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THE BEST BOUNDS IN GAUTSCHI'S INEQUALITY

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Abstract. Different approach to both Gautschi's inequalities (1) and (2) is given. This results in obtaining the best upper bound in (1) and the best lower bound in (2). The main result is the proof of the convexity of the function $[\Gamma(x+t)/\Gamma(x+s)]^{1/(t-s)}$ for $|t-s|<1$. Several new very simple inequalities for digamma function, like $\psi'(x)<\exp(-\psi(x))$ or $\psi(x+1)<\log(x+e^{-\gamma})$ are also proved.

1. Introduction

In this paper we shall discuss (again!) the following two Gautschi's inequalities:

$$\left(x+\frac{s}{2}\right)^{1-s}<\frac{\Gamma(x+1)}{\Gamma(x+s)}<\left(x-\frac{1}{2}+\sqrt{s+\frac{1}{4}}\right)^{1-s} \tag{1}$$

and

$$\exp[(1-s)\psi(x+\sqrt{s})]<\frac{\Gamma(x+1)}{\Gamma(x+s)}<\exp\left[(1-s)\psi\left(x+\frac{s+1}{2}\right)\right] \tag{2}$$

We shall call it the first and the second Gautschi's inequality. They are, in fact, Kershow's sharpening of original Gautschi's inequality for natural n [4]:

$$n^{1-s}<\frac{\Gamma(n+1)}{\Gamma(n+s)}<\exp\big[(1-s)\psi(n+1)\big]. \tag{3}$$

Kershow [8] proved these inequalities for $0<s<1$ and $x>0$. He defines

$$f(x)=\frac{\Gamma(x+1)}{\Gamma(x+s)}\cdot\exp[(s-1)\psi(x+\alpha)],$$

and study conditions for monotonicity of the function

$$F(x)=\frac{f(x)}{f(x+1)}=\left(\frac{x+s}{x+1}\right)\cdot\exp\left(\frac{1-s}{x+\alpha}\right),$$

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It turn out that F decreases for $\alpha = s^{1/2}$, and increases for $\alpha = \frac{1}{2}(s+1)$. Since $F(x) \to 1$ as $x \to \infty$, (2) follows. Similarly, he obtained (1).

In [5] it is proved that inequality (1) holds also for $2 < s$, and for $1 < s < 2$ we have the reversed sign in (1). Lower bound in (1) and upper bound in (2) are the best possible. But, the other two bounds can be improved.

2. The best bounds in the first Gautschi's inequality

Let us consider first inequality (1), written in the form

$$(x+\alpha)^{t-s} < \frac{\Gamma(x+t)}{\Gamma(x+s)} < (x+\beta)^{t-s} \tag{4}$$

where α and β depends on s and t. For the reasons which be clear later, it is more natural to consider the following form of this inequality:

$$\alpha < \left(\frac{\Gamma(x+t)}{\Gamma(x+s)}\right)^{\frac{1}{t-s}} - x < \beta \tag{5}$$

Of course, (4) and (5) are equivalent forms, but the sign of inequalities has to be changed for $t < s$, by passing from (4) to (5) and vice versa. We shall give the best bounds in (5), for all $s,t > 0$ and for x on some interval of $\mathbf{R}$.

Let us denote the Wallis function by $W(x) := \dfrac{\Gamma(x+t)}{\Gamma(x+s)}$, and the main function in (5) by $z(x) := W(x)^{1/(t-s)} - x$.

It will be shown that z is convex and decreasing (or concave and increasing), depending of the values of s and t. Theorem of such type is claimed in the old paper [10] but the proof of the main Theorem 2 is not correct. Namely, it is wrongly stated that $W(x) = \Gamma(x+1)/\Gamma(x+s)$ is a convex function (for $0 < s < 1$) from which it follows immediately that $z(x) = W(x)^{1/(1-s)} - x$ is also convex. In fact, W is concave for such s, but z is indeed a convex function, even for $0 < s < 2$.

Let us give main theorem.

THEOREM 1. *Let* $s,t > 0$, $r = \min(s,t)$. *Then*

$$z(x) := \left(\frac{\Gamma(x+t)}{\Gamma(x+s)}\right)^{\frac{1}{t-s}} - x \tag{6}$$

is convex and decreasing function on $(-r,+\infty)$ *for* $|t-s| < 1$, *and concave and increasing on the same interval for* $|t-s| > 1$.

We shall prove this theorem in the next secton. Let us see immediately the main consequences.

By means of the Stirling series, it is easy to prove that

$$\lim_{x\to\infty} z(x) = \frac{s+t-1}{2},$$

therefore, $(s+t-1)/2$ gives (the best) bound (at infinity) of the function z: it holds $z(x) > (s+t-1)/2$ for $|t-s| < 1$ and $z(x) < (s+t-1)/2$ for $|t-s| < 1$. Let us note also that we have $z(x) = 1$, for all x if $|t-s| = 1$.

The other bound in (5) on some interval $[x_0, +\infty]$ is given by the value $z(x_0)$ Hence, we can give the best bounds in the first Gautschi's inequality:

THEOREM 2. *Let $s, t > 0$. For all $x \geqslant x_0 > -\min(s,t)$, then*

$$\frac{s+t-1}{2} < \left(\frac{\Gamma(x+t)}{\Gamma(x+s)}\right)^{\frac{1}{t-s}} - x \leqslant \left(\frac{\Gamma(x_0+t)}{\Gamma(x_0+s)}\right)^{\frac{1}{t-s}} - x_0 \tag{7}$$

holds for all $|t-s| < 1$, and with reversed sign for $|t-s| > 1$. The bounds are the best possible.

For the sake of the convenience of the reader, we can put down this inequalities in the usual form:

THEOREM 3. *For all $x > 0$ and $s, t > 0$, the inequalities*

$$\left(x + \frac{s+t-1}{2}\right)^{t-s} < \frac{\Gamma(x+t)}{\Gamma(x+s)} < \left(x + \left[\frac{\Gamma(t)}{\Gamma(s)}\right]^{\frac{1}{t-s}}\right)^{t-s} \tag{8}$$

holds for all $s < t < s+1$ and $t < s-1$, and with reversed sign for $s+1 < t$ and $s-1 < t < s$. The bounds are the best possible.

COROLLARY 1. *Let ψ be digamma (psi) function, $s, t > 0$ and $r = \min(s,t)$. Then for all $x > -r$ inequality*

$$\left(\frac{\Gamma(x+t)}{\Gamma(x+s)}\right)^{\frac{1}{t-s}} < \frac{t-s}{\psi(x+t) - \psi(x+s)} \tag{9}$$

holds for $|t-s| < 1$ and with reversed sign if $|t-s| > 1$.

Proof. It holds

$$\begin{aligned} z'(x) &= \frac{1}{t-s} W(x)^{\frac{1-t+s}{t-s}} \left[\frac{\Gamma'(x+t)}{\Gamma(x+s)} - \frac{\Gamma(x+t)\Gamma'(x+s)}{\Gamma(x+s)^2}\right] - 1 \\ &= \frac{1}{t-s} W(x)^{\frac{1}{t-s}} [\psi(x+t) - \psi(x+s)] - 1 \end{aligned}$$

Let us supose $|t-s| < 1$. By Theorem 1, z is decreasing, therefore, it holds $z'(x) < 0$ for all $x > 0$ and the statement follows since ψ is increasing.

Let us note that for $|t-s| = 1$ the equality sign holds in (9).

COROLLARY 2. $\exp\{\psi(x+t)\} - x$ *is decreasing on $\mathbf{R}^+$, for all $t > 0$. Hence, for all $x > 0$ it holds*

$$\psi'(x) < \exp\{-\psi(x)\}. \tag{10}$$

Proof. $z(x)$ is decreasing for all $|t-s|<1$. Since

$$\lim_{s\to t}\left(\frac{\Gamma(x+t)}{\Gamma(x+s)}\right)^{\frac{1}{t-s}} = \exp\Big\{\lim_{s\to t}\frac{1}{t-s}\big[\log\Gamma(x+t)-\log\Gamma(x+s)\big]\Big\}$$
$$= \exp\{\psi(x+t)\}$$

it follows that $z(x)=\exp\{\psi(x+t)\}-x$ is decreasing on $\mathbf{R}^+$, for all $t>0$ and (10) follows.

COROLLARY 3. *For all $x>0$ and $t>0$ it holds*

$$\log\Big(x+\frac{2t-1}{2}\Big) < \psi(x+t) < \log\Big(x+\exp(\psi(t))\Big). \tag{11}$$

Hence,

$$\log\Big(x+\frac{1}{2}\Big) < \psi(x+1) < \log\Big(x+e^{-\gamma}\Big). \tag{12}$$

where $\gamma=0.57721566\cdots$, or, more generally

$$\log\Big(x+n-\frac{1}{2}\Big) < \psi(x+n) < \log\Big(x+e^{-\gamma+\sum_{k=1}^{n-1}1/k}\Big).$$

For all $t>0$ digamma function can be written in a way:

$$\psi(x+t)=\log(x+\delta(x)), \qquad x>0$$

where δ is decreasing convex function which maps $\mathbf{R}^+$ onto $[e^{\psi(t)},\frac{1}{2}(2t-1)]$.

Proof. We have

$$\lim_{s\to t} W(x)^{\frac{1}{t-s}} = \exp\psi(x+t).$$

Therefore, (7) implies

$$x+\frac{2t-1}{2} < \exp\psi(x+t) < x+\exp\psi(t)$$

and (11) holds true. Putting $t=1$ and $t=n$ the next two statements follows. The last one is a paraphrase of Corollary 2.

3. Proof of Theorem 1.

In this section we shall prove the main theorems. Let us first prove the folowing lemma which can be of independent interest.

LEMMA 1. *Let $s,t>0$ and β_0 be defined by*

$$\beta_0=-\frac{1}{2}+\sqrt{st+\frac{1}{4}} \tag{13}$$

Then we have

$$\frac{1}{x+r_1} < \frac{\psi(x+t)-\psi(x+s)}{t-s} < \frac{1}{x+r_2} \tag{14}$$

where

$$r_1 := \max\left\{\frac{s+t-1}{2}, \beta_0\right\}, \qquad r_2 := \min\left\{\frac{s+t-1}{2}, \beta_0\right\}. \tag{15}$$

Proof. Let us take the representation of the psi function [1]

$$\psi(x+1) = -\gamma + \sum_{k=1}^{\infty}\left(\frac{1}{k} - \frac{1}{x+k}\right). \tag{16}$$

Then,

$$\begin{aligned}\frac{\psi(x+t)-\psi(x+s)}{t-s} &= \frac{1}{t-s}\sum_{k=0}^{\infty}\left(\frac{1}{x+s+k} - \frac{1}{x+t+k}\right)\\ &= \sum_{k=0}^{\infty}\frac{1}{(x+s+k)(x+t+k)}\\ &\leqslant \sum_{k=0}^{\infty}\frac{1}{(x+k+\beta)(x+1+k+\beta)} = \frac{1}{x+\beta}\end{aligned}$$

The last inequality is satisfied if for all x and k we have

$$(x+k)(s+t-1-2\beta) + st - \beta - \beta^2 \geqslant 0$$

and this will be surely true if

$$\beta \leqslant \min\left\{\frac{s+t-1}{2}, \beta_0\right\} = r_2.$$

In the same way one can prove the left side in (14).

Proof of Theorem 1. It is sufficient to prove convexity (and concavity) of the function $h(x) = W(x)^{\frac{1}{t-s}}$. For its derivative ve have

$$\begin{aligned}h'(x) &= \frac{1}{t-s}h(x)[\psi(x+t)-\psi(x+s)],\\ h''(x) &= \frac{1}{t-s}h'(x)[\psi(x+t)-\psi(x+s)] + \frac{1}{t-s}h(x)[\psi'(x+t)-\psi'(x+s)]\\ &= h(x)\left[\left(\frac{\psi(x+t)-\psi(x+s)}{t-s}\right)^2 + \frac{1}{t-s}[\psi'(x+t)-\psi'(x+s)]\right].\end{aligned}$$

Taking the representation (16) of the psi function, we obtain

$$\begin{aligned}\frac{1}{t-s}[\psi(x+t)-\psi(x+s)] &= \sum_{k=0}^{\infty}\frac{1}{(x+s+k)(x+t+k)}\\ \frac{1}{t-s}[\psi'(x+t)-\psi'(x+s)] &= -\sum_{k=0}^{\infty}\frac{2x+s+t+2k}{(x+s+k)^2(x+t+k)^2}.\end{aligned}$$

Therefore, it is sufficient to prove that

$$\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\frac{1}{(x+s+k)(x+t+k)(x+s+j)(x+t+j)} - \sum_{n=0}^{\infty}\frac{2x+s+t+2k}{(x+s+k)^2(x+t+k)^2}.$$

is positive for $|t-s|<1$ and negative for $|t-s|>1$. Let us rearange the sumation in the double sum by summing all the members for which one index of sumation is equal to n, and the other is greater or equal to n. We shall compare this part of the sum with n-th member of the sum from the right side:

$$\sum_{j=n}^{\infty}\frac{1}{(x+s+n)(x+t+n)(x+s+j)(x+1+j)} + \sum_{k=n+1}^{\infty}\frac{1}{(x+s+k)(x+t+k)(x+s+n)(x+t+n)} - \frac{2x+s+t+2n}{(x+s+n)^2(x+t+n)^2}$$

i.e., after reducing identical positive terms,

$$\frac{1}{(x+s+n)(x+t+n)} + 2\sum_{k=n+1}^{\infty}\frac{1}{(x+s+k)(x+t+k)} - \frac{1}{x+s+n} - \frac{1}{x+t+n}$$

First part of this expression we can write in a way:

$$\sum_{k=n}^{\infty}\left(\frac{1}{(x+s+k)(x+t+k)} + \frac{1}{(x+1+s+k)(x+1+t+k)}\right)$$

and the last two fractions we shall replace with the sums:

$$\frac{1}{x+s+n}+\frac{1}{x+t+n} = \sum_{k=n}^{\infty}\left[\frac{1}{(x+s+k)(x+1+s+k)}+\frac{1}{(x+t+k)(x+1+t+k)}\right].$$

Therefore, it is sufficient to prove that

$$\frac{1}{(x+s+k)(x+t+k)} + \frac{1}{(x+1+s+k)(x+1+t+k)} - \frac{1}{(x+s+k)(x+1+s+k)} - \frac{1}{(x+t+k)(x+1+t+k)}. \qquad (17)$$

is positive for $|t-s|<1$ and negative for $|t-s|>1$.

This conclusion follows from the following elementary fact: if $a \leqslant b < c \leqslant d$, then

$$\frac{1}{ab}+\frac{1}{cd}>\frac{1}{ac}+\frac{1}{bd}.$$

Namely, we divide the first quadrant $s,t>0$ into four parts and identify in each of them the factors in (17):

	$s<t<1+s$	$1+s<t$	$t<s<1+t$	$1+t<s$
$x+s+k$	a	a	b	c
$x+t+k$	b	c	a	a
$x+1+s+k$	c	b	d	d
$x+1+t+k$	d	d	c	b
expr. (17)	>0	<0	>0	<0

Therefore, (17) is positive if $|t-s|<1$, and negative if $|t-s|>1$.

Note that claims of the theorem holds also for $s=t$, by taking limits on appropriate places. In this case $z(x)=\exp\psi(x+t)-x$, see Corollary 2.

Therefore, the convexity part of the theorem is proved. Let us prove monotonicity of z.

For the derivative of the function z it holds

$$z'(x)=\frac{1}{t-s}W(x)^{\frac{1}{t-s}}[\psi(x+t)-\psi(x+s)]-1.$$

Let us suppose $|t-s|<1$. Using Lemma 1, we have

$$z'(x)<W(x)^{\frac{1}{t-s}}\cdot\frac{1}{x+r_2}-1.$$

If $x\to+\infty$, $W(x)^{1/(t-s)}$ has asymptotic behaviour $x+\dfrac{s+t-1}{2}$. Therefore, it holds

$$\lim_{x\to\infty} z'(x)\leqslant 0.$$

Since z is convex, z' is increasing. Therefore, we have $z'(x)<0$ for all $x>0$, hence, z is decreasing.

If $|t-s|>1$, using similar arguments we can prove that z is increasing.

4. Kershow's bound

In [8] Kershow uses the functions

$$g(x)=\frac{\Gamma(x+1)}{\Gamma(x+s)}\cdot(x+\beta)^{s-1}$$

and

$$G(x)=\frac{g(x)}{g(x+1)}=\frac{x+s}{x+1}\cdot\left(\frac{x+\beta+1}{x+\beta}\right)^{1-s}$$

in order to obtain upper bound in (1). It is proved that $G'(x) > 0$ if $\beta \geqslant \beta_0$ (β_0 is given in (13)) and since G increases to 1, it follows $g(x) < g(x+1)$. By iteration, one concludes that $g(x) < \lim_{x\to\infty} g(x) = 1$.

Let us restrict for a moment our statements to the case $0 < s < 2$. Since $\beta = \Gamma(s)^{-\frac{1}{1-s}}$ is the best upper bound in (5) for $t = 1$, one may consider Kershow bound as a estimation of the value of gamma function:

$$\Gamma(s)^{-\frac{1}{1-s}} < -\frac{1}{2} + \sqrt{s + \frac{1}{4}}. \tag{18}$$

It may be of interest to obtain others bounds in the terms of elementary functions. As we see, the problems of such type has nothing more to do with Gautschi's inequality, but instead with estimation of the values of gamma function $\Gamma(s)$.

One still may wonder why (18) is satisfied. Is there any chance to prove this inequality 'directly'? The answer is definitely yes. We shall explain how such bound can be obtained from another, more natural, point of view.

In [6] Kershow's argument are used to obtain similar bounds in the domain $[x_0, \infty)$. Of course, the best upper bound for the function z in this domain is the value $\beta = z(x_0)$, since z is decreasing function.

Therefore, it is important to estimate the value $z(x)$ for a given $s, t > 0$. We shall restrict ourself to the case $|t - s| < 1$, identical analysis can be made in other case. By Corollary 1,

$$\begin{aligned} W(x)^{\frac{1}{t-s}} - x &< \frac{t-s}{\psi(x+t) - \psi(x+s)} - x \\ &= \frac{1}{\displaystyle\sum_{k=0}^{\infty} \frac{1}{(x+s+k)(x+t+k)}} - x \end{aligned} \tag{19}$$

Using the same arguments as in the proof of Lemma 1, each member of the sum in the denominator can be replaced by:

$$\frac{1}{(x+s+k)(x+t+k)} > \frac{1}{(x+\beta+k)(x+\beta+k+1)}$$

where $\beta = \max\{(s+t-1)/2, \beta_0\}$. In the case $|t-s| < 1$, it is easy to verify that we have $\beta = \beta_0$. Therefore, we obtain the upper bound in the form:

$$W(x)^{\frac{1}{t-s}} - x < \frac{1}{\displaystyle\sum_{k=0}^{\infty} \frac{1}{(x+\beta_0+k)(x+\beta_0+k+1)}} - x = (x+\beta_0) - x = \beta_0,$$

which shows the right side of the Kershow's form (1). But, if we leave the first term in the sum (19) *intact*, the better upper bound for Gautschi's inequality on the interval

(x,∞) will follows:

$$W(x)^{\frac{1}{t-s}}-x<\frac{1}{\frac{1}{(x+s)(x+t)}+\sum_{k=1}^{\infty}\frac{1}{(x+\beta_0+k)(x+\beta_0+k+1)}}-x$$
$$=\frac{1}{\frac{1}{(x+s)(x+t)}+\frac{1}{x+\beta_0+1}}-x.$$

For example, for $x=0$ we obtain a sequence of decreasing bounds of the Kershow's type:

$$\beta_0,\quad \frac{1}{\frac{1}{st}+\frac{1}{\beta_0+1}},\quad \frac{1}{\frac{1}{st}+\frac{1}{(t+1)(s+1)}+\frac{1}{\beta_0+2}},\quad \text{etc.}$$

For $x=1$, we obtain, among others, following bounds:

$$\beta_0,\quad \frac{1}{\frac{1}{(1+t)(1+s)}+\frac{1}{\beta_0+2}}-1,\quad \frac{1}{\frac{1}{(1+t)(1+s)}+\frac{1}{(2+t)(2+s)}+\frac{1}{\beta_0+3}}-1,\quad \text{etc.}$$

The second one, and of course the third one, gives upper bound which can be compared in the case $t=1$ with the bound $\frac{2}{3}s$ given by Laforgia in [9]. Those bounds are not completely comparable since they are worse for very small values of s, and gives better estimation for s close to 1.

5. The best bounds in the second Gautschi's inequality

Inequality (2) can be also written in more natural way:

$$\exp\Big[\psi(x+\sqrt{s})\Big]<\left(\frac{\Gamma(x+1)}{\Gamma(x+s)}\right)^{\frac{1}{1-s}}<\exp\Big[\psi\Big(x+\frac{s+1}{2}\Big)\Big]. \tag{20}$$

i.e.

$$\psi(x+\sqrt{s})<\frac{1}{1-s}\log\frac{\Gamma(x+1)}{\Gamma(x+s)}<\psi\Big(x+\frac{s+1}{2}\Big).$$

We shall obtain the best bounds for the following more general form of this inequality:

$$\psi(x+\alpha)<\frac{1}{t-s}\int_s^t\psi(x+u)du<\psi(x+\beta), \tag{21}$$

By Hermite-Hadamard inequality for concave function ψ, it holds

$$\frac{1}{t-s}\int_s^t\psi(x+u)du<\psi\Big(x+\frac{s+t}{2}\Big),$$

therefore the choice $\beta=(s+t)/2$ holds in (21). It is known (see e.g. Theorem 6 in [3]) that this is the best upper bound. Let us find the best lower bound.

Denote by I_ψ integral ψ-mean of s and t:

$$I_\psi=I_\psi(s,t)=\psi^{-1}\left[\frac{1}{t-s}\int_s^t\psi(u)du\right].$$

THEOREM 4. *For every* $x \geqslant 0$, $s,t > 0$ *it holds*

$$\psi\Big(x + I_\psi(s,t)\Big) < \frac{1}{t-s}\int_s^t \psi(x+u)du < \psi(x+\beta). \tag{22}$$

Proof. For the function

$$h(x) = I_\psi(x+s, x+t) - x$$

we have:

(*i*) h is increasing,

(*ii*) h is concave, and it holds

$$\lim_{x\to\infty} h(x) = \frac{s+t}{2},$$

see [3], Theorem 6. From the second property it follows again that $\beta = (s+t)/2$ is the best upper bound. From the first property, it follows

$$x + I_\psi(s,t) \leqslant I_\psi(x+s, x+t), \qquad \forall x \geqslant 0.$$

Therefore, since ψ is increasing

$$\psi(x + I_\psi(s,t)) \leqslant \psi(I_\psi(x+s,x+t)) = \frac{1}{t-s}\int_s^t \psi(x+u)du$$

and the theorem is proved.

6. Connection with previous results and open problems

Let us denote

$$u(x) := \frac{1}{t-s}\log\frac{\Gamma(x+t)}{\Gamma(x+s)} - \psi(x+\varphi) \tag{23}$$

where $\varphi = \varphi(s,t)$ is unknown function which has to be determined such that u has constant sign for all $x > 0$.

We shall use the following representation of gamma and digamma functions [1]:

$$\Gamma(x) = \lim_{n\to\infty}\frac{n!n^x}{x(x+1)\cdots(x+n)},$$
$$\psi(x+1) = \gamma + \sum_{k=1}^{\infty}\left[\frac{1}{k} - \frac{1}{x+k}\right] = \lim_{n\to\infty}\left[\log(n+1) - \sum_{k=1}^{n+1}\frac{1}{x+k}\right]. \tag{24}$$

Hence,

$$u(x) = \lim_{n\to\infty}\left[\log n - \frac{1}{t-s}\log\frac{(x+t)\cdots(x+t+n)}{(x+s)\cdots(x+s+n)}\right]$$
$$-\lim_{n\to\infty}\left[\log(n+1) - \sum_{k=0}^{n}\frac{1}{x+\varphi+k}\right]$$
$$= \lim_{n\to\infty}\sum_{k=0}^{n}\left[\frac{1}{x+\varphi+k} - \frac{1}{t-s}\log\frac{x+t+k}{x+s+k}\right]$$

Let us denote

$$f_k(x) = \frac{1}{x+\varphi+k} - \frac{1}{t-s}\log\frac{x+t+k}{x+s+k}. \tag{25}$$

It holds $f_n(x) = f_0(x+n)$ and

$$u(x) = \sum_{k=0}^{\infty} f_k(x) \implies u(x+n) = \sum_{k=n}^{\infty} f_k(x)$$

Therefore,

$$\lim_{x\to\infty} u(x) = 0. \tag{26}$$

We have from (25)

$$f_k'(x) = \frac{x[2\varphi(s)-s-t] + \varphi(s)^2 - st + k(2\varphi(s)-s-t)}{(x+\varphi+k)^2(x+s+k)(x+t+k)}$$

It is obvious that for $\varphi(s) > \frac{1}{2}(s+t)$ we have $f_k'(x) > 0$ for all $x > 0$, hence f_k and u are increasing functions. From (26), we conclude $u(x) < 0$. This gives again upper bound in Kershow's improvement (20). On the other hand, for $\varphi(s) \leqslant \sqrt{st}$ it holds $f_k'(x) < 0$ for all $x > 0$, therefore u is decreasing and positive. This gives (a generalization of) the second Kershow's bound in (20).

Let us choose φ such that it holds $f_0(0) = 0$:

$$\frac{1}{\varphi(s,t)} = \frac{1}{t-s}\log\frac{t}{s} \implies \varphi(s,t) = \frac{t-s}{\log t - \log s} = L(s,t).$$

Function f_0 satisfies $\lim_{x\to\infty} f_0(x) = 0$ and $f_0'(x) = 0$ for only one $x = x_0 > 0$. Therefore, f_0 is positive for $x > 0$. For $k > 0$, f_k has only one extrem in x_k (eventually, x_k becomes negative for k sufficient large), it is decreasing for $x > x_k$, and it holds $f_k(0) = f_0(k) > 0$. Therefore, $f_k(x) > 0$ for all $x > 0$, and it follows $u(x,s) > 0$ for all $x > 0$.

It holds $L(s,t) > \sqrt{st}$. Hence, the lower bound $\varphi = L(s,t)$ is better than the one in (20). Of course, it holds $L(s,t) \leqslant I_\psi(s,t)$, see [3], Theorem 6, and for the choice $\varphi = I_\psi(s,t)$ it holds $u(0) = 0$.

REMARK. The left side of Hermite-Hadamard inequality for the functin ψ is

$$\frac{\psi(x+s)+\psi(x+t)}{2} < \frac{1}{t-s}\int_s^t \psi(x+u)du. \tag{27}$$

An equivalent inequality is obtained in [11] using Schur convexity, and author states that this inequality is better than the known one:

$$\psi(x+\sqrt{st}) < \frac{\psi(x+s)+\psi(x+t)}{2}$$

but the statement and therefore also the proof are wrong.

In [7] J. Bustoz and M. E. H. Ismail showed that functions

$$\frac{\Gamma(x+s)}{\Gamma(x+1)}\exp\left[(1-s)\Psi\left(x+\frac{s+1}{2}\right)\right], \tag{28}$$

$$\frac{\Gamma(x+1)}{\Gamma(x+s)}\left(x+\frac{s}{2}\right)^{s-1}, \tag{29}$$

are completely monotonic on $(0,\infty)$, for $0 \leqslant s \leqslant 1$. (A function f is said to be completely monotonic on an interval I if $(-1)^n f^{(n)}(x) \geqslant 0$ on I, for every $n \in \mathbf{N}$.) See also [7] for a slight generalization. We shall give a more general result.

THEOREM 5. *Let* $s,t>0$, $r=\min(s,t)$. *Function*

$$v(x)=\psi\left(x+\frac{s+t}{2}\right)-\frac{1}{t-s}\log\frac{\Gamma(x+t)}{\Gamma(x+s)}$$

is completely monotonic on $(-r,+\infty)$.

REMARK. It is known that if v is completely monotonic then $-v'$ is completely monotonic, hence $\exp(v)$ is also completely monotonic. In [7] it is proved in fact that $-v'$ is completely monotonic, for $t=1$ and $0<s<1$.

Proof of Theorem 5. We shall use the following representation of gamma and digamma functions:

$$\log\Gamma(x)=\int_0^\infty\left[(x-1)e^{-u}-\frac{e^{-u}-e^{-xu}}{1-e^{-u}}\right]\frac{du}{u},$$

$$\psi(x)=\int_0^\infty\left[\frac{e^{-u}}{u}-\frac{e^{-xu}}{1-e^{-u}}\right]du$$

Hence, for the function u defined in (23), we have

$$\begin{aligned}-u(x)&=\psi(x+\varphi)-\frac{1}{t-s}\left[\log\Gamma(x+t)-\log\Gamma(x+s)\right]\\&=\int_0^\infty\frac{e^{-xu}}{1-e^{-u}}\left[-e^{-\varphi u}-\frac{e^{-tu}-e^{-su}}{u(t-s)}\right]\\&=\int_0^\infty\frac{e^{-xu}}{1-e^{-u}}g(u)du,\end{aligned}$$

where

$$g(u)=-e^{-\varphi u}+\frac{1}{t-s}\int_s^t e^{-\tau u}d\tau.$$

By Hermite-Hadamard inequality for convex function $\tau \mapsto e^{-\tau u}$, we have $g(u) > 0$ for $\varphi \geqslant A(s,t)$. Therefore, for this value of φ

$$(-1)^k v^{(k)}(x) = \int_0^\infty u^k \frac{e^{-xu}}{1-e^{-u}} g(u)du > 0$$

and the theorem is proved.

OPEN PROBLEM 1. Function

$$\exp\left[\psi\left(x+\frac{s+t}{2}\right)\right] - W(x)^{\frac{1}{t-s}} \tag{30}$$

is convex and decreasing.

OPEN PROBLEM 2. From the first Gautschi's inequality, it is known that

$$W(0)^{\frac{1}{t-s}} - \frac{t+s-1}{2} > W(x)^{\frac{1}{t-s}} - x - \frac{t+s-1}{2} \to 0 \text{ as } x \to \infty. \tag{31}$$

Fromm the second Gautschi's inequality, we have

$$\exp\left[\psi\left(x+\frac{s+t}{2}\right)\right] - x - \frac{s+t-1}{2} > W(x)^{\frac{1}{t-s}} - x - \frac{t+s-1}{2} \tag{32}$$

Upper bound for $x=0$ in (31) is better than the bound in (32). We conjecture that it hold a stronger inequality

$$\begin{aligned} &\exp\left[\psi\left(x+\frac{s+t}{2}\right)\right] - x - \frac{s+t-1}{2} \\ &\quad > \frac{\exp\left[\psi\left(\frac{s+t}{2}\right)\right] - \frac{s+t-1}{2}}{W(0)^{\frac{1}{t-s}} - \frac{t+s-1}{2}} \cdot W(x)^{\frac{1}{t-s}} - x - \frac{t+s-1}{2} \end{aligned} \tag{33}$$

which gives extremly good approximation of the function from the left and right side.

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