2}(\xi_2)}d\xi_2 \leqslant \int_0^{\theta_1^{\psi_1}(\varepsilon_1)}\int_0^{l_2} \frac{\overset{1,2}{\breve{\omega}_f}(\breve{\theta}_1^{\psi_1}(\xi_1),\breve{\theta}_2^{\psi_2}(\xi_2))}{\breve{\theta}_1^{\psi_1}(\xi_1)\cdot\breve{\theta}_2^{\psi_2}(\xi_2)}d\xi_1 d\xi_2. \quad (20)$$

Applying the estimate for the difference (9), we get

$$|\triangle \overset{1,0}{g}_{\psi_1}(t_{t_2'};t')| \leqslant \int_0^{\theta_1^{\psi_1}(\varepsilon_1)} \frac{\overset{1,0}{\breve{\omega}_f}(\breve{\theta}_1^{\psi_1}(\xi_1))}{\breve{\theta}_1^{\psi_1}(\xi_1)}d\xi_1 + \varepsilon_1 \int_{\theta_1^{\psi_1}(\varepsilon_1)}^{l_1} \frac{\overset{1,0}{\breve{\omega}_f}(\breve{\theta}_1^{\psi_1}(\xi_1))}{(\breve{\theta}_1^{\psi_1}(\xi_1))^2}d\xi_1 \quad (21)$$

Taking the estimates (19)-(21) into account, with $\varepsilon_i = |t_i - t_1'| < \delta_i (i = 1,2)$, we finally obtain:

$$\overset{1,0}{\omega}_{\tilde{f}_\psi}(\delta_1) \leqslant C \left[\int_0^{\theta_1^{\psi_1}(\delta_1)}\int_0^{l_2} \frac{\overset{1,2}{\breve{\omega}_f}(\breve{\theta}_1^{\psi_1}(\xi_1),\breve{\theta}_2^{\psi_2}(\xi_2))}{\breve{\theta}_1^{\psi_1}(\xi_1)\cdot\breve{\theta}_2^{\psi_2}(\xi_2)}d\xi_1 d\xi_2 \right.$$

$$+ \delta_1 \int_{\theta_1^{\psi_1}(\delta_1)}^{l_1}\int_0^{l_2} \frac{\overset{1,2}{\breve{\omega}_f}(\breve{\theta}_1^{\psi_1}(\xi_1),\breve{\theta}_2^{\psi_2}(\xi_2))}{(\breve{\theta}_1^{\psi_1}(\xi_1))^2\cdot\breve{\theta}_2^{\psi_2}(\xi_2)}d\xi_1 d\xi_2$$

$$\left. + \int_0^{\theta_1^{\psi_1}(\delta_1)} \frac{\overset{1,0}{\breve{\omega}_f}(\breve{\theta}_1^{\psi_1}(\xi_1))}{\breve{\theta}_1^{\psi_1}(\xi_1)}d\xi_1 + +\delta_1 \int_{\theta_1^{\psi_1}(\delta_1)}^{l_1} \frac{\overset{1,0}{\breve{\omega}_f}(\breve{\theta}_1^{\psi_1}(\xi_1))}{(\breve{\theta}_1^{\psi_1}(\xi_1))^2}d\xi_1 \right]. \quad (22)$$

In a similar way one may estimate $\Delta \tilde{f}_\psi(t_{t_1'};t') = \tilde{f}_\psi(t_1',t_2) - \tilde{f}_\psi(t_1',t_2')$. We have

$$\overset{0,2}{\omega}_{\tilde{f}_\psi}(\delta_2) \leqslant\leqslant C \left[\left(\int_0^{\theta_2^{\psi_2}(\delta_2)}\int_0^{l_1} \frac{\overset{1,2}{\breve{\omega}_f}(\breve{\theta}_1^{\psi_1}(\xi_1),\breve{\theta}_2^{\psi_2}(\xi_2))}{\breve{\theta}_1^{\psi_1}(\xi_1)\cdot\breve{\theta}_2^{\psi_2}(\xi_2)}d\xi_1 d\xi_2 \right.\right.$$

$$+ \delta_2 \int_{\theta_2^{\psi_2}(\delta_2)}^{l_2}\int_0^{l_1} \frac{\overset{1,2}{\breve{\omega}_f}(\breve{\theta}_1^{\psi_1}(\xi_1),\breve{\theta}_2^{\psi_2}(\xi_2))}{\breve{\theta}_1^{\psi_1}(\xi_1)\cdot(\breve{\theta}_2^{\psi_2}(\xi_2))^2}d\xi_1 d\xi_2$$

$$\left. + \int_0^{\theta_2^{\psi_2}(\delta_2)} \frac{\overset{0,2}{\breve{\omega}_f}(\breve{\theta}_2^{\psi_2}(\xi_2))}{\breve{\theta}_2^{\psi_2}(\xi_2)}d\xi_2 + +\delta_2 \int_{\theta_2^{\psi_2}(\delta_2)} \frac{\overset{0,2}{\breve{\omega}_f}(\breve{\theta}_2^{\psi_2}(\xi_2))}{\breve{\theta}_2^{\psi_2}(\xi_2)}d\xi_2 \right] \quad (23)$$

The last inequalities (18), (22) and (23) prove the theorem.

COROLLARY 1. *Let $d\psi(t) = F(t)dt$, where $F(t)$ is boundary value of a function analytic in $D^{\pm}$ and continuous up to the boundary. Suppose that*

$$f \in \Im_0(\triangle) = \left\{ f \in C_{\triangle} : \int_0^d \frac{\overset{1,2}{\omega_f}(\xi)}{\xi} d\theta(\xi) < \infty, \right.$$

$$\left. \int_0^{d_1} \frac{\overset{1,0}{\omega_f}(\xi_1)}{\xi_1} d\theta_1(\xi_1) < \infty, \int_0^{d_2} \frac{\overset{0,2}{\omega_f}(\xi_2)}{\xi_2} d\theta_2(\xi_2) < \infty \right\}.$$

Then the following inequalities are valid:

$$\overset{1,2}{\omega_{\tilde{f}}}(\delta) = O(Z(\overset{1,2}{\omega_f}; \delta, \theta)), 0 < \delta_1 \leqslant d_1, 0 < \delta_2 \leqslant d_2,$$

$$\overset{1,0}{\omega_{\tilde{f}}}(\delta_1) = O(Z(\overset{1,2}{\omega_f}; \delta_1, d_2, \theta)) + Z(\overset{1,0}{\omega_f}; \delta_1, \theta_1)), 0 < \delta_1 \leqslant d_1,$$

$$\overset{0,2}{\omega_{\tilde{f}}}(\delta_1) = O(Z(\overset{1,2}{\omega_f}; \delta_2, d_1, \theta)) + Z(\overset{0,2}{\omega_f}; \delta_2, \theta_2)), 0 < \delta_2 \leqslant d_2.$$

COROLLARY 2. *Let $\psi_k(t) = t, \gamma^k$ be c.j.e.c, $\theta_k(\delta) \sim \delta$ $(k = 1, 2)$ and f satisfy the assumption of corollary 1 with $F(t) = 1$. Then*

$$\overset{1,2}{\omega_{\tilde{f}}}(\delta) = O(Z(\overset{1,2}{\omega_f}; \delta, \theta)), 0 < \delta_1 \leqslant d_1, 0 < \delta_2 \leqslant d_2,$$

$$\overset{1,0}{\omega_{\tilde{f}}}(\delta_1) = O(Z(\overset{1,2}{\omega_f}; \delta_1, d_2) + Z(\overset{1,0}{\omega_f}; \delta_1)), 0 < \delta_1 \leqslant d_1,$$

$$\overset{0,2}{\omega_{\tilde{f}}}(\delta_2) = O(Z(\overset{1,2}{\omega_f}; \delta_2, d_1) + Z(\overset{0,2}{\omega_f}; \delta_2)), 0 < \delta_2 \leqslant d_2.$$

Let $\omega \in \Phi_{T^2}$. We introduce the linear space

$$K_{\omega} = \left\{ f \in C_{\triangle} : \overset{1,2}{\omega_f}(\delta_1, \delta_2) = O\left(\omega(\delta_1, \delta_2) \right), \right.$$

$$\left. \overset{1,0}{\omega_f}(\delta_1) = O\left(\omega(\delta_1, d_2) \right), \overset{0,2}{\omega_f}(\delta_2) = O\left(\omega(\delta_2, d_1) \right) \right\}$$

and equip it with the norm

$$\| f \|_{K_{\omega}} = max\{ \overset{1,2}{C_f}, \overset{1,0}{C_f}, \overset{0,2}{C_f}, \| f \|_{C_{\triangle}} \}$$

where

$$\overset{1,2}{C_f} = \sup_{\delta_1, \delta_2 > 0} \frac{\overset{1,2}{\omega_f}(\delta_1, \delta_2)}{\omega(\delta_1, \delta_2)}, \overset{1,0}{C_f} = \sup_{\delta_1 > 0} \frac{\overset{1,0}{\omega_f}(\delta_1)}{\omega(\delta_1, d_1)}, \overset{0,2}{C_f} = \sup_{\delta_2 > 0} \frac{\overset{0,2}{\omega_f}(\delta_2)}{\omega(d_1, \delta_2)}.$$

With respect to this norm, K_{ω} is a Banach space. It is easy to see that $K_{\omega} = K_{\omega_1}$ when $\omega_1 \sim \omega_2$ up to equivalence of norms. In the case $\omega(\delta_1, \delta_2) = \delta_1^{\alpha} \delta_2^{\beta}, 0 < \alpha, \beta \leqslant 1$, we denote this space by $K_{\alpha, \beta}$, this class being treated in [3], [6], [17].

THEOREM 3. *Let $d\psi(t) = F(t)dt$, where $F(t)$ is limiting value of a function analytic in $D^{\pm}$ and continuous up to the boundary) and let*

$$\omega \in \Im_0\Phi = \left\{ \omega \in \Phi_{T^2} \ : \ \int\limits_0^d \frac{\omega(\xi)}{\xi} d\theta^F(\xi) < \infty, \int\limits_0^{d_1} \frac{\omega(\xi_1, d_2)}{\xi_1} d\theta_1^F(\xi_1) < \infty, \right.$$
$$\left. \int\limits_0^{d_2} \frac{\omega(d_1, \xi_2)}{\xi_2} d\theta_2^F(\xi_2) < \infty \right\}.$$

Then $f \in K_\omega \Longrightarrow \tilde{f}_F \in K_z(\omega, \delta, \theta^F)$.

COROLLARY. *Let $\theta_k^F(\delta) \sim \delta$ $(k = 1, 2)$. Then under the assumption of Theorem 3 on ω we have $f \in K_\omega \Longrightarrow \tilde{f}_F \in K_z(\omega, \delta)$.*

Finally, let

$$\theta_k^F(\delta) \sim \delta \ (k = 1, 2), \omega \in \left\{ \omega \in \Im_0\Phi : Z(\omega; \delta_1, \delta_2) = O(\omega(\delta_1, \delta_2)) \right\}. \qquad (24)$$

Then the following theorem is valid.

THEOREM 4. *Let $f \in K_\omega$. Then, under the assumptions (24), $\tilde{f} \in K_\omega$ as well and $\|\tilde{f}\|_{K_{\omega|}} \leqslant \|f\|_{K_{\omega|}}$.*

The proof of the theorem follows from Theorem 2, Corollary to Theorem 3 and the estimate:

$$\| \tilde{f} \|_{K_\omega} \leqslant C \| f \|_{K_\omega} \left(\int\limits_0^l \frac{\omega(\xi)}{\xi} d\xi + \int\limits_0^{l_1} \frac{\omega(\xi_1, d_2)}{\xi_1} d\xi_1 + \int\limits_0^{l_2} \frac{\omega(d_1, \xi_2)}{\xi_2} d\xi_2 \right) = C_1 \| \tilde{f} \|_{K_\omega}$$

REFERENCES

[1] CESARI, L., *Sulle serie di Fourier delle funzioni lipschitziane di piu variabili* (Italian), Ann. Scuola Norm. Sup. Pisa, II. Ser. 7, 279–295 (1938).
[2] LEKISHVILI, M.M., *Conjugate functions of several variables in the class Lip α* (Russian), Mat. Zametki 23, 361–372 (1978).
[3] JAK, I.E., *On a theorem of L. Cesari. On conjugate functions of two variables* (Russian), Dokl. Akad. Nauk. SSSR, 1952, v.87, no 6, 87–880.
[4] MUSAEV, V.I. AND SALAEV, V.V., *On conjugate functions of many variables* (Russian), Uchen. Zapiski M.V. i SSO Azerb. SSR, ser.fiz-mat. nauk, 1979, no 4, 5–17.
[5] JAK, I.E., *On conjugate double trigonometric series* (Russian), Matem. sbornik, 1952, v. 31, no 3.
[6] DZHVARSHEISHVILI, A.G., *On an inequality of A.Zygmund for functions of two variables* (Russian), Soobsch. Akad. Nauk Gruz. SSR, 1954, v.15, no 9, 561–568.
[7] KOKILASHVILI, V.M., *On some properties of conjugate functions of two variables* (Russian), Soobsch. Akad. Nauk. Gruz. SSR, 1965, v.40, no 2.
[8] MAGNARAGZE L.G., *On a generalization of Plemeli-Privalov theorem* (Russian), Soobsch. Akad. Nauk. Gruz. SSR, 1947, v.4, no 8, 509–516.
[9] ASHUROV, R.A. AND SALAEV V.V., *Double singular integral with a continuous density* (Russian), Nauch. Trudy. M.V. i SSO Azerb. SSR, ser. fiz.-mat. nauk, 1979, no 6, 29–43.
[10] SALAEV, V.V., *Direct and inverse estimates for the Cauchy type singular integral along a closed curve* (Russian), Mat. Zametki, 1976, 19, 365–380.

[11] GAZIEV, A.G., *The singular operator with the Cauchy-Stiltjes integral along a closed curve* (Russian), Izv.AN Uzb SSR,1981, N 1, no 8, 3–9.

[12] GAZIEV, A.G., *Study of properties of some linear singular operators*, Izv. AN Uzb SSR,1967, no. 2, 7–13.

[13] GAZIEV, A.G. AND BUBNOV, E.A., *On Cauchy singular integrals with a continuous density for functions of many variables* (Russian), Samarkand State Univ., Samarkand, 1985, Deponierted in UZNIINTI 08.07.85., N 318 Uz - 85 Dep.

[14] KAMKE, E., *Das Lebesgue-Stieltjes-Integral*, Leipzig, Teubner Verlagsgesellschaft VI, 1956, 226 S., 24.

[15] BABA-ZADE, M.A., *On a class of continuous functions of two variables* (Russian), Uch. Zam. MV i SSO AZSSR, ser. fiz-mat nauk., 1979, no 3, 32–41.

[16] BABAEV, A.A. AND SALAEV, V.V., *A one-dimensional singular operator with continuous density along a closed curve* (English, Russian original) Sov. Math., Dokl. 14, 598–602 (1973); translation from Dokl. Akad. Nauk SSSR 209, 1257–1260 (1973).

[17] CHELIDZE, V.G., *On the absolute convergence of double Fourier series* (Russian), Dokl. Acad. Nauk. SSSR, v. 54, no 2, 117–119 (1946).

[18] BEZLYUDNEY, A.S., *Approximation of periodic functions of two variables by interpolating trigonometric polynomials* (Russian), Dokl. Akad. Nauk SSSR, 1949, 65, no 3, 257–260.

A. G. Gaziev
Samarkand University
Dept. of Mathematics
Bul'var M. Gor'kogo, 15
703004, Samarkand
Uzbekistan

Mathematical Inequalities & Applications
www.ele-math.com
mia@ele-math.com