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SIMPLE PROOF OF THE CONCAVITY OF OPERATOR ENTROPY $f(A) = -A\log A$

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Dedicated to Professor M. Nakamura and Professor H. Umegaki with respect and affection

(*communicated by J. Pečarić*)

Abstract. A simple proof of the concavity of operator entropy $f(A) = -A\log A$ is given.

A capital letter means a bounded linear and *strictly positive* operator on a Hilbert space. Here we shall give a simple proof of the following well known and excellent result obtained by [1] and [2] independently.

THEOREM A. *$f(A) = -A\log A$ is concave function for any $A > 0$.*

Proof. Firstly we recall the following obvious result

$$\lim_{n\to\infty}(T^{\frac{-1}{n}} - I)n = -\log T \qquad \text{for any } T > 0. \tag{$*$}$$

As $g(t) = t^q$ is operator concave for $q \in [0,1]$, then for $A > 0$, $B > 0$ and $\alpha, \beta \in [0,1]$ such that $\alpha + \beta = 1$

$$\Big(\alpha A + \beta B\Big)^{1-\frac{1}{n}} \geqslant \alpha A^{1-\frac{1}{n}} + \beta B^{1-\frac{1}{n}}$$

for any natural number n, so we obtain

$$(\alpha A + \beta B)\Big((\alpha A + \beta B)^{-\frac{1}{n}} - I\Big)n \geqslant \alpha A(A^{-\frac{1}{n}} - I)n + \beta B(B^{-\frac{1}{n}} - I)n.$$

Tending $n \to \infty$, we have

$$-(\alpha A + \beta B)\log(\alpha A + \beta B) \geqslant (-\alpha A\log A - \beta B\log B) \qquad \text{by } (*)$$

that is,

$$f(\alpha A + \beta B) \geqslant \alpha f(A) + \beta f(B)$$

so the proof is complete.

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