Mathematical
Inequalities
& Applications
Volume 3, Number 2 (2000), 269–293

DESCRIPTION OF PSEUDOCHARACTERS' SPACE ON FREE PRODUCT OF GROUPS

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(*communicated by T. Rassias*)

Abstract. Let $G = A * B$ be a free product of groups A and B. A description is given of the space of real-valued functions φ on the group G satisfying the following conditions:

1) the set $\{\varphi(xy) - \varphi(x) - \varphi(y);\ x, y \in G\}$ is bounded;

2) $\varphi(x^n) = n\varphi(x)$ for any $x \in G$ and any $n \in Z$

Introduction

In 1940 S. M. Ulam posed the following problem. Given a group G_1, a metric group (G_2, d) and a positive number ε, does there exist a $\delta > 0$ such that if $f : G_1 \to G_2$ satisfies $d(f(xy), f(x)f(y)) < \delta$ for all $x, y \in G_1$, then a homomorphism $T : G_1 \to G_2$ exists with $d(f(x), T(x)) < \varepsilon$ for all $x, y \in G_1$?

See S. M. Ulam (1960) or (1974) for a discussion of such problems, as well as D. H. Hyers (1941, 1983), D. H. Hyers and S. M. Ulam (1945, 1947), Th. M. Rassias (1978, 1991), J. Aczèl and J. Dhombres (1989).

The first affirmative answer was given by D. H. Hyers [16] in 1941.

THEOREM OF HYERS. *Let E_1, E_2 be Banach spaces and let $f : E_1 \to E_2$ satisfies to the following condition: there is $\varepsilon > 0$ such that*

$$||f(x+y) - f(x) - f(y)|| < \varepsilon \quad \textit{for all} \quad x, y \in E_1. \tag{1}$$

Then there exists $T : E_1 \to E_2$ such that

$$T(x+y) - T(x) - T(y) = 0 \quad \textit{for all} \quad x, y \in E_1 \tag{2}$$

and

$$||f(x) - T(x)|| < \varepsilon \quad \textit{for all} \quad x \in E_1. \tag{3}$$

The subject rested there until Th. M. Rassias [30] considered a generalized version of the previous result which permitted the Cauchy difference to become unbounded. That is, he assumed that

$$||f(x+y) - f(x) - f(y)|| \leqslant \varepsilon \cdot (||x||^p + ||y||^p) \quad \text{for all} \quad x, y \in E_1 \ ,$$

Mathematics subject classification (1991): Primary 47H15 Secondary 20M30.

Key words and phrases: Free semigroup, free product of groups, pseudocharacter, linear space.

where ε and p are constant with $\varepsilon > 0$ and $0 \leqslant p < 1$.

By making use of a direct method, Th. M. Rassias proved in this case too, that there is an additive function T from E_1 into E_2 given by the formula

$$T(x) = \lim_{n\to\infty} \frac{1}{2^n} f(2^n x) \;;$$

such that

$$||T(x) - f(x)|| \leqslant k \cdot \varepsilon \cdot ||x||^p,$$

where k depends on p as well as ε.

Th. M. Rassias (1990) during the 27^{th} International Symposium on Functional Equations asked the question whether such a theorem can also be proved for $p \geqslant 1$.

Z. Gajda [14], following the same approach as in [30], gave an affirmative solution to this question for $p > 1$.

For the generalization of these results several papers were published c.f. [20–24, 30–33].

In connection with these results the following question arises.

Let S be an arbitrary semigroup or group and let a mapping $f : S \to \mathbb{R}$ satisfies the following condition:

$$\text{the set } \{f(xy) - f(x) - f(y)\ ,\quad x, y \in S\} \text{ is bounded.}$$

Is it true that there is $T : S \to \mathbb{R}$ satisfying the following conditions:

$$T(xy) - T(x) - T(y) = 0\ ,\quad x, y \in S, \text{ and}$$
$$\text{the set } \{T(x) - f(x)\,;\ x \in S\} \text{ is bounded.}$$

A negative answer was given by G. L. Forti [13] by means of the following example. Let $F(\alpha,\beta)$ be the free group generated by the two elements α,β. Let each word $x \in F(\alpha,\beta)$ be written in reduced form , i.e., x does not contain pairs of the forms $\alpha\alpha^{-1}$, $\alpha^{-1}\alpha$, $\beta\beta^{-1}$, $\beta^{-1}\beta$ and has no exponents different from 1 and -1. Define the function $f : F(\alpha,\beta) \to \mathbb{R}$ as follows. If $r(x)$ is the number of pairs of the form $\alpha\beta$ in x and $s(x)$ is the number of pairs of the form $\beta^{-1}\alpha^{-1}$ in x , put $f(x) = r(x) - s(x)$.

It is easily seen that for all $x, y \in F(\alpha,\beta)$ we have $f(xy) - f(x) - f(y) \in \{-1, 0, 1\}$. Now, assume that there is $T : F(\alpha,\beta) \to \mathbb{R}$ such that the relations (2), (3) hold.

However T is completely determined by its values $T(\alpha)$ and $T(\beta)$, while f is identically zero on the subgroups A and B generated by α and β , respectively. For $\alpha \in A$ we have $T(\alpha^n) = nT(\alpha)$ and $f(\alpha^n) = 0$ for $n \in N$. Since $T(\alpha^n) - f(\alpha^n) = nT(\alpha)$ for $n \in N$, it follows that $T(\alpha) = 0$. Similarly we have $T(\beta) = 0$, so that T is identically zero on $F(\alpha,\beta)$. Hence, $f - T = f$ on $F(\alpha,\beta)$ where f is unbounded. This contradiction proves that there is no homomorphism $T : F(\alpha,\beta) \to \mathbb{R}$ such that the relation (3) holds.

It turns out that the existence of mappings that are “almost homomorphism” but are not small perturbations of homomorphisms has an algebraic nature.

DEFINITION. A *quasicharacter* of a semigroup S is a real-valued function f on S satisfying the condition: the set $\{f(xy) - f(x) - f(y) | x, y \in S\}$ is bounded .

DEFINITION. By a *pseudocharacter* on a semigroup S (group S) we mean its quasicharacter f that satisfies the following condition:

$f(x^n) = nf(x)$, $\forall x \in S$ and $\forall n \in \mathbb{N}$ (and $\forall n \in Z$, if S is group).

The set of quasicharacters of a semigroup S is a vector space (with respect to the usual operations of addition of functions and their multiplication by numbers), which will be denoted by $KX(S)$. The subspace of $KX(S)$ consisting of pseudocharacters will be denoted by $PX(S)$ and the subspace consisting of real additive characters of the semigroup S, will be denoted by $X(S)$.

We say that a pseudocharacter φ of the group G is *nontrivial* if $\varphi \notin X(G)$.

In connection with the example of Forti note that his function is a quasicharacter of the free group $F(\alpha, \beta)$ but not a pseudocharacter of $F(\alpha, \beta)$. In [6, 10] the set of all pseudocharacters of free groups was described.

In [5–7,9–11] a description of the spaces of pseudocharacters on free groups and semigroups, semidirect and free products of semigroups was given. In [34] a pseudocharacter on the group $SL(2,\mathbb{Z})$ was constructed. In [8] a description of the spaces of all pseudocharacters on the group $SL(2,\mathbb{Z})$ was given.

For a mapping f of the group G into the semigroup of linear transformations of a vector space, in the papers [2–4] sufficient conditions for the coincidence of the solution of the functional inequality $\|f(xy) - f(x)\cdot f(y)\| < c$ with the solution of the corresponding functional equation $f(xy) - f(x)\cdot f(y) = 0$ were studied. In the papers [15,28], it was independently shown that if a continuous mapping f of a compact group G into the algebra of endomorphisms of a Banach space satisfies the relation $\|f(xy) - f(x)\cdot f(y)\| \leqslant \delta$ for all $x, y \in G$ with a sufficiently small $\delta > 0$, then f is ε-close to a continuous representation g of the same group in the same Banach space (i.e., we have $\|f(x) - g(x)\| < \varepsilon$ for all $x \in G$).

Let H be a Hilbert space and let $U(H)$ be the group of unitary operators of H endowed by operator-norm topology. If H is n - dimensional $n \in N$ the group $U(H)$ we denote by $U(n)$.

DEFINITION. Let $0 < \varepsilon < 2$. Let T be a mapping of a group G into $U(H)$. We say that T is an ε*–representation* if for any x, y from group G the relation

$$\|T(xy) - T(x)T(y)\| < \varepsilon$$

holds.

V. Milman raised the following question: Let $\rho : G \to U(H)$ be an ε – representation with small ε. Is it true that ρ is near to an actual representation π of the group G in H , i.e., does there exist some small $\delta > 0$ such that $\|\rho(x) - \pi(x)\| < \delta$ for all $x \in G$?

Answering this question Kazhdan in [28] obtained the following result.

THEOREM OF KAZHDAN. *There is a group Γ with the following property. For any $0 < \varepsilon < 1$ and any natural number $n > \frac{3}{\varepsilon}$ there exists an ε – representation ρ such that for any homomorphism $\pi : G \to U(n)$ the relation*

$$\|\rho - \pi\| = \sup\{\|\rho(x) - \pi(x)\|; \quad x \in \Gamma\} > \frac{1}{10}$$

holds.

Note that the group Γ has the following presentation in terms of generations and defining relations: $\Gamma = < x, y, a, b \parallel x^{-1}y^{-1}xya^{-1}b^{-1}ab >$.

In [12] by using pseudocharacters a strengthen of Kazhdan Theorem was established as follows.

We say that a group G belongs to the class $\mathcal{K}$ if every nonunit quotient group of G has an element of order two.

THEOREM FAĬZIEV. *Let H be a Hilbert space and let $U(H)$ be its group of unitary operators. Suppose that a groups A and B belongs to the class $\mathcal{K}$ and the order of B is more than two. Then the free product $G = A * B$ has the following property. For any $\varepsilon > 0$ there exists a mapping $T : G \to U(H)$ satisfying the following conditions :*

1) $\|T(xy) - T(x) \cdot T(y)\| \leqslant \varepsilon, \quad \forall x, \forall y \in G$;

2) for any representation $\pi : G \to U(H)$ the relation

$$\sup\{\|T(x) - \pi(x)\| \,, x \in G\} = 2$$

holds.

There is a following connection of quasicharacters and pseudocharacters with the theory of Banach algebra cohomology. The definition of quasicharacters coincides with a bounded $2-$cocycle on semigroup. Hence, if a semigroup S has nontrivial pseudocharacter, i.e., $PX(S) \setminus X(S) \neq \emptyset$, then arguing as [25, Proposition 2.8] we obtain $H^2(S, \mathbb{C}) \neq 0$.

The aim of the present paper is to give a description of the space $PX(G)$, where the group G is the free product $G = A * B$.

The paper contains two sections. In §1 a description of the space $PX(G)$ in term of pseudocharacters of free factors and some space of pseudocharacters $PX(D, -1)$ of free semigroup D is given.

In §2 a description of $PX(D, -1)$ and $PX(G)$ is given.

§1 Decomposition of the space $PX(A * B)$ into direct sum of the spaces $PX(A), PX(B), PX(D, -1)$

DEFINITION. By a *pseudocharacter* on a semigroup S (group S) we mean a real-valued function f on S satisfies the following conditions:

1) the set $\{f(xy) - f(x) - f(y) | x, y \in S\}$ is bounded ;

2) $f(x^n) = f(x)$, $\quad \forall x \in S$ and $\forall n \in \mathbb{N}$ (and $\forall n \in Z$, if S is group).

Let G be an arbitrary group and $\tau : G \to H$ its an epimorphism onto a group H.

Denote by τ^* the mapping that takes each element $\varphi \in PX(H)$ to $\varphi \circ \tau \in PX(G)$. It is evidently that τ^* is an embedding $PX(H)$ into $PX(G)$.

Let $G = A * B$ be the free product of nontrivial groups A and B, and let τ_A^*, τ_B^* be embedding of the spaces $PX(A)$, $PX(B)$ into $PX(G)$ respectively. Below we shall identify the spaces $PX(A)$ and $PX(B)$ with their τ_A^* and τ_B^* isomorphic images respectively.

Set $A_0 = A \setminus 1, B_0 = B \setminus 1$ and $M = \{a \cdot b \,|\, a \in A_0, b \in B_0\}$. It is clear that subsemigroup $\tilde{D}$ of group G generated by the set M is free and M is the system of free generators for $\tilde{D}$. By D denote a semigroup generated by $\tilde{D}$ and 1.

Let $v \in D$. By $|v|$ denote the length of the word v in alphabet M. If $v = 1$ we set $|v| = 0$.

DEFINITION. By *canonical* form of nonunit element g from $G = A * B$ we mean its presentation in the form $g = c_1c_2 \cdots c_n$, where $c_i \in A_0 \cup B_0$, and $c_ic_{i+1} \notin A \cup B$.

Let $v = a_1b_1 \cdots a_nb_n \in \tilde{D}$. By $\overline{v}$ denote the element $b_1a_2b_2 \cdots a_nb_na_1$. Let $PX(D,-1)$ be the subspace of $PX(D)$ consisting of the pseudocharacters φ of D satisfying the following conditions:

1) the set $\varphi(M)$ is bounded,

2) $\varphi((\overline{v})^{-1}) = -\varphi(v), \forall v \in D$.

REMARK 1. We recall that by the Lemma 2 from [5] for any pseudocharacter φ of arbitrary semigroup S the following relation $\varphi(xy) = \varphi(yx), \quad \forall x,y \in S$ holds.

Let $\varphi \in PX(D,-1)$. Denote by $\overline{\varphi}$ the function on the group G defining as follows. If element v from G is conjugate to some element $a \in A$ or some element $b \in B$, then we set $\overline{\varphi}(v) = 0$. Otherwise we set $\overline{\varphi}(v) = \varphi(t)$, where $t \in D$ and elements v and t are conjugate in G. Remark 1 implies that the function $\overline{\varphi}$ is well defined . It is clear that the function $\overline{\varphi}$ is constant on the classes of conjugacy in G.

Denote by $\sim$ the relation of conjugacy in the group G.

Below in the statements of Lemmas 1–5 we shall assume that φ is an element from $PX(D,-1)$ and $c > 0$ such that

$$|\varphi(xy) - \varphi(x) - \varphi(y)| \leqslant c \quad \forall x,y \in D; \quad \text{and} \quad |\varphi(x)| \leqslant c \quad \forall x \in M.$$

LEMMA 1. *For any $g \in G$ and any $n \in Z$ we have*

$$\overline{\varphi}(g^n) = n\varphi(g).$$

Proof. 1) Suppose that g is conjugate to some element $z \in A \bigcup B$. Then it is clear that for any $n \in Z$ element g^n is conjugate to z^n and the relation $\overline{\varphi}(g^n) = \varphi(g) = 0$ holds.

2) Now let $g \sim t, t \in D$. It is clear that $g^n \sim t^n$ for any $n \in Z$. Therefore , if $n \in N$, then $\overline{\varphi}(g^n) = \overline{\varphi}(t^n) = \varphi(t^n) = n\varphi(t) = n\overline{\varphi}(g)$. It is obvious that $g^{-1} \sim t^{-1}$. And for any positive integer n we have $\overline{\varphi}(g^{-n}) = \overline{\varphi}(t^{-n}) = \overline{\varphi}(\overline{t^{-n}}) = \varphi(\overline{(t^n)^{-1}}) = -\varphi(t^n) = -n\varphi(t) = -n\overline{\varphi}(g)$.

The Lemma is proved.

LEMMA 2. *Let $w = a_1b_1 \cdots a_nb_n, \quad a_i \in A_0, b_i \in B_0$. Then for any $a \in A$ such that $aa_1 \neq 1$ and for any $b \in B$ such that $bb_1 \neq 1$ the following relations $|\overline{\varphi}(wa) - \overline{\varphi}(w)| \leqslant 4c$; $\quad |\overline{\varphi}(b \cdot w) - \overline{\varphi}(w)| \leqslant 4c$ hold.*

Proof. If $n = 1$ we have $w = a_1b_1$, $|\varphi(a \cdot w)| \leqslant c$. Hence $|\overline{\varphi}(a \cdot w) - \overline{\varphi}(w)| \leqslant 2c$. Now if $n \geqslant 2$, then $wa \sim aw = aa_1b_1w_1, \quad a_1 \cdot b_1, w_1 \in D$, where $w_1 = a_2b_2 \cdots a_nb_n$. Hence

$$|\overline{\varphi}(wa) - \overline{\varphi}(aa_1b_1) - \overline{\varphi}(w_1)| \leqslant c. \tag{1.1}$$

Now assume that $w = a_1b_1w_1$ .Then

$$|\overline{\varphi}(w) - \overline{\varphi}(a_1b_1) - \overline{\varphi}(w_1)| \leqslant c. \tag{1.2}$$

From (1.1) and (1.2) we obtain $|\overline{\varphi}(wa) - \overline{\varphi}(w)| = |\overline{\varphi}(wa) - \overline{\varphi}(aa_1b_1) - \overline{\varphi}(w_1) + \overline{\varphi}(aa_1b_1) + \overline{\varphi}(w_1) - \overline{\varphi}(w) + \overline{\varphi}(a_1b_1) - \overline{\varphi}(a_1b_1)| \leqslant |\overline{\varphi}(wa) - \overline{\varphi}(aa_1b_1) - \varphi(w_1)| + |\overline{\varphi}(w) - \overline{\varphi}(a_1b_1) - \overline{\varphi}(w_1)| + |\varphi(aa_1b_1) - \overline{\varphi}(a_1b_1)| \leqslant 2c + |\varphi(aa_1b_1) - \varphi(a_1b_1)| \leqslant 2c + |\varphi(aa_1b_1)| + |\varphi(a_1b_1)| \leqslant 4c$.

Similarly,if $b_nb \neq 1$, we get $|\overline{\varphi}(wb) - \overline{\varphi}(w)| \leqslant 4c$

The Lemma is proved.

REMARK 2. Let $a, a_i \in A_0$; $b, b_i \in B_0$; $z = b_1a_1 \cdots b_ma_m$; then

$$|\varphi(azb) - \overline{\varphi}(z)| \leqslant 6c.$$

Indeed, by definition we have $\overline{\varphi}(z) = \varphi(a_1b_2 \cdots b_ma_mb_1)$, hence,

$$|\overline{\varphi}(z) - \varphi(a_1b_2 \cdots a_{m-1}b_m) - \varphi(a_mb_1)| \leqslant c.$$

Furthermore, we obtain

$$|\varphi(azb) - \varphi(ab_1) - \varphi(a_1b_2 \cdots a_{m-1}b_m) - \varphi(a_mb)| \leqslant 2c$$

and

$$|\varphi(azb) - \overline{\varphi}(z)| \leqslant 2c + |\varphi(ab_1)| + |\varphi(a_mb)| + |\varphi(a_mb_1)| \leqslant 6c.$$

DEFINITION. If $v = c_1c_2 \cdots c_n$ is canonical form, then we set $\dot{v} = c_1$, $\ddot{v} = c_n$.

LEMMA 3. *Let $d \in A \bigcup B$. Then for any $v \in D$ the following relation $|\overline{\varphi}(d \cdot v) - \overline{\varphi}(v)| \leqslant 20c$ holds.*

Proof. Let $v = a_1b_1 \cdots a_nb_n$ be canonical form of the element v. Suppose that $n = 1$. It is clear that either element dv is conjugate in G to some element from the set $A \bigcup B$, or element dv is conjugate to some element from the set M.

Hence we have $|\overline{\varphi}(dv)| \leqslant c$ and $|\overline{\varphi}(dv) - \overline{\varphi}(v)| \leqslant 2c$.

Now suppose that $n \geqslant 2$.

Consider two cases: the case I, when $d = a \in A$; the case II, when $d = b \in B$.

I. If $aa_1 \neq 1$, then from Lemma 2 it follows that conclusion of the lemma is true. Now we assume $aa_1 = 1$. Let $n = 2$, $v = a_1b_1a_2b_2$. Then element av is conjugate to some element from the set $A \bigcup B \bigcup M$. Hence, $|\overline{\varphi}(av)| \leqslant c$. Furthermore, inequality $|\varphi(v) - \varphi(a_1b_1) - \varphi(a_2b_2)| \leqslant c$ implies $|\varphi(v)| \leqslant 3c$ and $|\overline{\varphi}(av) - \overline{\varphi}(v)| \leqslant 4c$. Now we can assume that $n \geqslant 3$.

If $b_nb_1 \neq 1$; then

$$\overline{\varphi}(av) = \overline{\varphi}(b_1a_2b_2 \cdots a_nb_n) = \varphi(a_2b_2 \cdots a_{n-1}b_{n-1}a_nb_nb_1).$$

Hence, from Lemma 2 we obtain

$$|\overline{\varphi}(av) - \overline{\varphi}(a_2b_2 \cdots a_nb_n)| \leqslant 4c. \tag{1.3}$$

From $|\varphi(v)-\varphi(a_1b_1)-\varphi(a_2b_2\cdots a_nb_n)|\leqslant c$ we get

$$|\overline{\varphi}(v)-\varphi(a_2b_2\cdots a_nb_n)|\leqslant 2c.$$

Combining this with (1.3), we get $|\overline{\varphi}(av)-\overline{\varphi}(v)|\leqslant 6c$. Now let $aa_1=1, b_nb_1=1,\ldots$ Then the canonical form of $w=av$ is

$$w=av=b_1a_2b_2\cdots a_nb_n=t^{-1}ut, \tag{1.4}$$

where the word u satisfyes the conditions: $\ddot{u}\cdot\dot{u}\neq 1$, $\ddot{u}\cdot\dot{u}\in A_0\cup B_0$.

From (1.4) we have $\overline{\varphi}(av)=\overline{\varphi}(u)$. It is clear that there are two cases:

a) $u=u_1\gamma,\quad u_1\in D,\ \gamma\in A$, or

b) $u=\beta u_2\quad u_2\in D, \beta\in B.$

Note that the elements γ, β are different from 1 and the following relations $\ddot{u}_2\beta\neq 1$, $\gamma\dot{u}_1\neq 1$ hold. Lemma 2 imply that in the case a) we have $|\overline{\varphi}(u)-\varphi(u_1)|\leqslant 4c$ and in the case b) we have $|\overline{\varphi}(u)-\varphi(u_2)|\leqslant 4c$.

Consider the case a).

The Canonical form of element v is $v=a_1b_1\cdots a_nb_n=a_1t^{-1}ut$, where $t=\beta_1\alpha_1\cdots\beta_r\alpha_r\beta_{r+1}$; $\beta_i\in B$, $\alpha_i\in A$. If $r=1$, then $t=\beta_1\alpha_1\beta_2$, $w=\beta_2^{-1}\alpha_1^{-1}\beta_1^{-1}u\beta_1\alpha_1\beta_2$. Lemma 2 implies

$$|\varphi(av)-\varphi(u_1)|\leqslant 4c. \tag{1.5}$$

Let $r>1$. Since $v=a_1\beta_{r+1}^{-1}\alpha_r^{-1}\beta_r^{-1}\cdots\alpha_1^{-1}\beta_1^{-1}u_1\gamma\beta_1\alpha_1\cdots\beta_r\alpha_r\beta_{r+1}$ is the canonical form of v we obtain

$$|\varphi(v)-\varphi(a_1t^{-1})-\varphi(u_1)-\varphi(\gamma t)|\leqslant 2c. \tag{1.6}$$

Furthermore, since $t=\beta_1\alpha_1\cdots\beta_r\alpha_r\beta_{r+1}=\beta_1t_1,\quad t_1=\alpha_1\cdots\beta_r\alpha_r\beta_{r+1}\in D$, we get

$$|\varphi(a_1t)-\varphi(a_1\beta_1)-\varphi(\alpha_1\beta_2\cdots\beta_r\alpha_r\beta_{r+1})|\leqslant c,$$

$$|\varphi(a_1t)-\varphi(t_1)|\leqslant 2c.$$

The latter inequality and Lemma 1 imply

$$|\overline{\varphi}(t_1)-\varphi(\gamma t)|=|\varphi(t_1)-\varphi(\gamma\beta_1t_1)|\leqslant 2c. \tag{1.7}$$

Furthermore, since $a_1t^{-1}=a_1t_1^{-1}\beta_1^{-1}$, Remark 2 implies

$$|\varphi(a_1t^{-1})-\overline{\varphi}(t_1^{-1})|\leqslant 6c. \tag{1.8}$$

Now from (1.7) and (1.8) we have

$$|\varphi(a_1t^{-1})+\varphi(\gamma t)|=|\varphi(a_1t^{-1})-\overline{\varphi}(t_1^{-1})+\overline{\varphi}(t_1^{-1})+\varphi(\gamma t)|\leqslant$$

$$|\varphi(a_1t^{-1})-\overline{\varphi}(t_1^{-1})|+|\overline{\varphi}(t_1^{-1})+\varphi(\gamma t)|\leqslant 6c+2c=8c.$$

From (1.6) it follows that

$$|\varphi(v)-\varphi(u_1)|\leqslant 10c. \tag{1.9}$$

And from (1.5), (1.9) we get $\quad|\varphi(av)-\varphi(v)|\leqslant 14c.$

Now consider the case b) $u=\beta u_2$; $\quad u_2\in D,\quad\beta\in B.$

It is clear that $av = t^{-1}\beta u_2 t$, $\quad a_1 t^{-1}\beta u_2 t$ are canonical forms such that $\dot{t} \in A_0$, $\ddot{t} \in B_0$. From Remark 1 we obtain

$$|\varphi(v) - \varphi(a_1 t^{-1}\beta) - \varphi(u_2) - \varphi(t)| \leqslant 2c. \tag{1.10}$$

Let $t \in M$; then $|\varphi(a_1 t^{-1}\beta)| \leqslant 3c, \quad |\varphi(t)| \leqslant c$. By (1.10) we obtain

$$|\varphi(v) - \varphi(u_2)| \leqslant 6c. \tag{1.11}$$

From (1.11), $\overline{\varphi}(w) = \overline{\varphi}(\beta u_2)$ and Lemma 2 we get $|\overline{\varphi}(av) - \varphi(v)| \leqslant 10c$. Now suppose that the length of the word t in alphabet M is more than 1. Then there is element τ in D such that $t = \tilde{a}_2\tilde{b}_2\tau, \tilde{a}_2 \in A_0, \tilde{b}_2 \in B_0$. Hence , $v = a_1\tau^{-1}\tilde{b}_2^{-1}\tilde{a}_2^{-1}\beta u_2\tilde{a}_2\tilde{b}_2\tau$ and

$$|\varphi(v) - \varphi(a_1\tau^{-1}\tilde{b}_2) - \varphi(\tilde{a}_2^{-1}\beta) - \varphi(u_2) - \varphi(\tilde{a}_2\tilde{b}_2) - \varphi(\tau)| \leqslant 4c. \tag{1.12}$$

Suppose that $\tau = \alpha_1\beta_1\alpha_2\beta_2\cdots\alpha_k\beta_k, \quad k \geqslant 1$; then

$$a_1\tau^{-1}\tilde{b}_2 = a_1\beta_k^{-1}\alpha_k^{-1}\cdots\beta_2^{-1}\alpha_2^{-1}\beta_1^{-1}\alpha_1^{-1}\tilde{b}_2.$$

If $k = 1$, then $|\varphi(a_1\tau^{-1}\tilde{b}_2) - \varphi(\alpha_1\beta_1^{-1}) - \varphi(\alpha_1^{-1}\tilde{b}_2)| \leqslant c$. If $k > 1$, then

$$|\varphi(a_1\tau^{-1}\tilde{b}_2) - \varphi(\alpha_1\beta_k^{-1}) - \varphi(\alpha_k^{-1}\cdots\beta_2^{-1}\alpha_2^{-1}\beta_1^{-1}) - \varphi(\alpha_1^{-1}\tilde{b}_2)| \leqslant 2c$$

Hence, for any $k \geqslant 1$ we have

$$|\varphi(a_1\tau^{-1}\tilde{b}_2) - \varphi(\alpha_k^{-1}\beta_{k-1}^{-1}\cdots\beta_2^{-1}\alpha_2^{-1}\beta_1^{-1})| \leqslant 4c.$$

Let $\delta = \beta_1\alpha_2\beta_2\cdots\beta_{k-1}\alpha_k$.

Then the latter inequality implies

$$|\varphi(a_1\tau^{-1}\tilde{b}_2) - \varphi(\delta^{-1})| \leqslant 4c. \tag{1.13}$$

Since $\tau = \alpha_1\delta\beta_k$ we get $\overline{\varphi}(\tau) = \overline{\varphi}(\delta\beta_k\alpha_1)$ and

$$|\varphi(\tau) - \varphi(\alpha_1\beta_1) - \varphi(\alpha_2\beta_2\cdots\alpha_{k-1}\beta_{k-1}) - \varphi(\alpha_k\beta_k)| \leqslant 2c.$$

Hence,

$$|\varphi(\tau) - \varphi(\alpha_2\beta_2\cdots\alpha_{k-1}\beta_{k-1}| \leqslant 4c. \tag{1.14}$$

By definition we have $\overline{\varphi}(\delta) = \varphi(\alpha_2\beta_2\cdots\alpha_{k-1}\beta_{k-1}\alpha_k\beta_1)$. Therefore,

$$|\overline{\varphi}(\delta) - \overline{\varphi}(\alpha_2\beta_2\cdots\alpha_{k-1}\beta_{k-1}) - \overline{\varphi}(\alpha_k\beta_1)| \leqslant c$$

and

$$|\overline{\varphi}(\delta) - \varphi(\alpha_2\beta_2\cdots\alpha_{k-1}\beta_{k-1})| \leqslant 2c. \tag{1.15}$$

From (1.14) and (1.15) we obtain

$$|\overline{\varphi}(\tau) - \overline{\varphi}(\delta)| \leqslant 6c. \tag{1.16}$$

Using lemma 1, (1.13),(1.16) and (1.12) we get

$$|\varphi(v) - \varphi(\delta^{-1}) - \varphi(\tilde{a}_2^{-1}\beta) - \varphi(u_2) - \varphi(\tilde{a}_2\tilde{b}_2) - \overline{\varphi}(\delta)| \leqslant 14c, \quad \text{i.e.,}$$

$$|\varphi(v) - \varphi(u_2)| \leqslant 16c. \tag{1.17}$$

Now from $\varphi(av) = \varphi(\beta u_2)$ and Lemma 2 we obtain $|\varphi(av) - \varphi(u_2)| \leqslant 4c$. Hence, from (1.17) it follows $|\varphi(av) - \varphi(v)| \leqslant 20c$.

The case II, when $d = b \in B$ is considered similarly. The Lemma is proved.

DEFINITION. Let v be a word in alphabet $A \cup B$. By *regular subdivision* of v we mean its presentation in the form $v = v_1 v_2$, where v_1, v_2 are canonical forms such that $\ddot{v}_1 \cdot \dot{v}_2 \notin A_0 \cup B_0$.

LEMMA 4. *Let $t = c_1 c_2 \cdots c_k$ be the canonical form and $z \in A_0 \cup B_0$. Then*

$$|\overline{\varphi}(zt) - \overline{\varphi}(t)| \leqslant 40c. \tag{1.18}$$

Proof. If $t \in D$, then Lemma 3 implies the inequality (1.18).

A. Let $t = ba_1b_1 \cdots a_nb_na$, $v = a_1b_1 \cdots a_nb_n$.

Then

$$|\varphi(t) - \varphi(v)| \leqslant 2c. \tag{1.19}$$

Consider two cases: 1) $z \in A_0$ 2)$z \in B_0$.

In the case 1) we have $|\varphi(zbv) - \varphi(v)| \leqslant 2c$. From the Lemma 3 we obtain inequality $|\overline{\varphi}(zt) - \varphi(zbv)| \leqslant 20c$. Hence, from (1.19) it follows that $|\varphi(zt) - \varphi(t)| \leqslant 24c$.

In the case 2) we have $zt \sim va \cdot zb$.

If $zb = 1$, then $zt \sim va$ and by Lemma 3 we have $|\overline{\varphi}(zt) - \varphi(v)| \leqslant 20c$. Therefore, (1.19) implies $|\overline{\varphi}(zt) - \varphi(t)| \leqslant 22c$.

If $zb \neq 1$, then $|\overline{\varphi}(zt) - \varphi(v)| \leqslant 2c$. And from (1.19) it follows that $|\overline{\varphi}(zt) - \varphi(t)| \leqslant 4c$.

B. Let $t = a_1b_1 \cdots a_nb_na$, $v = a_1b_1 \cdots a_nb_n$.

Lemma 3 implies that

$$|\overline{\varphi}(t) - \varphi(v)| \leqslant 20c. \tag{1.20}$$

Consider two cases $a)$ $z \in A_0$, $b)$ $z \in B_0$.

In the case a) we have $zt = zv \sim vaz$, and by Lemma 3 we obtain

$$|\overline{\varphi}(zt) - \varphi(v)| \leqslant 20c. \tag{1.21}$$

From (1.20,1.21) it follows that $|\overline{\varphi}(zt) - \varphi(t)| \leqslant 40c$.

In the case b) we have $zt = zva \sim vaz \in D$, therefore $|\overline{\varphi}(zt) - \varphi(v)| \leqslant 2c$. From this inequality and (1.20) we get $|\overline{\varphi}(zt) - \overline{\varphi}(t)| \leqslant 22c$.

C. $t = ba_1b_1 \cdots a_nb_n = bv$.

The Lemma 3 implies

$$|\varphi(t) - \varphi(v)| \leqslant 20c. \tag{1.22}$$

a) $z \in A_0$. It is clear that $zt \sim vzb$. Hence, $|\overline{\varphi}(zt) - \varphi(v)| \leqslant 2c$. From (1.22) we have $|\overline{\varphi}(zt) - \varphi(t)| \leqslant 22c$.

b) $z \in B_0$. Lemma 3 implies that $|\varphi(zt) - \varphi(v)| \leqslant 20c$. Hence, from (1.22) we obtain $|\overline{\varphi}(zt) - \overline{\varphi}(t)| \leqslant 40c$. The Lemma is proved.

LEMMA 5. *Let $v = c_1c_2 \cdots c_n$ be canonical form of a word v in alphabet $A \cup B$, and let $v = v_1v_2$ be its regular subdivision. Then the following inequality*

$$|\overline{\varphi}(v) - \overline{\varphi}(v_1) - \overline{\varphi}(v_2)| \leqslant 47c$$

holds.

Proof. 1) Let $\ddot{c}_1 \cdot \dot{c}_n \notin A \cup B$.

Consider the case when $c_1 \in A, \quad c_n \in B$.

We have $v_1 = c_1c_2\cdots c_k, \quad v_2 = c_{k+1}c_{k+2}\cdots c_n$. By the Lemma 3 we can assume that $n \geqslant 2$ and $2 \leqslant k \leqslant n-1$.

a) If $c_k \in B$, then from the relation $\varphi \in PX(D,-1)$, we have

$$|\overline{\varphi}(v) - \overline{\varphi}(v_1) - \overline{\varphi}(v_2)| \leqslant c.$$

b) If $c_k \in A$, then $c_k = a, c_{k+1} = b$ and

$$v_1 = \tilde{v}a,\ v_2 = b\tilde{v}_2; \quad \tilde{v}_1,\ \tilde{v}_2 \in D.$$

Hence, $\overline{\varphi}(v) = \overline{\varphi}(\tilde{v}_1 a \cdot b\tilde{v}_2)$. Since $\overline{\varphi}\big|_D \in PX(D)$, we get

$$|\overline{\varphi}(v) - \overline{\varphi}(\tilde{v}_1) - \overline{\varphi}(ab) - \overline{\varphi}(\tilde{v}_2)| \leqslant 2c. \tag{1.23.}$$

From Lemma 3 it follows that

$$|\varphi(v_1) - \varphi(\tilde{v}_1)| \leqslant 20c, \quad |\varphi(v_2) - \varphi(\tilde{v}_2)| \leqslant 20c. \tag{1.24}$$

Therefore from (1.23) and (1.24) we obtain

$$|\overline{\varphi}(v) - \overline{\varphi}(v_1) - \overline{\varphi}(v_2)| \leqslant 43c.$$

Now let $c_1 \in B$, $c_n \in A$, then v has the form $v = ba_1b_1a_2b_2\cdots a_mb_ma$.

In the case a) for some positive integer r we have

$$v_1 = ba_1b_1a_2b_2\cdots a_rb_r, \quad v_2 = a_{r+1}b_{r+1}b_1\cdots a_mb_ma.$$

It is clear that

$$|\overline{\varphi}(v) - \varphi(a_1b_1a_2b_2\cdots a_mb_m) - \varphi(ab)| \leqslant c,$$

and

$$|\overline{\varphi}(v) - \varphi(a_1b_1a_2b_2\cdots a_mb_m)| \leqslant 2c. \tag{1.25}$$

Lemma 3 implies the following inequalities

$$|\overline{\varphi}(v_1) - \varphi(a_1b_1a_2b_2\cdots a_rb_r)| \leqslant 20c$$

and

$$|\overline{\varphi}(v_2) - \varphi(a_{r+1}b_{r+1}b_1\cdots a_mb_m)| \leqslant 20c.$$

From the relation $\varphi \in PX(D,-1)$ we obtain

$$|\varphi(a_1b_1\cdots a_mb_m) - \varphi(a_1b_1\cdots a_rb_r) - \varphi(a_{r+1}b_{r+1}b_1\cdots a_mb_m)| \leqslant c.$$

By the three latter inequalities and (1.25) we get

$$|\overline{\varphi}(v) - \overline{\varphi}(v_1) - \overline{\varphi}(v_2)| \leqslant 43c.$$

b) In this case we have

$$v_1 = ba_1b_1a_2b_2\cdots a_rb_ra_{r+1}, \quad v_2 = b_{r+1}a_{r+2}b_{r+2}\cdots a_mb_ma.$$

Therefore, the following relations

$$|\overline{\varphi}(v) - \varphi(a_1b_1a_2b_2\cdots a_mb_m) - \varphi(ab)| \leqslant c,$$
$$|\overline{\varphi}(v) - \varphi(a_1b_1a_2b_2\cdots a_mb_m)| \leqslant 2c$$

and

$$|\varphi(a_1b_1a_2b_2\cdots a_mb_m) - \varphi(a_1b_1a_2b_2\cdots a_rb_r) - \varphi(a_{r+1}b_{r+1}) - \varphi(a_{r+2}b_{r+2}b_1\cdots a_mb_m)| \leqslant 2c.$$

hold. Hence,

$$|\varphi(a_1b_1a_2b_2\cdots a_mb_m) - \varphi(a_1b_1a_2b_2\cdots a_rb_r) - \varphi(a_{r+2}b_{r+2}b_1\cdots a_mb_m)| \leqslant 3c.$$

Now, taking into account (1.25), we obtain

$$|\overline{\varphi}(v) - \varphi(a_1b_1a_2b_2\cdots a_rb_r) - \varphi(a_{r+2}b_{r+2}b_1\cdots a_mb_m)| \leqslant 5c. \tag{1.26}$$

By definition of $\overline{\varphi}$ we have

$$|\overline{\varphi}(v_1) - \varphi(a_1b_1a_2b_2\cdots a_rb_r)| \leqslant 2c. \tag{1.27}$$

Similarly,

$$|\overline{\varphi}(v_2) - \varphi(a_{r+2}b_{r+2}b_1\cdots a_mb_m)| \leqslant 2c. \tag{1.28}$$

Therefore, from (1.26),(1.27),(1.28) it follows that

$$|\varphi(v) - \varphi(v_1) - \varphi(v_2)| \leqslant 9c.$$

2) $\ddot{c}_1 \cdot \dot{c}_n \in A \cup B$.

I. Let $c_1, c_n \in A$, $v = a_1b_1a_2b_2\cdots a_mb_ma$;

a) $v_1 = a_1b_1a_2b_2\cdots a_kb_k$, $v_2 = a_{k+1}b_{k+1}\cdots a_mb_ma$.

Since the function φ is a pseudocharacter of D, we have

$$|\varphi(a_1b_1a_2b_2\cdots a_mb_m) - \varphi(v_1) - \varphi(a_{k+1}b_{k+1}\cdots a_mb_m)| \leqslant c. \tag{1.29}$$

The Lemma 3 implies the following relations

$$|\overline{\varphi}(v) - \varphi(a_1b_1a_2b_2\cdots a_mb_m)| \leqslant 20c, \tag{1.30}$$

$$|\overline{\varphi}(a_{k+1}b_{k+1}\cdots a_mb_ma) - \varphi(a_{k+1}b_{k+1}\cdots a_mb_m)| \leqslant 20c. \tag{1.31}$$

Now from (1.29),(1.30),(1.31) we obtain

$$|\varphi(v) - \varphi(v_1) - \varphi(v_2)| \leqslant 41c.$$

Consider the case b). We have

$$v_1 = a_1b_1a_2b_2\cdots a_kb_ka_{k+1}, \quad v_2 = b_{k+1}\cdots a_mb_ma.$$

From Lemma 3 we get

$$|\overline{\varphi}(v) - \varphi(a_1b_1a_2b_2\cdots a_mb_m)| \leqslant 20c \tag{1.32}$$

By (1.32) and inequality

$$|\varphi(a_1b_1a_2b_2\cdots a_mb_m)-\varphi(a_1b_1a_2b_2\cdots a_kb_k)-\varphi(a_{k+1}b_{k+1}\cdots a_mb_m)|\leqslant c$$

we have

$$|\overline{\varphi}(v)-\varphi(a_1b_1a_2b_2\cdots a_kb_k)-\varphi(a_{k+1}b_{k+1}\cdots a_mb_m)|\leqslant 21c. \tag{1.33}$$

Lemma 3 implies that

$$|\overline{\varphi}(v_1)-\varphi(a_1b_1a_2b_2\cdots a_kb_k)|\leqslant 20c. \tag{1.34}$$

Furthermore, it is clear that

$$|\varphi(v_2)-\varphi(ab_{k+1})-\varphi(a_{k+2}b_{k+2}\cdots a_mb_m)|\leqslant c,$$

hence

$$|\varphi(v_2)-\varphi(a_{k+2}b_{k+2}\cdots a_mb_m)|\leqslant 2c$$

and

$$\varphi(v_2)-\varphi(a_{k+1}b_{k+1}\cdots a_mb_m)|\leqslant 4c. \tag{1.35}$$

From (1.33),(1.34),(1.35) it follows inequality

$$|\overline{\varphi}(v)-\overline{\varphi}(v_1)-\overline{\varphi}(v_2)|\leqslant 45c.$$

II. Let $c_1,\ c_n\in B$. Suppose that $v=c_1c_2\cdots c_kc_{k+1}\cdots c_n$ is canonical form and $v_1=c_1c_2\cdots c_k,\ v_2=c_{k+1}\cdots c_n$.

Consider two cases : a) $c_k\in A$ and b) $c_k\in B$.

In the case a) for some positive integers m and l we get

$$v_1=ba_1b_1a_2b_2\cdots a_mb_ma_{m+1},\quad v_2=b_{m+1}a_{m+2}\cdots b_{l-1}a_lb_l,\quad b_l\in B.$$

From definition of $\overline{\varphi}$ we have

$$\overline{\varphi}(v_1)=\varphi(a_1b_1a_2b_2\cdots a_mb_ma_{m+1}b).$$

Therefore

$$|\overline{\varphi}(v_1)-\varphi(a_1b_1a_2b_2\cdots a_mb_m)|\leqslant 2c. \tag{1.36}$$

By Lemma 3

$$|\overline{\varphi}(v_2)-\varphi(a_{m+2}b_{m+2}\cdots a_{l-1}b_{l-1}a_lb_l)|\leqslant 20c.$$

Therefore,

$$|\overline{\varphi}(v_2)-\varphi(a_{m+2}b_{m+2}\cdots a_{l-1}b_{l-1})|\leqslant 22c. \tag{1.37}$$

Since

$$v=ba_1b_1a_2b_2\cdots a_mb_ma_{m+1}b_{m+1}a_{m+2}\cdots b_{l-1}a_lb_l,$$

Lemma 3 implies that

$$|\overline{\varphi}(v)-\varphi(a_1b_1a_2b_2\cdots a_lb_l)|\leqslant 20c. \tag{1.38}$$

Since $\varphi\in PX(D,-1)$, we obtain

$$|\varphi(a_1b_1a_2b_2\cdots a_lb_l)-\varphi(a_1b_1a_2b_2\cdots a_mb_m)-$$

$$\varphi(a_{m+1}b_{m+1}a_{m+2}\cdots b_{l-1}a_lb_l)| \leqslant c \tag{1.39}$$

and

$$|\varphi(a_{m+1}b_{m+1}a_{m+2}\cdots b_{l-1}a_lb_l) - \varphi(a_{m+2}b_{m+2}a_{m+2}\cdots b_{l-1}a_lb_l)| \leqslant 2c. \tag{1.40}$$

From (1.39),(1.40) we have

$$|\varphi(a_1b_1a_2b_2\cdots a_lb_l) - \varphi(a_1b_1a_2b_2\cdots a_mb_m) - \varphi(a_{m+2}b_{m+2}\cdots b_{l-1}a_lb_l)| \leqslant 3c.$$

From this inequality by (1.36), (1.37) and (1.38) we get

$$|\overline{\varphi}(v) - \overline{\varphi}(v_1) - \overline{\varphi}(v_2)| \leqslant 47c.$$

The case b) $c_k \in B$.

In this case for some positive integers m and l we have the following relations $v_1 = ba_1b_1a_2b_2\cdots a_mb_m$ and $v_2 = a_{m+1}b_{m+1}\cdots b_{l-1}a_lb_l$.

By Lemma 3

$$|\overline{\varphi}(v) - \varphi(a_1b_1a_2b_2\cdots a_lb_l)| \leqslant 20c, \tag{1.41}$$

$$|\overline{\varphi}(v_1) - \varphi(a_1b_1a_2b_2\cdots a_mb_m)| \leqslant 20c. \tag{1.42}$$

Now from (1.39),(1.41),(1.42) we obtain

$$|\overline{\varphi}(v) - \overline{\varphi}(v_1) - \overline{\varphi}(v_2)| \leqslant 41c.$$

The Lemma is proved.

THEOREM 1. *Let $\varphi \in PX(D,-1)$ and $c > 0$ such that $|\varphi(xy) - \varphi(x) - \varphi(y)| \leqslant c \quad \forall x, y \in D$. Then the function $\overline{\varphi}$ is a pseudocharacter of group G such that for any u, v from G the following inequality*

$$|\overline{\varphi}(uv) - \overline{\varphi}(u) - \overline{\varphi}(v)| \leqslant 261c.$$

holds.

Proof. The canonical forms of elements u and v are $u = u_1\gamma_1t$ and $v = t^{-1}\gamma_2v_1$ respectively, where γ_1 and γ_2 belong to the same of free factor and $\gamma_1\,\gamma_2 \neq 1$. The cases when among elements $t, \gamma_1, \gamma_2, u_1, v_1$ there are equal to 1 are possible.

It is clear that $uv = u_1(\gamma_1\gamma_2)v_1$ is canonical form of element uv in free product $G = A * B$.

Hence,

$$|\overline{\varphi}(uv) - \overline{\varphi}(u_1) - \overline{\varphi}(\gamma_1\gamma_2v_1)| \leqslant 47c, \quad \text{by Lemma 5}$$

and

$$|\overline{\varphi}(\gamma_1\gamma_2v_1) - \overline{\varphi}(\gamma_1\gamma_2) - \overline{\varphi}(v_1)| \leqslant 40c, \quad \text{by Lemma 4}$$

Therefore,

$$|\overline{\varphi}(uv) - \overline{\varphi}(u_1) - \overline{\varphi}(v_1)| \leqslant 87c. \tag{1.43}$$

Furthermore,

$$|\overline{\varphi}(u_1\gamma_1t) - \overline{\varphi}(u_1) - \overline{\varphi}(\gamma_1t)| \leqslant 45c, \quad \text{by Lemma 5;}$$

and
$$|\overline{\varphi}(u_1\gamma_1 t) - \overline{\varphi}(\gamma_1) - \overline{\varphi}(t)| \leqslant 40c, \quad \text{by Lemma 4}$$
Hence,
$$|\overline{\varphi}(u_1\gamma_1 t) - \overline{\varphi}(u_1) - \overline{\varphi}(t)| \leqslant 87c. \tag{1.44}$$
Similarly,
$$|\overline{\varphi}(t^{-1}\gamma_2 v_1) - \overline{\varphi}(t^{-1}) - \overline{\varphi}(v_1)| \leqslant 87c. \tag{1.45}$$
Now from (1.43), (1.44), (1.45) and Lemma 1 we obtain
$$\begin{aligned}
&|\overline{\varphi}(uv) - \overline{\varphi}(u) - \overline{\varphi}(v)| \\
&\quad = |\overline{\varphi}(uv) - \overline{\varphi}(u_1) - \overline{\varphi}(v_1) + \overline{\varphi}(u_1) + \overline{\varphi}(v_1) - \overline{\varphi}(u) - \overline{\varphi}(v)| \\
&\quad \leqslant |\overline{\varphi}(uv) - \overline{\varphi}(u_1) - \overline{\varphi}(v_1)| + |\varphi(u_1) + \overline{\varphi}(t) - \overline{\varphi}(u)| + |\varphi(v_1) \\
&\qquad\qquad + \overline{\varphi}(t^{-1}) - \overline{\varphi}(v)| \\
&\quad \leqslant 3 \cdot 87c = 261c.
\end{aligned}$$

The Theorem is proved.

THEOREM 2. 1) *The mapping* $\lambda : \varphi \to \overline{\varphi}$ *is an embedding* $PX(D,-1)$ *into* $PX(G)$;

2) $PX(G) = PX(A) \dotplus PX(B) \dotplus PX(D,-1)$

Proof. It is obvious that that the map $\lambda : \varphi \to \overline{\varphi}$ is linear. Let us check that $\ker\lambda = 0$. Indeed, if $\overline{\varphi} \equiv 0$ on group G, then $\varphi = \overline{\varphi}\big|_D \equiv 0$. Now we claim that subspace of $PX(G)$ generated by $PX(A), PX(B)$, and $PX(D,-1)$ is their direct sum.

From definition of the mappings τ_A^* and τ_B^*, we have that the subspace of $PX(G)$ generated by $PX(A)$ and $PX(B)$ is their direct sum.

Indeed, if $\varphi \in PX(A) \bigcap PX(B)$, then $\varphi\big|_{A\cup B} \equiv 0$, on the other hand, if F is cartesian subgroup of $G = A * B$, then $\varphi\big|_F \equiv 0$.

Each element $g \in G$ is uniquely representable in the form
$$g = abv, \quad a \in A, b \in B, v \in F.$$

Hence,if $c > 0$ such that for any $x, y \in G$ the following inequality holds $|\varphi(xy) - \varphi(x) - \varphi(y)| \leqslant c$, then $|\varphi(g) - \varphi(a) - \varphi(b) - \varphi(v)| \leqslant 2c$ and $|\varphi(g)| \leqslant 2c \quad \forall g \in G$. Furthemore, $\varphi \equiv 0$. Now let us show that $(PX(A) \dotplus PX(B)) \bigcap PX(D,-1) = 0$. Indeed, if $\psi \in (PX(A) \dotplus PX(B)) \bigcap PX(D,-1)$, then from the relation $\psi \in PX(D,-1)$ we obtain $\psi\big|_{A\cup B} \equiv 0$, and from the relation $\psi \in PX(A) \dotplus PX(B)$ we get $\psi\big|_F \equiv 0$. Hence, as above we have the inequality $|\varphi(g)| \leqslant 2c \quad \forall g \in G$. The latter implies that $\psi \equiv 0$. Thus, the subspace of $PX(G)$, generated by $PX(A), PX(B), PX(D,-1)$ is their direct sum. We claim that $PX(G) = PX(A) \dotplus PX(B) \dotplus PX(D,-1)$. Indeed, suppose that $f \in PX(G)$, $f\big|_A = \varphi_1$, $f\big|_B = \psi_1$. Then $\varphi = \tau_A^* \circ \varphi_1 \in PX(G)$, $\psi = \tau_B^* \circ \varphi_2 \in PX(G)$.

And we have $\gamma = (f - \varphi - \psi)\big|_{A\cup B} \equiv 0$. Let us check that $\gamma \in PX(D,-1)$. Let $\eta = \gamma\big|_D$ be pseudocharacter from $PX(D,-1)$ and let $\overline{\eta}$ be its λ -image. Then we have $(\gamma - \overline{\eta})\big|_{A\cup B\cup D} \equiv 0$. Hence, $\gamma - \overline{\eta} \equiv 0$ and $\gamma = \overline{\eta}$, $f = \varphi + \psi + \gamma$.

The Theorem is proved.

§2 Description of the space $PX(D,-1)$

Let A and B be a groups of order two generated by a and b respectively, then $M=\{a\cdot b\}$ and any pseudocharacter of D is a real additive character of D. Obviously, in this case for any $v\in D$ we have $\overline{v}=v^{-1}$. This implies that if $\varphi\in PX(D,-1)$, then $\varphi\equiv 0$.

Hence, we may assume that either the order of the group A or the order of the group B is more than two.

Let D^* be a free subsemigroup of the group G generated by the set $M^*=\{ba\,|\,b\in B_0, a\in A_0\}$.

For any word v in alphabet M we introduce the set of "beginnings" $H(v)$ and the set of "ends" $K(v)$ as follows. $H(v)=K(v)=\emptyset$, if $|v|\leqslant 1$ and

$$H(v)=\{x_{i_1},x_{i_1}x_{i_2},\ldots,x_{i_1}x_{i_2}\ldots x_{i_{n-1}}\},$$
$$K(v)=\{x_{i_2}\ldots x_{i_n},x_{i_3}\ldots x_{i_n},\ldots,x_{i_{n-1}}x_{i_n},x_{i_n}\},$$

if $v=x_{i_1}\cdot\ldots\cdot x_{i_n}, n>1$.

Denote by $\sim_D$ and $\sim_{D^*}$ respectively the restrictions of the relation $\sim$ to D and D^* respectively.

It is evident that $\sim_D$ and $\sim_{D^*}$ are an equivalence relations.

For any element w from D such that $H(w)\cap K(w)=\emptyset$ in [10] the functions $\eta_w(v)$ and $e_w(v)$ were defined as follows. If $v\in D$, then $\eta_w(v)$ is equal to the number of occurrences of w in the word v;

$$e_w(v)=\max\{\eta_w(v')\,|\,v'\sim_D v\}.$$

An element v from the free semigroup D is called *simple* if it is not a nontrivial power of another element $u\in D$.

The set of simple elements of semigroup D will be denoted by $\mathcal{P}$. Obviously, that if $u\sim_D v$, then $u\in\mathcal{P}$ if and only if $v\in\mathcal{P}$.

By Lemma 8 from [10] we have that in any class of $\sim_D$ conjugate elements belonging to the set $\mathcal{P}$ there is representative w satisfies to the condition

$$H(w)\cap K(w)=\emptyset. \tag{2.1}$$

Let us fix some of a system P of representatives w of classes $\sim_D$ conjugate elements belonging to the set $\mathcal{P}$ such that the relation (2.1) holds.

It is clear that if w is a word in alphabet M such that $H(w)\cap K(w)=\emptyset$, then the word w^{-1} in alphabet M^* satisfies the condition $H(w^{-1})\cap K(w^{-1})=\emptyset$. By Lemma 13 from [10] we have that for any $w\in P$ the function e_w is the pseudocharacter of semigroup D such that for any u,v from D the relation

$$|e_w(uv)-e_w(u)-e_w(v)|\leqslant 2$$

holds. A similar pseudocharacter of semigroup group D^* which corresponds to the word w^{-1} denote by $e_{w^{-1}}$. Denote by P_0 a subset of P consisting of elements w such that $w\sim w^{-1}$ in the group G. Let $Q=P\setminus P_0$. Let us check that $Q\neq\emptyset$. Let $a\in A_0$; $b_1,b_2\in B_0$ and $b_1\neq b_2$.

Then the criterion of conjugacy in the free product of groups (see [29]) implies that if $a^2 \neq 1$, then element $w = ab_1$ is not conjugate to $w^{-1} = b_1^{-1}a^{-1}$.

Now assume that the group A and B satisfy to the identity relation $x^2 = 1$.

Since G is not infinite dihedral group we may assume that there are elements $b_1, b_2, b_3 \in B$ such that $b_i \neq b_j$, if $i \neq j$. Consider a words of the form $w_{i,j,k} = (ab_1)^i(ab_2)^j(ab_3)^k$. Then $w_{i,j,k}^{-1} = (b_3a)^k(b_2a)^j(b_1a)^i \sim (ab_3)^k(ab_2)^j(ab_1)^i$. The criterion of conjugacy in the free product of groups implies that for any positive integers i,j,k the elements $w_{i,j,k}$ and $w_{i,j,k}^{-1}$ are not conjugate.

Note that $H(w_{i,j,k}) \cap K(w_{i,j,k}) = \emptyset$.

Let $w \in Q$; then from definition of the set Q we obtain $w \not\sim w^{-1}$ in group G. Therefore, there exists $w_1 \in Q$ such that $w^{-1} \sim w_1$.

Obviously, $w_1^{-1} \sim w$. Define a relation $\sim_1$ on the set Q as follows. Set $w_1 \sim_1 w_2$ if and only if either $w_1 = w_2$ or $w_1^{-1} \sim w_2$.

It is clear that $\sim_1$ is the relation of equivalence such that there are only two elements in each class of $\sim_1$ equivalency.

Let us choose in each of these classes a representative. Denote by Q^+ the set of these representatives.

By Q_n^+ denote subset of Q^+ consisting of element of length n in alphabet M.

LEMMA 6. *Let $\varphi \in PX(D,-1)$. Then φ by its restriction on the set Q^+ completely defined.*

Proof. Suppose that there is element ψ in $PX(D,-1)$ such that $\psi\big|_{Q^+} \equiv \varphi\big|_{Q^+}$; then $f = (\varphi - \psi)\big|_{Q^+} \equiv 0$.

By Lemma 1 φ uniquely defined by its restriction on P. Let $v \in P$. If $v \sim v^{-1}$, then $f(v) = f(v^{-1}) = -f(v)$. Hence, $f(v) = 0$.

If $v \not\sim v^{-1}$, then there is $w \in Q^+$ such that either v is conjugate to w or v is conjugate to element w^{-1}.

Therefore, $f(v) = 0$. Then $f \equiv 0$ on D and $\varphi \equiv \psi$.

The Lemma is proved.

Let $\overline{H}(w) = H(w) \cup \{w\}$, $\overline{K}(w) = K(w) \cup \{w\}$. Define measures $\mu_{u,v}, \mu_{a,b,c}$ $u,v,a,b,c \in D$ on Q^+ as follows.

We set $\mu_{u,v}(w) = 1$ if there exist x and y such that $x \in \overline{K}(w)$, $y \in \overline{H}(w)$, and $w = xy$; otherwise we set $\mu_{u,v}(w) = 0$.

We set $\mu_{a,b,c}(w) = 1$ if there exist x,y such that $x \in \overline{K}(u)$, $y \in \overline{H}(w)$, and $w = xby$; otherwise we set $\mu_{a,b,c}(w) = 0$.

Similarly the measures $\mu_{u,v}$, $\mu_{a,b,c}(w)$ $u,v,a,b,c \in D^*$ on Q^{-1} are defined.

Set

$$\nu_{u,v}(w) = \mu_{u,v}(w) + \mu_{v,u}(w) + \mu_{u,v,u}(w) + \mu_{v,u,v}(w) - \mu_{u,u}(w) - \mu_{v,v}(w).$$

It is easy to prove that

$$\nu_{u,v}(w) = \mu_{u,v}(w) + \mu_{uv,uv}(w) - \mu_{u,u}(w) - \mu_{v,v}(w).$$

Hence, the measure $\nu_{u,v}$ take values from the set $\{-2,-1,0,1,2\}$.

Let $v = c_1c_2\cdots c_kc_{k+1}$ be canonical form.

We set $\tilde{v} = 1$,if $k = 1$ and $\tilde{v} = c_2\cdots c_k$, if $k > 1$.

For $u = a_1b_1a_2b_2\cdots a_nb_n \in D\setminus\{1\}$ we have $\dot{u} = a_1$, $\ddot{u} = b_n$; and $\tilde{u} = 1$, if $n = 1$; $\tilde{u} = b_1a_2b_2\cdots a_n$, if $n > 1$. Therefore,

$$\overline{u} = \tilde{u}\ddot{u}\dot{u}, \quad \overline{v} = \tilde{v}\ddot{v}\dot{v}, \quad \overline{uv} = \tilde{u}\ddot{u}\dot{v}\tilde{v}\ddot{v}\dot{u}. \tag{2.2}$$

For any $t \in M^*$ and any $w \in Q^+$ we set $\delta_t(w^{-1}) = 1$, if $t = w^{-1}$; and $\delta_t(w) = 0$, if $t \neq w^{-1}$. Now for u, v from $D\setminus\{1\}$ define measure $\zeta_{u,v}(w)$ on Q^+ by setting

$$\begin{aligned}\zeta_{u,v}(w) &= \delta_{\ddot{u}\dot{v}}(w) + \delta_{\ddot{v}\dot{u}}(w) - \delta_{\ddot{u}\dot{u}}(w) - \delta_{\ddot{v}\dot{v}}(w)\\ &+\nu_{\tilde{u}\ddot{u}\dot{v},\tilde{v}\ddot{v}\dot{u}}(w^{-1}) + \nu_{\tilde{u},\ddot{u}\dot{v}}(w^{-1}) + \nu_{\tilde{v},\ddot{v}\dot{u}}(w^{-1})\\ &-\nu_{\tilde{u},\ddot{u}\dot{u}}(w^{-1}) - \nu_{\tilde{v},\ddot{v}\dot{v}}(w^{-1}).\end{aligned} \tag{2.3}$$

By Lemma 12 from [10] we have

$$e_w(uv) - e_w(u) - e_w(v) = \nu_{u,v}(w). \tag{2.4}$$

Hence, for any $a, b, c, d \in D$ we obtain

$$\begin{aligned}e_w(abcd) - e_w(ab) - e_w(cd) &= \nu_{ab,cd}(w),\\ e_w(ab) - e_w(a) - e_w(b) &= \nu_{a,b}(w),\\ e_w(cd) - e_w(c) - e_w(d) &= \nu_{c,d}(w).\end{aligned}$$

These equalities imply

$$e_w(abcd) = e_w(a) + e_w(b) + e_w(c) + e_w(d) + \nu_{ab,cd}(w) + \nu_{a,b}(w) + \nu_{c,d}(w). \tag{2.5}$$

LEMMA 7. *For any $w \in Q^+$ and any $u, v \in D$ the following equality holds*

$$e_{w^{-1}}(\overline{uv}) - e_{w^{-1}}(\overline{u}) - e_{w^{-1}}(\overline{v}) = \zeta_{u,v}(w).$$

Proof. Taking into account (2.2), (2.4), (2.5), we obtain

$$\begin{aligned}e_{w^{-1}}(\overline{uv}) - e_{w^{-1}}(\overline{u}) - e_{w^{-1}}(\overline{v}) &= e_{w^{-1}}(\tilde{u}\ddot{u}\dot{v}\tilde{v}\ddot{v}\dot{u}) - e_{w^{-1}}(\tilde{u}\ddot{u}\dot{u}) - e_{w^{-1}}(\tilde{v}\ddot{v}\dot{v})\\ &= e_{w^{-1}}(\tilde{u}) + e_{w^{-1}}(\ddot{u}\dot{v}) + e_{w^{-1}}(\tilde{v}) + e_{w^{-1}}(\ddot{v}\dot{u})\\ &\qquad + \nu_{\tilde{u}\ddot{u}\dot{v},\tilde{v}\ddot{v}\dot{u}}(w^{-1}) + \nu_{\tilde{u},\ddot{u}\dot{v}}(w^{-1}) + \nu_{\tilde{v},\ddot{v}\dot{u}}(w^{-1})\\ &- e_{w^{-1}}(\tilde{u}) - e_{w^{-1}}(\ddot{u}\dot{u}) - \nu_{\tilde{u},\ddot{u}\dot{u}}(w^{-1}) - e_{w^{-1}}(\ddot{v}\dot{u}) - e_{w^{-1}}(\tilde{v}) - \nu_{\tilde{v},\ddot{v}\dot{v}}(w^{-1})\\ &= e_{w^{-1}}(\ddot{u}\dot{v}) + e_{w^{-1}}(\ddot{v}\dot{u}) - e_{w^{-1}}(\ddot{u}\dot{u}) - e_{w^{-1}}(\ddot{v}\dot{v})\\ &\qquad + \nu_{\tilde{u}\ddot{u}\dot{v},\tilde{v}\ddot{v}\dot{u}}(w^{-1}) + \nu_{\tilde{u},\ddot{u}\dot{v}}(w^{-1}) + \nu_{\tilde{v},\ddot{v}\dot{u}}(w^{-1}) - \nu_{\tilde{u},\ddot{u}\dot{u}}(w^{-1}) - \nu_{\tilde{v},\ddot{v}\dot{v}}(w^{-1})\\ &= \delta_{\ddot{u}\dot{v}}(w) + \delta_{\ddot{v}\dot{u}}(w) - \delta_{\ddot{u}\dot{u}}(w) - \delta_{\ddot{v}\dot{v}}(w) + \nu_{\tilde{u}\ddot{u}\dot{v},\tilde{v}\ddot{v}\dot{u}}(w^{-1})\\ &\qquad + \nu_{\tilde{u},\ddot{u}\dot{v}}(w^{-1}) + \nu_{\tilde{v},\ddot{v}\dot{u}}(w^{-1}) - \nu_{\tilde{u},\ddot{u}\dot{u}}(w^{-1}) - \nu_{\tilde{v},\ddot{v}\dot{v}}(w^{-1}) = \zeta_{u,v}(w).\end{aligned}$$

The Lemma is proved.

If $w \in M$, it is clear that

$$\nu_{\tilde{u}\ddot{u}\dot{v},\tilde{v}\ddot{v}\dot{u}}(w^{-1}) = \nu_{\tilde{u},\ddot{u}\dot{v}}(w^{-1}) = \nu_{\tilde{v},\ddot{v}\dot{u}}(w^{-1}) = \nu_{\tilde{u},\ddot{u}\dot{u}}(w^{-1}) = \nu_{\tilde{v},\ddot{v}\dot{v}}(w^{-1}) = 0.$$

Hence, we obtain

$$\zeta_{u,v}(w) = \delta_{\ddot{u}\dot{v}}(w) + \delta_{\ddot{v}\dot{u}}(w) - \delta_{\ddot{u}\dot{u}}(w) - \delta_{\ddot{v}\dot{v}}(w) \quad \forall w \in M. \tag{2.6}$$

If $w \notin M$, then we have

$$\delta_{\ddot{u}\dot{v}}(w) = \delta_{\ddot{v}\dot{u}}(w) = \delta_{\ddot{u}\dot{u}}(w) = \delta_{\ddot{v}\dot{v}}(w) = 0.$$

Therefore, from (2.3) we obtain

$$\begin{aligned} \zeta_{u,v}(w) = \nu_{\tilde{u}\ddot{u}\dot{v},\tilde{v}\ddot{v}\dot{v}}(w^{-1}) + \nu_{\tilde{u},\ddot{u}\dot{v}}(w^{-1}) \\ + \nu_{\tilde{v},\ddot{v}\dot{u}}(w^{-1}) - \nu_{\tilde{u},\ddot{u}\dot{u}}(w^{-1}) - \nu_{\tilde{v},\ddot{v}\dot{v}}(w^{-1}). \end{aligned} \tag{2.7}$$

For any $w \in Q^+$ and any $u, v \in D$ define $\Theta_{u,v}(w)$ by setting

$$\Theta_{u,v}(w) = \nu_{u,v}(w) - \zeta_{u,v}(w). \tag{2.8}$$

Note that if either $\dot{u} = \dot{v}$ or $\ddot{u} = \ddot{v}$,then $\Theta_{u,v}(w) = 0 \quad \forall w \in M.$

Indeed, for example if $\dot{u} = \dot{v}$, then we obtain $\ddot{u}\dot{v} = \ddot{u}\dot{u}, \quad \ddot{v}\dot{u} = \ddot{v}\dot{v}$. Hence, (2.8) implies $\Theta_{u,v}(w) = 0$. Thus,if $\Theta_{u,v}(w) \neq 0$, then $\dot{u} \neq \dot{v}, \ddot{u} \neq \ddot{v}$. From the latter we obtain that the words $\ddot{u}\dot{v}, \ddot{v}\dot{u}, \ddot{u}\dot{u}, \ddot{v}\dot{v}$ are pare wise different. Hence, the measure $\Theta_{u,v}$ on Q_1^+ take values from the set $\{ -1, 0, 1 \}$.

LEMMA 8. *Suppose that* $w \in Q^+$ *and* $H(w) \cap K(w) = \emptyset$. *Then a function*

$$v \to \pi_w(v) = e_w(v) - e_{w^{-1}}(\overline{v})$$

is an element from $PX(D, -1)$.

Proof. Obviously, for any $v \in D$ we have

$$e_w(v) = e_{w^{-1}}(v^{-1}). \tag{2.9}$$

It easy to see that for any $v \in D$ the following relations hold:

$$(\overline{v^{-1}})^{-1} \sim \overline{v} \quad \text{and} \quad \overline{v^{-1}} \sim (\overline{v})^{-1}. \tag{2.10}$$

Hence, (2.9) implies that for any $v \in D$ the equality $e_w(\overline{v^{-1}}) = e_{w^{-1}}(\overline{v^{-1}})^{-1}$ holds. From (2.10) we obtain the following relations

$$e_w(\overline{v^{-1}}) = e_{w^{-1}}(\overline{v}), \tag{2.11}$$

$$e_{w^{-1}}(\overline{\overline{v^{-1}}}) = e_w(v). \tag{2.12}$$

From (2.11) and (2.12) we get

$$\pi_w(\overline{v^{-1}}) = e_w(\overline{v^{-1}}) - e_{w^{-1}}(\overline{\overline{v^{-1}}}) = e_w(\overline{v^{-1}}) - e_{w^{-1}}(v) = \pi_w(v),$$

and

$$\pi_w(\overline{v^{-1}}) = -\pi_w(v).$$

Let us check that for any $n \in N$ and any $v \in D$ the relation

$$\pi_w(v^n) = n\pi_w(v)$$

holds. Indeed, it easy to see that $\overline{v^n} \sim (\overline{v})^n$, hence we obtain

$$\pi_w(v^n) = e_w(v^n) - e_{w^{-1}}(\overline{v^n}) = ne_w(v) - e_{w^{-1}}((\overline{v})^n) = ne_w(v) - ne_{w^{-1}}(\overline{v}) = n\pi_w(v).$$

Suppose that $u = a_1b_1a_2b_2\cdots a_nb_n$, $v = \alpha_1\beta_1\alpha_2\beta_2\cdots\alpha_k\beta_k$ are an arbitrary elements from $D\setminus\{1\}$.

Let us check that $|\pi_w(uv) - \pi_w(u) - \pi_w(v)| \leqslant 12$. In fact, for any $w \in Q$ we have $|\nu_{u,v}(w)| \leqslant 2$. Hence, from (2.7) and (2.8) we obtain the following estimations $|\zeta_{u,v}(w)| \leqslant 10$ and

$$|\Theta_{u,v}(w)| \leqslant 12. \tag{2.13}$$

Furthermore, we get

$$\begin{aligned}|\pi_w(uv) - \pi_w(u) - \pi_w(v)| &= |e_w(uv) - e_w(u) - e_w(v) - e_w^{-1}(\overline{uv}) + e_w^{-1}(\overline{u}) + e_w^{-1}(\overline{v})|\\ &= |\nu_{u,v}(w) - \zeta_{u,v}(w)| \leqslant |\nu_{u,v}(w)| + |\zeta_{u,v}(w)| \leqslant 12.\end{aligned}$$

The Lemma is proved.

LEMMA 9. *Let $w \in P$. Then we have:*
1) $e_w(u) = 0$, for any $u \in D$ such that $|u| < |w|$;
2) $e_w(u) = 0$, if $|u| = |w|$ and $u \not\sim w$;
3) $e_w(u) = 1$, if $u \sim w$.

Proof. See [10, Lemma 13].

COROLLARY A. *For each $w \in Q^+$, and each $u \in D$ we have*
1) $\pi_w(u) = 0$, if $|u| < |w|$;
2) $\pi_w(u) = 0$, if $|u| = |w|$ but $u \not\sim w$, $u \not\sim w^{-1}$;
3) $\pi_w(u) = \varepsilon$, if $u \sim w^\varepsilon$, $\varepsilon \in \{+1,-1\}$.

Proof. 1) Obviously. 2). From Lemma 9 if follows $e_w(u) = 0$, $e_{w^{-1}}(\overline{u}) = 0$. Hence, $\pi_w(u) = 0$.

3). Let $u \sim w$. Since $w \in Q^+$, we have $u \not\sim w^{-1}$. Therefore, $e_{w^{-1}}(\overline{u}) = 0$ and $\pi_w(u) = 1$. Similarly, if $u \sim w^{-1}$, then $e_w(u) = 0$, $e_w(u^{-1}) = 1$. Therefore, $\pi_w(u) = e_{w^{-1}}(\overline{u}) = -1$.

The corollary is proved.

From the definition of the measures $\nu_{u,v}$ it follows that for each elements u and v from D there are at most $4(n-1)$ words in P_n such that $\nu_{u,v}(w) \neq 0$. Hence, we have

$$|P_n \cap supp\ \Theta_{u,v}| \leqslant 24(n-1). \tag{2.14}$$

LEMMA 10. *Suppose that $n \in \mathbb{N}$ and λ is a bounded function on Q_n^+. Then the function*

$$\psi_\lambda = \sum_{w \in Q_n^+} \lambda(w)\pi_w \qquad (*)$$

belongs to the space $BPX(D,-1)$.
And for any $w_0 \in Q_n^+$ the following equality $\psi_\lambda(w_0) = \lambda(w_0)$ holds.

Proof. Let $\lambda_0 = \sup\{\lambda(w)|w \in Q_n^+\}$. First consider the case $n = 1$.
It is clear that $e_w(uv) - e_w(u) - e_w(v) = 0$ and

$$\pi_w(uv) - \pi_w(u) - \pi_w(v) = -[\delta_{\ddot{u}\dot{v}}(w) + \delta_{\ddot{v}\dot{u}}(w) - \delta_{\ddot{u}\dot{u}}(w) - \delta_{\ddot{v}\dot{v}}(w)].$$

Hence, we get

$$|\psi(uv) - \psi(u) - \psi(v)| = |\sum_{w \in Q_1^+} \lambda(w)[\pi_w(uv) - \pi_w(u) - \pi_w(v)]|$$
$$= |\sum_{w \in Q_1^+} \lambda(w)[\delta_{\ddot{u}\dot{v}}(w) + \delta_{\ddot{v}\dot{u}}(w) - \delta_{\ddot{u}\dot{u}}(w) - \delta_{\ddot{v}\dot{v}}(w)]|.$$

It is easy to see that for any pair u, v there are at most four elements $w \in M$ such that

$$\delta_{\ddot{u}\dot{v}}(w) + \delta_{\ddot{v}\dot{u}}(w) - \delta_{\ddot{u}\dot{u}}(w) - \delta_{\ddot{v}\dot{v}}(w) \neq 0.$$

Hence, taking into account the Remark 3, we obtain

$$|\psi(uv) - \psi(u) - \psi(v)| \leqslant |\sum_{w \in Q_1^+} \lambda(w)| \leqslant 4 \cdot \lambda_0.$$

Now let $n > 1$. From (2.13), (2.14) we get

$$|\psi_\lambda(xy) - \psi_\lambda(x) - \psi_\lambda(y)| = |\sum_{w \in Q_n^+} \lambda(w)[\pi_w(xy) - \pi_w(x) - \pi_w(y)]|$$
$$= |\sum_{Q_n^+} \lambda(w)\Theta_{u,v}(w)| \leqslant \sum_{w \in Q_n^+} |\lambda(w)||\Theta_{u,v}(w)|$$
$$\leqslant \lambda_0 \cdot 12 \cdot 24(n-1) = 288\lambda_0 \cdot (n-1).$$

The Corollary A implies that for any $w_0 \in Q_n^+$ the equality $\psi_\lambda(w_0) = \lambda(w_0)$ holds.
The Lemma is proved.

Denote by $K(D)$ the set of functions φ on the semigroup D satisfying the following relations :
1) $\varphi(x^n) = n\varphi(x) \quad \forall n \in N, \quad \forall x \in D$;
2) $\varphi(xy) = \varphi(yx) \quad \forall x, y \in D$;
3) $\varphi((\overline{v})^{-1}) = -\varphi(v) \quad \forall v \in D$;
4) $\varphi\big|_{Q_i}$ is a bounded function $\forall i \in N$.

It is clear that $K(D)$ is a linear space (with respect to ordinary operations).

LEMMA 11. *Let $\varphi \in PX(D,-1)$; then φ is bounded on Q_n^+ $\quad \forall n \in N$.*

Proof. From the condition of the Lemma it follows that the function $\varphi\big|_M$ is bounded. Let $c > 0$ such that for any $x, y \in D$ the inequality $|\varphi(xy) - \varphi(x) - \varphi(y)| \leqslant c$ holds.

By induction on n we obtain $|\varphi(x_1 x_2 \ldots x_{n+1}) - \sum_{i=1}^{n} \varphi(x_i)| \leqslant n \cdot c$. The latter implies that the function $\varphi\big|_{Q_n^+}$ is bounded $\forall n \in N$.

The Lemma is proved.

From Lemma 11 it follows that $PX(D,-1)$ is subspace of $K(D)$.

Denote by $L(Q^+)$ the space of real-valued functions α on Q^+ satisfying the following condition: $\alpha\big|_{Q_n^+}$ is bounded for any $n \in N$.

Let us construct an isomorphism Δ between the spaces $K(D)$ and $L(Q^+)$.

Let $\varphi \in K(D)$. For each $i \in N$ we define the function $\alpha_i : Q_i^+ \to R$ by induction as follows : $\alpha_1 \equiv \varphi\big|_{Q_1^+}$, and if the values $\alpha_1, \ldots, \alpha_n$ have already been defined, then we set

$$\alpha_{n+1} = (\varphi - \sum_{i=1}^{n} \varphi_{\alpha_i})\big|_{Q_{n+1}^+}(w), \quad w \in Q_{n+1}^+. \tag{2.15}$$

Here φ_{α_i} are pseudocharacters introduced in the Lemma 10 by (*).

Now we define the function $\alpha = \Delta(\varphi)$ via its restriction to Q_i^+ by setting $\alpha\big|_{Q_i^+} = \alpha_i$.

Let us show that $\Delta(\varphi)$ belongs to $L(Q^+)$. Indeed, we have $\alpha\big|_{Q_1^+} = \alpha_1$. Furthermore, suppose that we have already established that the functions $\alpha\big|_{Q_1^+}, \ldots, \alpha\big|_{Q_n^+}$ are bounded. Let us prove that $\alpha\big|_{Q_{n+1}^+}$ is also bounded. Indeed, Lemma 11 implies that all pseudocharacters $\varphi_{\alpha_1}, \varphi_{\alpha_2}, \ldots, \varphi_{\alpha_n}$ are bounded functions on Q_{n+1}^+. Hence, $\Delta(\varphi) \in L(Q^+)$.

Now let us show that the mapping Δ is linear. First we note that if τ, τ_1, τ_2 are bounded functions on Q_{n+1}^+ and λ_1, λ_2 are reals such that $\tau = \lambda_1\tau_1 + \lambda_2\tau_2$, then we have

$$\varphi_\tau = \lambda_1\varphi_{\tau_1} + \lambda_2\varphi_{\tau_2}, \tag{2.16}$$

where $\varphi_\tau, \lambda_1\varphi_{\tau_1}, \lambda_2\varphi_{\tau_2}$ are the pseudocharacters defined in Lemma 10.

Suppose that the function $\varphi \in K(D)$ satisfies formula (2.15). Let $\lambda \in R$. Assume that we have already established that the restrictions of the function $\Delta(\varphi)$ to $Q_1^+, \ldots, Q_n^+$ are equal to $\lambda\alpha_1, \lambda\alpha_2, \ldots, \lambda\alpha_n$ respectively. Then from formulas (2.15) and (2.16) we obtain:

$$\Delta(\lambda\varphi)\big|_{Q_{n+1}^+} = \big(\lambda\varphi - \sum_{i=2}^{n} \varphi_{\lambda\alpha_i}\big)\big|_{Q_{n+1}^+} = \big(\lambda\varphi - \sum_{i=2}^{n} \lambda\varphi_{\alpha_i}\big)\big|_{Q_{n+1}^+} = \lambda\alpha_{n+1},$$

i.e., $\Delta(\lambda\varphi) = \lambda\Delta(\varphi)$.

Let $\psi \in K(D)$ and $\Delta(\psi)\big|_{Q_n^+} = \beta_n \in N$. Then it is clear that $(\varphi + \psi)\big|_{Q_1^+} \equiv 0, (\varphi + \psi)\big|_{Q_2^+} = \varphi\big|_{Q_2^+} + \psi\big|_{Q_2^+} = \alpha_2 + \beta_2$.

Suppose that we have already established that $(\varphi + \psi)\big|_{Q_i^+} = \varphi\big|_{Q_i^+} + \psi\big|_{Q_i^+}$ $i = 1, \dots, n$. Then for $n+1$ the formulas (2.15) and (2.16) imply $(\varphi+\psi)\big|_{Q_{n+1}^+} = \big((\varphi+\psi) - \sum_{i=2}^{n} \varphi_{(\alpha_i+\beta_i)}\big)\big|_{Q_{n+1}^+} = \big((\varphi+\psi) - \sum_{i=2}^{n}(\varphi_{\alpha_i} + \varphi_{\beta_i})\big)\big|_{Q_{n+1}^+} = \big(\varphi - \sum_{i=2}^{n} \varphi_{\alpha_i}\big)\big|_{Q_{n+1}^+} + \big(\psi - \sum_{i=2}^{n} \varphi_{\beta_i}\big)\big|_{Q_{n+1}^+} = \alpha_{n+1} + \beta_{n+1} = \varphi\big|_{Q_{n+1}^+} + \psi\big|_{Q_{n+1}^+}$.

Thus, the mapping Δ is linear.

Let us show that Δ maps $K(D)$ onto $L(Q^+)$. Indeed, let $\alpha \in L(Q^+)$ and $\alpha\big|_{Q_i^+} = \alpha_i$. Since for each $w \in Q^+$ there is only a finite set of nonzero numbers of the form $\varphi_{\alpha_n}(w)$, $\quad n \in N$, the function $\varphi = \sum_{i=2}^{\infty} \varphi_{\alpha_i}$ on D is well defined and belongs to the space $K(D)$. Let us show that $\Delta(\varphi) = \alpha$. We set $\Delta(\varphi)\big|_{Q_i^+} = \beta$; let us verify that $\beta_i = \alpha_i \quad \forall i \in N$. Since $\beta_1 = \varphi\big|_{Q_1^+}$, we have $\beta_1 = \alpha_1$. Suppose that we have already established the relations $\beta_i = \alpha_i$ for $i \leqslant n$; then for all $w \in Q_{n+1}^+$ we have

$$\beta_{n+1}(w) = \big(\varphi(w) - \sum_{i=2}^{n} \varphi_{\alpha_i}(w)\big) = \sum_{i=2}^{\infty} \varphi_{\alpha_i}(w) - \sum_{i=2}^{n} \varphi_{\alpha_i}(w) = \sum_{i=n+1}^{\infty} \varphi_{\alpha_i}(w)$$

by formula (2.15).

Furthermore,since $\varphi_{\alpha_i}\big|_{Q_{n+1}^+} \equiv 0, \quad \forall i > n+1$, we obtain

$$\sum_{i=n+1}^{\infty} \varphi_{\alpha_i}(w) = \varphi_{\alpha_{n+1}}(w) = \alpha_{n+1}(w),$$

where the latter equality follows from Lemma 10. Thus, $\beta_{n+1} = \alpha_{n+1}$, and Δ is onto.

Now let us verify that $\ker\Delta = 0$. Indeed,let $\varphi \in K(D)$ and $\Delta(\varphi) = 0$. This means $\alpha_i = \Delta(\varphi)_{Q_i^+} \equiv 0 \quad \forall i \in N$. Formula (2.15) implies $\varphi\big|_{Q_1^+} \equiv 0, \varphi\big|_{Q_2^+} \equiv 0$ and for any $w \in Q_{n+1}^+$ we get $\varphi(w) = \sum_{i=2}^{n} \varphi_{\alpha_i}(w) + \alpha_{n+1}(w) = \sum_{i=2}^{n+1} \varphi_{\alpha_i}(w)$. Since $\alpha_i \equiv 0 \quad \forall i \in N$,we have $\varphi_{\alpha_i}(w) \equiv 0$,therefore, the latter equality implies $\varphi\big|_{Q_{n+1}^+} \equiv 0$. Now from properties 1) and 2) of functions from the space $K(D)$ it follows that $\varphi\big|_{Q_1^+} \equiv 0$. Therefore, $\ker\Delta = 0$. Hence, as was shown above, for a function α from $L(Q^+)$ such that $\alpha\big|_{Q_i^+} = \alpha_i \quad \forall i \in N$, we have

$$\Delta^{-1}(\alpha) = \sum_{i=2}^{\infty} \varphi_{\alpha_i}(w). \tag{2.17}$$

Thus, Δ is an isomorphism between linear spaces $K(D)$ and $L(Q^+)$.

Furthermore, if $\Delta(\varphi) = \alpha$, then φ by formula (2.17) is defined. Denote by $L(Q^+, \Theta)$ a subspace of $L(Q^+)$, consisting of functions $\alpha \in L(Q^+)$ such that the quantities

$$\Big|\int_{Q^+} \alpha d\Theta_{u,v}\Big| \quad u, v \in D$$

are uniformly bounded.

THEOREM 3. 1) *The mapping* Δ *establish an isomorphism between linear spaces* $PX(D,-1)$ *and* $L(Q^+,\Theta)$.

2) *Each element* φ *from the space* $PX(D,-1)$ *is uniquely representable in the form*

$$\varphi = \sum_{w\in Q^+} \alpha(w)\pi_w, \quad \textit{where} \quad \alpha \in L(Q^+,\Theta)$$

Proof. Let us show that under the isomorphism $\Delta : K(D) \to L(Q^+)$ we assign to the element of $PX(D,-1)$ functions α from $L(Q^+)$ such that $\left|\int_{Q^+} \alpha d\Theta_{u,v}\right| \leqslant \varepsilon$ for some $\varepsilon > 0$ and any u and v from D.

Let $\varphi \in PX(D,-1)$. Let us choose $\varepsilon > 0$ such that $|\varphi(uv)-\varphi(u)-\varphi(v)| \leqslant \varepsilon$ for each u and v from D. Suppose that $\Delta(\varphi) = \alpha$, then

$$\begin{aligned}
\Big|\int_{Q^+} \alpha d\Theta_{u,v}\Big| &= \Big|\sum_{i=2}^{\infty}\int_{Q_i^+} \alpha d\Theta_{u,v}\Big| = \Big|\sum_{i=2}^{\infty}\int_{Q_i^+} \alpha_i d\Theta_{u,v}\Big| = \Big|\sum_{i=2}^{\infty}\sum_{w\in Q_i^+} \alpha_i(w) d\Theta_{u,v}(w)\Big| \\
&= \Big|\sum_{i=2}^{\infty}\sum_{w\in Q_i^+} \alpha_i(w)\big(\pi_w(uv)-\pi_w(u)-\pi_w(v)\big)\Big| \\
&= \Big|\sum_{i=2}^{\infty}\Big(\sum_{w\in Q_i^+} \alpha_i(w)\pi_w(uv) - \sum_{w\in Q_i^+} \alpha_i(w)\pi_w(u) - \sum_{w\in Q_i^+} \alpha_i(w)\pi_w(v)\Big)\Big| \\
&= (\text{by Lemma 10}) = \Big|\sum_{i=2}^{\infty}\big(\varphi_{\alpha_i}(uv)-\varphi_{\alpha_i}(u)-\varphi_{\alpha_i}(v)\big)\Big| \\
&= |\varphi(uv)-\varphi(u)-\varphi(v)| \leqslant \varepsilon.
\end{aligned}$$

Now let $\alpha \in L(Q^+)$ and $\left|\int_{Q^+} \alpha d\Theta_{u,v}\right| \leqslant \varepsilon$ for some $\varepsilon > 0$ and each u and v from D. If $\alpha_i = \alpha\big|_{Q_i^+}$ and $\varphi = \Delta^{-1}(\alpha_i)$, then $\varphi = \sum_{i=2}^{\infty} \varphi_{\alpha_i}$. Therefore, we have

$$\begin{aligned}
|\varphi(uv)-\varphi(u)-\varphi(v)| &= \Big|\sum_{i=2}^{\infty}\big(\varphi_{\alpha_i}(uv)-\varphi_{\alpha_i}(u)-\varphi_{\alpha_i}(v)\big)\Big| \\
&= \Big|\sum_{i=2}^{\infty}\Big(\sum_{w\in Q_i^+} \alpha_i(w)\pi_w(uv) - \sum_{w\in Q_i^+} \alpha_i(w)\pi_w(u) - \sum_{w\in Q_i^+} \alpha_i(w)\pi_w(v)\Big)\Big| \\
&= \Big|\sum_{i=2}^{\infty}\sum_{w\in Q_i^+} \alpha_i(w)\big(\pi_w(uv)-\pi_w(u)-\pi_w(v)\big)\Big| \\
&= \Big|\sum_{i=2}^{\infty}\sum_{w\in Q_i^+} \alpha_i(w) d\Theta_{u,v}(w)\Big| = \Big|\int_{Q^+} \alpha(w) d\Theta_{u,v}(w)\Big| \leqslant \varepsilon.
\end{aligned}$$

The Theorem is proved.

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(Received October 18, 1999)

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