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NEW CONVERGENCE RESULTS OF ITERATIVE METHODS FOR SET–VALUED MIXED VARIATIONAL INEQUALITIES

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Abstract. An iterative method for solving set-valued variational inequalities is considered and its convergence properties are studied under strong monotonicity and coercivity conditions. The results obtained in this paper include, as a special case, some known results in this field.

1. Introduction

In recent years, variational inequalities have been extended and generalized in several directions using new, innovative, and novel techniques to study a wide class of problems arising in pure and applied sciences. Inspired and motivated by the research going on this field, we propose an iterative method for solving a class of variational inequalities, which is called the generalized mixed variational inequality. We study the convergence analysis of this algorithm relying on an equivalent fixed point formulation of the given problem. We show a strong convergence result on strong monotonicity and Lipschitz continuity assumptions, as well as a weak convergence result on a weaker condition which is called " co-coercivity". The latter implies that the operator is Lipschitz continuous with respect to the Hausdorff excess which ensures, in particular, the boundedness of the operator on bounded sets.

To begin with, let X be a real Hilbert space, and let the associated inner product and norm be denoted by $\langle \cdot, \cdot \rangle$ and $|\cdot|$ respectively. Troughout, we use the following concepts which are of the common use in the context of convex and nonlinear analysis. A mulifunction $T : X \to 2^X$ is said to be a *monotone* operator if

$$\langle x - x', y - y' \rangle \geqslant 0 \quad \text{whenever} \quad y \in T(x), y' \in T(x'). \tag{1}$$

It said to be *maximal monotone* if, in addition, the *graph* $\{(x, y) \in X \times X; y \in T(x)\} :=$ graph T is not properly contained in the graph of any other monotone operator. Such operators have been studied extensively because of their role in the modelization of unilateral problems, nonlinear dissipative systems, convex optimization ...
Let T^{-1} be the *inverse* of T, i.e., $T^{-1}(y) = \{x \in X; y \in T(x)\}$. Obviously T^{-1} is maximal monotone if and only if T is maximal monotone. The effective *domain* of

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T is the set $\{x \in X; T(x) \neq \emptyset\} := \text{Dom}\, T$. The *resolvent* of parameter $\lambda > 0$ of T is given by $J_\lambda^T := (I + \lambda T)^{-1}$. It is a contraction which is everywhere defined. The *subdifferential*, $\partial\varphi(x_0)$, of a convex function $\varphi : X \to \mathbb{R} \cup \{+\infty\}$ at x_0, is defined as the set of $y_0 \in X$ satisfying

$$\varphi(x) \geqslant \varphi(x_0) + \langle y_0, x - x_0 \rangle \quad \forall x \in X. \tag{2}$$

Remember that the subdifferential of a proper convex and lower semicontinuous function is a maximal monotone operator, see Brézis [4].
Finally, for a subset D of X, we denote the distance from $x \in X$ to D by $dist(x, D) = inf_{y \in D}|x - y|$ and for C and D subsets of X, the Hausdorff excess of C over D by $e(C, D) = sup_{x \in C} d(x, D)$. Let us remark That if C is bounded, then $e(C, D) < +\infty$.

2. Formulation and Basic results

Let $\varphi : X \to \mathbb{R} \cup \{+\infty\}$ be a proper, convex and lower semicontinuous function and $T, V : X \to 2^X$ two given monotone operators with closed values. We consider the problem of finding $x \in X, y \in T(x), z \in V(x)$ such that

$$(\mathscr{M}VI) \quad \langle y + z, u - x \rangle + \varphi(u) - \varphi(x) \geqslant 0 \quad \forall u \in X.$$

The problem $(\mathscr{M}VI)$ is called the set-valued mixed variational inequality. This problem has many important and significant applications and a wide class of obstacle, unilateral, contact and free boundary problem arising in pure and applied sciences can be studied in the general framework of mixed variational inequalities.

EXAMPLE 2.1. To convey an idea of the applications of the set-valued variational inequality ($\mathscr{M}\mathscr{V}\mathscr{I}$), we consider an elastoplasticity problem, which is mainly due to Panagiotopoulos and Stavroulakis [14]. For simplicity, it is assumed that a general hyper-elastic material law holds for the elastic behaviour of the elastoplastic material under consideration. Let us assume the decomposition

$$E = E^e + E^p$$

where E^e denotes the elastic and E^p denotes the plastic deformation of the three-dimentional elasto-plastic body. We write the complementary virtual work expression for the body in the form

$$\langle E^e, \tau - \sigma \rangle + \langle E^p, \tau - \sigma \rangle = \langle f, \tau - \sigma \rangle, \quad \forall \tau \in Z.$$

Here we have assumed that the body on a part Γ_U of its boundary Γ has given displacements, that is, $\mu_i = U_i$ on Γ_U and that on the rest of its boundary $\Gamma_F = \Gamma - \Gamma_U$, the boundary tractions are given, that is, $S_i = F_i$ on Γ_F, where

$$\langle E, \sigma \rangle = \int_\Omega \varepsilon_{ij}\sigma_{ij} d\Omega$$

$$\langle f, \sigma \rangle = \int_{\Gamma_U} U_i S_i d\Gamma$$

$$Z = \{\tau : \tau_{ij,j} + f_i = 0 \text{ on } \Omega, i,j = 1,2,3, T_i = F_i \text{ on } \Gamma_F, i = 1,2,3\},$$

is the set of statically admissible stresses and Ω is the structure of the body. Let us assume that the material of the structure Ω is hyperelastic such that

$$\langle E^e(\sigma), \tau - \sigma \rangle \leqslant \langle W_m'(\sigma), \tau - \sigma \rangle \quad \forall \tau \in \mathbb{R}^6,$$

where W_m is the superpotential which produces the constitutive law of the hyperelastic material and is assumed to be quasi-differentiable, that is, there exist convex and compact subsets $\underline{\partial'} W_m$ and $\overline{\partial'} W_m$ such that

$$\langle W_m'(\sigma), \tau - \sigma \rangle = \max_{W_1^e \in \underline{\partial'} W_m} \langle W_1^e(\sigma), \tau - \sigma \rangle + \min_{W_2^e \in \overline{\partial'} W_m} \langle W_2^e(\sigma), \tau - \sigma \rangle$$

Here W_m is a generally nonconvex and nonsmooth, but is quasi-differentiable function for the case of plasticity with convex yield surface and hyperelasticity. Combining these facts and using the technique of Panagiotopoulos and Stavroulakis [14] we obtain the following multivalued variational inequality problem: Find $\sigma \in Z, W_1^e \in \underline{\partial'} W_m(\sigma), W_2^e \in \overline{\partial'} W_m(\sigma)$ such that

$$\langle W_1^e + W_2^e, \tau - \sigma \rangle \geqslant t \int_\Gamma h(|\tau| - |\sigma|) d\Omega \geqslant \langle f, \tau - \sigma \rangle, \quad \forall \tau \in Z$$

which is exactly the problem $(\mathscr{M}VI)$, with $y = W_1^e$, $z = W_2^e$, $\varphi(\sigma) = \int_F h(|\sigma|) d\Omega$, h is a positive bounded function on the friction, $T(\sigma) = \underline{\partial'} W_m(\sigma)$, $V(\sigma) = \overline{\partial'} W_m(\sigma)$ and $X = Z$. For further applications, see [14].

It worth noting that if φ is the indicator function of a closed convex set C, $V = 0$ and T a single-valued operator, then problem ($\mathscr{M}$VI) reduces to finding $x \in C$ such that

$$(\mathscr{V}I) \quad \langle Tx, y - x \rangle \quad \forall y \in C,$$

which is the classical variational inequality problem introduced by Stampacchia [18] in 1964. For a suitable choice of the operators $T, V, \partial\varphi$ and the space X, one can obtain a number of classes of variational inequalities and related complementarity problems studied by many authors including Noor [12], Adly and Oettli [1] and Moudafi and Théra [10] from the problem $(\mathscr{M}VI)$.

3. Algorithm and Convergence Analysis

3.1. Equivalence and an iterative algorithm

In this section, we establish the equivalence between $(\mathscr{M}VI)$ and a fixed point formulation which will be used to suggest our iterative method.

PROPOSITION 1. *$x \in X$ is a solution of $(\mathscr{M}VI)$ if and only if, $x \in X, y \in T(x), z \in V(x)$ satisfy the relation*

$$x = J_\lambda^\varphi (x - \lambda(y + z)),$$

where $\lambda > 0$ is a constant and $J_\lambda^\varphi(x) = (I + \lambda\partial\varphi)^{-1}$ is the proximal mapping.

Proof. Let $x \in X, y \in T(x), z \in V(x)$ be a solution of $(\mathscr{M}VI)$, then

$$\langle y+z, u-x\rangle + \varphi(u) - \varphi(x) \geqslant 0 \quad \forall u \in X. \tag{3}$$

Definition of the subdifferential yields,

$$-(y+z) \in \partial\varphi(x)$$

which can be rewriten as

$$x - \lambda(y+z) \in (I + \lambda\partial\varphi)(x),$$

from which we deduce the desired result, i.e.,

$$x = J_\lambda^\varphi\left(x - \lambda(y+z)\right),$$

where $\lambda > 0$ is a constant.

□

This alternate formulation, which is more flexible, is very important from numerical and approximation point of views. We use it to suggest and analyze an iterative algorithm. It is worth mentioning that this equivalente formulation can be applied to study the sensitivity analysis of variational inequality problems.

ALGORITHM.
For a given $x_0 \in X, \gamma > 0$, compute the sequences $(x_n)_{n\in\mathbb{N}}, (y_n)_{n\in\mathbb{N}}$ and $(z_n)_{n\in\mathbb{N}}$ by the iterative schemes

$$\begin{cases} \forall n \in \mathbb{N}^* & x_n = (1-\gamma)x_{n-1} + \gamma J_\lambda^\varphi\left(x_{n-1} - \lambda(y_{n-1} + z_{n-1})\right) \\ y_{n-1} \in T(x_{n-1}), & |y_n - y_{n-1}| \leqslant e\left(T(x_n), T(x_{n-1})\right) \\ z_{n-1} \in V(x_{n-1}), & |z_n - z_{n-1}| \leqslant e\left(V(x_n), V(x_{n-1})\right). \end{cases} \tag{4}$$

3.2. A strong convergence result

In this subsection, we study conditions under which the approximate solution obtained from our Algorihm converges strongly to the exact solution of $(\mathscr{M}VI)$. The next theorem improves a convergence result by M. A. Noor [11, 12].

THEOREM 1. *In addition to conditions on φ, V and T, we assume that T is strongly monotone over X (with modulus α), namely, there exists a positive constant α such that $\forall x_1, x_2 \in domT, \forall y_1 \in T(x_1), y_2 \in T(x_2)$,*

$$\langle y_1 - y_2, x_1 - x_2\rangle \geqslant \alpha|x_1 - x_2|^2, \tag{5}$$

Lipschitz continuous (with constant β), this means, there exists a positive constant $\beta > 0$ such that

$$e(T(x_1), T(x_2)) \leqslant \beta|x_1 - x_2| \quad \forall x_1, x_2 \in domT. \tag{6}$$

If we suppose that V is also a Lipschitz continuous operator with constant $\eta > 0$ and

$$0 < \lambda < 2\frac{\alpha - \eta}{\beta^2 - \eta^2}, \quad \eta < \alpha, \quad and \quad \lambda\eta < 1, \tag{7}$$

then, the sequence $(x_n)_{n\in\mathbb{N}}$ strongly converges to the unique solution of $(\mathscr{M}\mathscr{V}\mathscr{I})$.

Proof. The first equality of relation (4) gives

$$\begin{aligned}|x_{n+1}-x_n| &= \left|(1-\gamma)(x_n-x_{n-1})+\gamma\left(J_\lambda^\varphi(x_n-\lambda(y_n+z_n))-J_\lambda^\varphi(x_{n-1}-\lambda(y_{n-1}+z_{n-1}))\right)\right| \\ &\leqslant (1-\gamma)\,|x_n-x_{n-1}|+\gamma\,|x_n-x_{n-1}-\lambda(y_n-y_{n-1})|+\gamma\lambda\,|z_n-z_{n-1}|\,.\end{aligned}$$

On the other hand, we have

$$\begin{aligned}|(x_n-x_{n-1})-\lambda(y_n-y_{n-1})|^2 &= |x_n-x_{n-1}|^2+\lambda^2|y_n-y_{n-1}|^2 \\ &- 2\lambda\langle y_n-y_{n-1},x_n-x_{n-1}\rangle \\ &\leqslant |x_n-x_{n-1}|^2+\lambda^2 e(Tx_n,Tx_{n-1})^2 \\ &- 2\lambda\alpha\,|x_n-x_{n-1}|^2 \\ &\leqslant \left(1-2\lambda\alpha+\lambda^2\beta^2\right)|x_n-x_{n-1}|^2\,.\end{aligned}$$

Since

$$|z_n-z_{n-1}|\leqslant e(V(x_n),V(x_{n-1}))\leqslant \eta\,|x_n-x_{n-1}|\,,$$

we infer

$$|x_{n+1}-x_n|\leqslant \theta\,|x_n-x_{n-1}|\,, \tag{8}$$

with $\theta=1-\gamma+\gamma\lambda\eta+\gamma\sqrt{1-2\lambda\alpha+\lambda^2\beta^2}$.
Assumptions of the theorem ensure that $\theta<1$. Hence the sequence $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in X. We then obtain that $(y_n)_{n\in\mathbb{N}}$ and $(z_n)_{n\in\mathbb{N}}$ are also Cauchy sequences thanks to the next inequalities

$$|y_n-y_{n-1}|\leqslant e(Tx_n,Tx_{n-1})\leqslant \beta\,|x_n-x_{n-1}|\,. \tag{9}$$

and

$$|z_n-z_{n-1}|\leqslant e(Vx_n,Vx_{n-1})\leqslant \eta\,|x_n-x_{n-1}|\,. \tag{10}$$

Thus there exist (x,y,z) such that $(x_n,y_n,z_n)\to(x,y,z)$ as $n\to+\infty$.

Now we shall show that $y\in T(x)$. In fact,

$$\begin{aligned}dist(y,T(x)) &\leqslant |y-y_n|+dist\,(y_n,T(x)) \\ &\leqslant |y-y_n|+e(T(x_n),T(x)) \\ &\leqslant |y-y_n|+\beta|x_n-x|,\end{aligned}$$

that is $dist(y,T(x))=0$ as $n\to+\infty$. This implies that $y\in T(x)$ since $T(x)$ is closed. Similarly we obtain that $z\in V(x)$.
Passing to the limit in the first equality of (4) and taking into account the continuity of the proximal mapping, J_λ^φ, we obtain $x=J_\lambda^\varphi\,(x-\lambda(y+z))$, which with proposition 1 complete the proof.

□

3.3. A weak convergence result

For simplicity we take $V = 0$ which amounts to setting $T := T + V$. In our analysis we allow λ and γ to vary from one iteration to the next and we assume that T is an operator with closed and convex values and λ_n, γ_n are sequences such that

$$\underline{\gamma} := \inf_{n\geqslant 1} \gamma_n > 0, \overline{\gamma} := \sup_{n\geqslant 1} \gamma_n < 1, \underline{\lambda} := \inf_{n\geqslant 1} \lambda_n > 0, \overline{\lambda} := \sup_{n\geqslant 1} \lambda_n < 2\beta.$$

The first equality defining algorithm (4) takes the following form

$$\begin{cases} \forall n \in \mathbb{N}^* & x_n = x_{n-1} - \gamma_n R(x_{n-1}) \\ \text{where} & R(x) = x - J^{\varphi}_{\lambda_n}(x - \lambda_n y) \quad \text{with} \quad y \in T(x). \end{cases} \tag{11}$$

The following proposition gives sufficient conditions in order the iterative method be convergent.

THEOREM 2. *In addition to hypothesis on T and φ, we assume that the graph of $T + \partial\varphi$ is weakly-strongly closed in $X \times X$ and that T is co-coercive over X (with modulus $\beta > 0$), namely, there exists a positive constant β such that $\forall x_1, x_2 \in domT, \forall y_1 \in T(x_1), y_2 \in T(x_2)$,*

$$\langle y_1 - y_2, x_1 - x_2 \rangle \geqslant \beta e^2 \left(T(x_1), T(x_2)\right). \tag{12}$$

If $0 < \lambda_n < 2\beta$ and $0 < \gamma_n < 1$, then, the sequence $(y_n)_{n\in\mathbb{N}}$ is bounded and the sequence $(x_n)_{n\in\mathbb{N}}$ weakly converges to a solution of $(\mathscr{MVI})$.

Proof. Let $\bar{x}$ be a solution of $(\mathscr{MVI})$ and choose $\bar{y} \in T(\bar{x})$ and $y_n \in T(x_n)$ such that

$$|y_n - \bar{y}| \leqslant e\left(T(x_n), T(\bar{x})\right).$$

Let us first show that $P(x) := J^{\varphi}_{\lambda_n}(x - \lambda_n y)$ is nonexpansive. Indeed,

$$\begin{aligned} |P(x_n) - P(\bar{x})|^2 &= |J^{\varphi}_{\lambda_n}(x_n - \lambda_n y_n) - J^{\varphi}_{\lambda_n}(\bar{x} - \lambda_n \bar{y})|^2 \\ &\leqslant |x_n - \bar{x} + \lambda_n(y_n - \bar{y})|^2 \\ &= |x_n - \bar{x}|^2 - 2\lambda_n \langle y_n - \bar{y}, x_n - \bar{x} \rangle + \lambda_n^2 |y_n - \bar{y}|^2 \\ &\leqslant |x_n - \bar{x}|^2 - \lambda_n(2\beta - \lambda_n) e^2\left(T(x_n), T(\bar{x})\right)|^2 \\ &\leqslant |x_n - \bar{x}|^2. \end{aligned}$$

We have use the fact that $J^{\varphi}_{\lambda_n}$ is nonexpansive, T is co-coercive and $\lambda_n < 2\beta$.

On the other hand, since $(I - R)(x) = P(x)$ and $R(\bar{x}) = 0$, a direct computation yields

$$\begin{aligned} \langle R(x_n) - R(\bar{x}), x_n - \bar{x} \rangle - \frac{1}{2}|R(x_n) - R(\bar{x})|^2 &= |x_n - \bar{x}|^2 - \langle P(x_n) - P(\bar{x}), x_n - \bar{x} \rangle \\ &\quad - \frac{1}{2}\left(|x_n - \bar{x}|^2 + |P(x_n) - P(\bar{x})|^2 - 2\langle P(x_n) - P(\bar{x}), x_n - \bar{x} \rangle\right) \\ &= \frac{1}{2}\left(|x_n - \bar{x}|^2 - |P(x_n) - P(\bar{x})|^2\right) \geqslant 0. \end{aligned}$$

Thus R is co-coercive with constant $\frac{1}{2}$ with respect to $\bar{x}$, in other words

$$\langle R(x_n), x_n - \bar{x}\rangle \geqslant \frac{1}{2}|R(x_n)|^2. \tag{13}$$

Finally, we obtain

$$\begin{aligned} |x_{n+1} - \bar{x}|^2 &= |x_n - \bar{x} - \gamma_n R(x_n)|^2 \\ &= |x_n - \bar{x}|^2 + \gamma_n^2 |R(x_n)|^2 - 2\gamma_n \langle x_n - \bar{x}, R(x_n)\rangle , \end{aligned}$$

which combined with (13) yields

$$|x_{n+1} - \bar{x}|^2 \leqslant |x_n - \bar{x}|^2 - \underline{\gamma}(1 - \overline{\gamma})|R(x_n)|^2. \tag{14}$$

The sequence $\{x_n\}$ is bounded which implies, in turn, that $\{y_n\}$ is bounded. Likewise

$$\mu(\bar{x}) := \lim_{n\to+\infty} |x_n - \bar{x}| \tag{15}$$

exists and is finite. In view of (15) and (12), we get easily that

$$\lim_{n\to+\infty} R(x_n) = \lim_{n\to+\infty} \frac{|x_n - x_{n+1}|}{\underline{\gamma}} = 0. \tag{16}$$

This means that $\{x_n\}$ is asymptotically regular.
Now let x^* be a weak cluster point of $\{x_n\}$, there exists a subsequence $\{x_{n_k}\}$ which converges weakly to x^* and according (12), we have

$$\frac{x_{n_k} - x_{n_{k+1}}}{\gamma_{n_k}\lambda_{n_k}} \in T(x_{n_k}) + \partial\varphi(\tilde{x}_{n_k}), \tag{17}$$

where $\tilde{x}_{n_k} = x_{n_k} + \frac{x_{n_k+1} - x_{n_k}}{\gamma_{n_k}\lambda_{n_k}}$.
Setting $\tilde{y}_{n_k} = argmin\{|y - y_{n_k}|,\ y \in T(\tilde{x}_{n_k})\}$ (the minimum is attained uniquely since $T(\tilde{x}_{n_k})$ is a closed convex set), the last inclusion can be rewritten as

$$\frac{x_{n_k} - x_{n_k+1}}{\gamma_{n_k}\lambda_{n_k}} + \tilde{y}_{n_k} - y_{n_k} \in T(\tilde{x}_{n_k}) + \partial\varphi(\tilde{x}_{n_k}). \tag{18}$$

Since

$$|\tilde{y}_{n_k} - y_{n_k}| = dist(y_{n_k}, T(\tilde{x}_{n_k}) \leqslant e\left(Tx_{n_k}, T\tilde{x}_{n_k}\right) \leqslant \frac{1}{\beta}|x_{n_k} - \tilde{x}_{n_k}|, \tag{19}$$

and the graph of $T + \partial\varphi$ is weakly-strongly closed, we get at the limit

$$0 \in T(x^*) + \partial\varphi(x^*),$$

that is, x^* is a solution of the given problem. It remains to prove that there is no more than one weak cluster point, our argument follows that given in ([16], p. 885) and is presented here for completeness.

Let $\tilde{x}$ be another weak cluster point of $\{x_n\}$, we will show that $\tilde{x} = x^*$. This is a consequence of (16). Indeed, since $\mu(\tilde{x}) := \lim_{n\to+\infty} |x_n - \tilde{x}|$ and $\mu(x^*) := \lim_{n\to+\infty} |x_n - x^*|$, from

$$|x_n - \tilde{x}|^2 = |x_n - x^*|^2 + |x^* - \tilde{x}|^2 + 2\langle x_n - x^*, x^* - \tilde{x}\rangle ,$$

we see that the limit of $\langle x_n - x^*, x^* - \tilde{x}\rangle$ as $n \to +\infty$ must exists. This limit has to be zero because x^* is a weak cluster point of $\{x_n\}$. Hence at the limit

$$\mu(\tilde{x}) = \mu(x^*) + |x^* - \tilde{x}|^2.$$

Reversing the role of $\tilde{x}$ and x^* we also have

$$\mu(x^*) = \mu(\tilde{x}) + |x^* - \tilde{x}|^2.$$

From these we infer that $x^* = \tilde{x}$, which completes the proof.

□

Note that the weak convergence result still hold true if we allow the resolvent to be approximately evaluated, so long as the sum of all errors is finite.

REMARK. It worth mentioning that if T is weakly closed, then every weak-cluster point of the sequence $(y_n)_{n\in\mathbb{N}}$ belongs in $T(x^*)$. Moreover, if T is assumed to be maximal monotone, then $\forall x \in$ dom $T, T(x)$ is a closed convex set. Furthermore, to ensure the maximal monotonicity of $T+\partial\varphi$ which implies that $gph(T+\partial\varphi)$ is weakly-strongly closed, it suffices to assume, for example, that $int(domT) \cap dom\partial\varphi \neq \emptyset$ or $0 \in int(domT - dom\partial\varphi)$.
We would like to mention that the results obtained in this paper are still true when replacing the subdifferential of the function φ by any maximal monotone operator. In the case when T is single-valued, we recover the forward-backward splitting method proposed by Lions and Mercier [9], by Gabay [6] and in a dual form by Han and Lou [7]. In the case where φ is the indicator function of a nonempty closed convex set, this method reduces to a projection method proposed by Sibony [17] for monotone variational inequalities and, in the further case where T is the gradient of a differentiable convex function, it reduces to a gradient projection method of Goldstein and Levitin and Polyak see [3].

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