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WEIGHTED INEQUALITY FOR SOME CLASSICAL INTEGRAL OPERATORS: $0 < p < 1$

QINSHENG LAI

(*communicated by L. Maligranda*)

Abstract. Suppose $0 < p < 1$ and $0 < q < \infty$. In this note, we prove that the weighted (p,q) inequality

$$\Big(\int_0^\infty (Tf(x))^q w(x)dx\Big)^{1/q} \leqslant C\Big(\int_0^\infty (f(x))^p v(x)dx\Big)^{1/p}$$

has no nontrivial solution if Tf is a Hardy type operator, the Hardy-Littlewood maximal operator or an one-sided maximal operator.

1. Introduction

We are interested in weighted norm inequalities for the integral operators defined as follows.

DEFINITION 1. (see [1]) *The Hardy type operator T is defined by*

$$Tf(x) = \int_0^x K(x,y)f(y)dy, \quad (x > 0),$$

where the kernel $K(x,y)$ on $R^+ \times R^+$ satisfies
(i) $K(x,y) > 0$ if $x > y$;
(ii) $K(x,y)$ is nondecreasing in x and nonincreasing in y;
(iii) There exists a constant $D > 0$ such that

$$K(x,y) \leqslant D(K(x,z) + K(z,y))$$

for all $0 \leqslant y \leqslant z \leqslant x$.

DEFINITION 2. (see [17]) *Suppose f is locally integrable on R^n. The Hardy-Littlewood maximal operator Mf is given by*

$$Mf(x) = \sup_{Q \ni x} \frac{1}{|Q|} \int_Q |f(y)|dy,$$

where the supremum is taken over all cubes Q, containing x.

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DEFINITION 3. *Given a positive and locally integrable function* $g(x)$ *on* $(-\infty,\infty)$. *Define the one-sided maximal operator* M_g^+f *as the following:*

$$M_g^+f(x) = \sup_{h>0} \frac{1}{g(x,x+h)} \int_x^{x+h} |f(y)|g(y)dy,$$

for all locally integrable f *, where* $g(x,x+h) = \int_x^{x+h} g(y)dy$. *(*see [11], [22]*)*

In past decades, weighted norm inequalities for these operators have been intensely investigated.

For example, suppose $1 \leqslant p < \infty$, $0 < q < \infty$ and Tf is a Hardy type operator. Let w and v be weight functions, that is, nonnegative and measurable functions. The characterizations of weight functions (w,v) for which the weighted (p,q) inequality

$$\Big(\int_0^{\infty} (Tf(x))^q w(x)dx\Big)^{1/q} \leqslant C\Big(\int_0^{\infty} (f(x))^p v(x)dx\Big)^{1/p} \tag{1}$$

is satisfied have been established. (see [1], [2], [10], [14], [15], [16], [23], [24]).

Let Tf be the Hardy-Littlewood operator or an one-sided maximal operator. The weight conditions for the weighted (p,q) inequality (1) have been obtained in the index range $1 \leqslant p \leqslant q < \infty$ and $1 < p < \infty$, $0 < q < p$. For these two operators, the weak type (p,q) inequality

$$[\lambda^q w(\{x : \ |Tf(x)| > \lambda\})]^{1/q} \leqslant C\Big(\int_{R^n} |f(x)|^p v(x)dx\Big)^{1/p}$$

have been characterized in the case $1 \leqslant p < \infty$, $0 < q < \infty$. (see [3], [17], [18], [21], [26], [11], [22], [9].)

In order to complete these weighted (p,q) inequalities, it is interesting to provide the answers to the case $0 < p < 1$, $0 < q < \infty$. That is the purpose of this note. It turns out that the weighted (p,q) inequality for these three operators has no non-trivial solution in the case $0 < p < 1$, $0 < q < \infty$.

The idea stems from the classical Day's theorem. He proved in [4] that if μ is a nonatomic measure and $0 < p < 1$, then there is no nonzero bounded linear funcional on $L_p(\mu)$ (cf. [8], [12]). Later on, completing a partial result of Williamson [27], Pallaschke [20] and Turpin [25] showed that there is no compact endomorphism T : $L_p(\mu) \to L_p(\mu)$ other than zero. Further investigations in this direction can be found in [6], [7] and many others. In [12, p. 150] there is a proof of the following result: If (Ω,Σ,μ) is a nonatomic measure, ν is another measure, $T : L_p(\mu) \to L_q(\nu)$ is a bounded linear operator and $0 < p < 1$, $p < q \leqslant \infty$, then $T = 0$.

Recently, Maligranda extended the above result and Turpin's work [25] and had the following general theorem.

THEOREM. ([13]) *If* (Ω,Σ,μ) *is a nonatomic measure space and* (K,Γ,ν) *another measure space. Let* $0 < p < 1$ *and* $p < q \leqslant \infty$. *Then there is no nozero bounded sublinear operator from* $L_p(\mu)$ *into* $L_q(\nu)$.

Instead of the general operators, we focus on these three specific operators, therefore our results cover more general range, and our theorem 2 and 3 work on weighted weak type inequalities. The proofs are based on the classical Day's theorem.

For a weight function w and a measurable set E, we write $w(E) = \int_E w(x)dx$, and $w(a,b) = w((a,b))$ for $E = (a,b)$. The Lebesgue measure of E is denoted by $|E|$, and the characteristic function of E by χ_E. We shall adopt the conventions of arithmetic operations in $R\cup\{\infty\}$: $t\cdot\infty = \infty\cdot t = \infty$ (for $t \in (0,\infty]$), $0\cdot\infty = \infty\cdot 0$ $=$ 0, $\frac{1}{\infty} = 0$ and $\frac{1}{0} = \infty$.

2. Theorems and proofs

If $w(x) \equiv 0$ *a.e.* or $v(x) \equiv \infty$ *a.e.*, the weighted (p,q) inequality (1) holds automatically. In addition, suppose Tf is a Hardy type operator or an one-sided maximal operator, the weighted inequality (1) is satisfied trivially, provided that there is a number $c \in (0,\infty)$ such that $w(x) \equiv 0$ *a.e.* on $(0,c)$ and $v(x) \equiv \infty$ *a.e.* on (c,∞). These are trivial solutions for the weighted inequality (1).

THEOREM 1. *Suppose $0 < p < 1$, $0 < q < \infty$, and Tf is a Hardy type operator associated with kernel K and w, v are weight funcitons. There is no nontrivial solution for the weighted (p,q) inequality* (1).

Proof. Let

$$c = \inf\{\tau : w(x) \equiv 0 \; a.e. x \in (\tau,\infty)\}.$$

The trivial case $c = 0$ should be excluded. When $c = \infty$. It is easy to see that for any $b \in (0,\infty)$ there exists $b < c < \infty$ such that $w(b,c) > 0$, and the following argument applies to this case obvoiusly. Now, suppose $c \in (0,\infty)$. Then $w(x) \equiv 0$ in (c,∞) and $w(b,c) > 0$, for any $b \in (0,c)$. We shall prove that the weight inequality (1) forces $v(x) \equiv \infty$ for $x \in (0,c)$.

First of all, we may assume $v(x) > 0$ *a.e.* on $(0,c)$. Indeed, suppose $|\{x \in (0,c) : v(x) = 0\}| = |E| > 0$. Then $x_0 = \operatorname{ess\,inf}\{x\chi_E(x)\} < c$. Set $f(x) = \chi_E(x)$. We have $Tf(x) > 0$ on (x_0,∞) and the right side of the weighted inequality is zero. This fact forces $w(x) = 0$ on (x_0,∞), which contradicts to the definition of c.

Given $0 < p < \infty$ and a measure space (X,μ). Define

$$\mathcal{N}_p(f) = \Big(\int_X |f(x)|^p d\mu\Big)^{1/p}$$

and

$$\mathcal{N}_p'(g) = \sup\{|\int_X f(x)g(x)d\mu| : \mathcal{N}_p(f) \leqslant 1\}.$$

These notations will be used throughout. We shall not specify the (X,μ) in each case, and it wouldn't cause any ambiguity.

We claim that for each interval (a,b) with $0 \leqslant a < b < c$, $\mathcal{N}_p'(\frac{\chi_{(a,b)}(\cdot)}{v(\cdot)}) < \infty$. Otherwise, assume there exist $\{f_n\}$ of nonnegative functions such that $\mathcal{N}_p(f_n) \leqslant 1$ and

$$\int_a^b f_n \frac{1}{v} v = \int_a^b f_n > n.$$

It follows from the weighted (p,q) inequality (1) that

$$C \geqslant C\Big(\int_0^\infty f_n^p v\Big)^{1/p} \geqslant \Big(\int_0^\infty (Tf)^q w\Big)^{1/q}$$
$$\geqslant \Big(\int_b^c (\int_a^b f_n)^q K(x,b)^q w(x)dx\Big)^{1/q} > n\Big(\int_b^c K(x,b)^q w(x)dx\Big)^{1/q}.$$

This is a contradiction, since $w(b,c) > 0$, and $K(x,b)$ is positive for $x \in (b,c)$.

$\mathscr{N}_p'(\frac{\chi_{(a,b)}(\cdot)}{v(\cdot)}) < \infty$ means that $f \mapsto \int_a^b f\,\frac{1}{v}v$ determines a continuous linear functional on the topological vector space $L^p(v)$. It must be zero functional (see [4] or [8]). That happens only if $v \equiv \infty$ on (a,b), since $v > 0$. This proves Theorem 1.

REMARK 1. *In the proof of Theorem 1, the condition (iii) in the definition of Hardy type operator is not used.*

THEOREM 2. *Let $0 < p < 1$ and $0 < q < \infty$. Then the weak type (p, q) inequality*

$$[w(\{x \in R^n : Mf(x) > \lambda\})\lambda^q]^{1/q} \leqslant C\Big(\int_{R^n} |f(x)|^p v(x)dx\Big)^{1/p} \tag{2}$$

has no nontrivial solution.

Proof. With some obvious modifications, the argument in [5] (p. 388) shows that the weak type (p,q) inequality (2) implies that $v(x) > 0$ a.e. unless $w(x) \equiv 0$ a.e..

Our argument is based on a simple fact:

$$Q \subset \{x : Mf(x) > \lambda_0\} \quad \text{for any } \lambda_0 < \tfrac{1}{|Q|}\int_Q |f(y)|dy.$$

Suppose $w(x)$ is not equivalent to 0. Choose Q_0 such that $w(Q_0) > 0$. Following the same idea as in Theorem 1, we need only prove $\mathscr{N}_p'(\frac{\chi_Q(\cdot)}{v(\cdot)}) < \infty$ for all $Q \supset Q_0$.

To make a contradiction, assume for some $Q \supset Q_0$ there exist $\{f_n \geqslant 0\}$ such that $\mathscr{N}_p(f_n) \leqslant 1$ and

$$\int_Q f_n \frac{1}{v} v = \int_Q f_n > n.$$

Then the weak type inequality (2) yields

$$w(Q) \leqslant w(\{Mf_n > \frac{n}{|Q|}\}) \leqslant \frac{C|Q|}{n}\mathscr{N}_p(f_n) \leqslant \frac{C|Q|}{n}.$$

This is a contradiction, since $w(Q) > 0$. The proof of Theorem 2 is completed.

THEOREM 3. *Let $0 < p < 1$ and $0 < q < \infty$. Then the weak type (p, q) inequality*

$$[w(\{x \in (-\infty,\infty) : M_g^+ f(x) > \lambda\})\lambda^q]^{1/q} \leqslant C\Big(\int_{-\infty}^\infty |f(x)|^p v(x)dx\Big)^{1/p} \tag{3}$$

has no nontrivial solution.

Proof. As in Theorem 1, suppose there is $a \in (-\infty, \infty]$ such that $w(x) \equiv 0$ on $(-\infty, a)$ and $w(a,b) > 0$ for any $b > a$.

We shall use the elementery estimate:

$$(a,b) \subset \{x : M_g^+ f(x) > \lambda_0\} \quad \text{for all } \lambda_0 < \tfrac{1}{g(a,c)} \int_b^c |f| g,$$

for arbitrary $a < b < c$.

Once again, the weak type inequality (3) implies $v(x) > 0$ a.e. on (a, ∞). Otherwise, let $|E| = |\{x \in (a,\infty) : v(x) = 0\}| > 0$. Then $c = esssup\{x : x\chi_E(x)\} > a$. Choose $b \in (a,c)$, we have $|S| = |E \cap (b,c)| > 0$. Set $f(x) = \chi_S(x)$ in the weighted weak type inequality (3), it is easy to see that

$$\frac{g(S)}{g(a,c)} w(a,b)^{1/q} \leqslant C v(S)^{1/p} = 0,$$

which impies $w(a,b) = 0$, since $|S| > 0$ and g is positive. This violates our choice of a. Therefore, $v(x)$ must be positive on (a, ∞).

Now we prove $\mathcal{N}_p{}'(\frac{g(\cdot)\chi_{(b,c)}(\cdot)}{v(\cdot)}) < \infty$ for all intervals (b,c) with $a < b$. Similar to the argument in Theorem 2, suppose we can choose $\{f_n\}$, which satisfy $\mathcal{N}_p(f_n) \leqslant 1$ and

$$\int_b^c f_n g \frac{1}{v} v = \int_b^c f_n g > n.$$

Then it follows from the weak type inequality (3) that

$$w(a,b) \leqslant w(\{x : M_g^+ f_n(x) > \frac{n}{g(a,c)}\}) \leqslant \frac{Cg(a,c)}{n} \mathcal{N}_p(f_n) \leqslant \frac{Cg(a,c)}{n}.$$

This is a contradiction, and Theorem 3 is proved.

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Qinsheng Lai
Department of Mathematics & Statistics
McMaster University
Hamilton, Ontario
Canada
L8S 4K1