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TRACE INEQUALITIES FOR MULTIPLE PRODUCTS OF TWO MATRICES

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Abstract. Trace inequalities for multiple products of powers of two matrices are discussed via the method of log majorization. For instance, the trace inequality $|\mathrm{Tr}(A^{p_1}B^{q_1}A^{p_2}B^{q_2}\cdots A^{p_K}B^{q_K})| \leqslant \mathrm{Tr}(AB)$ is obtained for positive semidefinite matrices A,B and $p_i,q_i \geqslant 0$ with $p_1+\cdots+p_K = q_1+\cdots+q_K = 1$ under some additional condition.

1. Introduction and notations

First of all, recall the notion of majorization of multiplicative type. Let $\vec{a} = (a_1,a_2,\ldots,a_n)$ and $\vec{b} = (b_1,b_2,\ldots,b_n)$ be real (row) vectors such that $a_1 \geqslant a_2 \geqslant \cdots \geqslant a_n \geqslant 0$ and $b_1 \geqslant b_2 \geqslant \cdots \geqslant b_n \geqslant 0$. The log majorization $\vec{a} \prec_{(\log)} \vec{b}$ is said to hold if

$$\prod_{i=1}^{k} a_i \leqslant \prod_{i=1}^{k} b_i \quad \text{for} \quad 1 \leqslant k \leqslant n$$

with equality for $k = n$. It is well-known that $\vec{a} \prec_{(\log)} \vec{b}$ implies the weak majorization (of additive type) $\vec{a} \prec_w \vec{b}$, that is,

$$\sum_{i=1}^{k} a_i \leqslant \sum_{i=1}^{k} b_i \quad \text{for} \quad 1 \leqslant k \leqslant n.$$

For an $n \times n$ positive semidefinite matrix A let

$$\vec{\lambda}(A) = (\lambda_1(A),\lambda_2(A),\ldots,\lambda_n(A))$$

denote the vector of the eigenvalues of A arranged in decreasing order $\lambda_1(A) \geqslant \lambda_2(A) \geqslant \cdots \geqslant \lambda_n(A) \geqslant 0$ with multiplicities. We use the same notation $\vec{\lambda}(X)$ for a matrix X having the nonnegative eigenvalues; for instance, this is the case if X is the product of two positive semidefinite matrices. Moreover, for an arbitrary $n \times n$ matrix X write

$$|\vec{\lambda}(X)| = (|\lambda_1(X)|,|\lambda_2(X)|,\ldots,|\lambda_n(X)|),$$

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where $\lambda_1(X),\dots,\lambda_n(X)$ are the eigenvalues of X arranged as $|\lambda_1(X)| \geqslant |\lambda_2(X)| \geqslant \cdots \geqslant |\lambda_n(X)|$.

The majorization method applied to the vectors of eigenvalues (also singular values) of matrices supplies a powerful tool in studying matrix norm inequalities and trace inequalities (see [1, 4, 10] for example). The well-known majorization theorem of Weyl ([4, p. 42]) is

$$|\vec{\lambda}(X)| \prec_{(\log)} \vec{\lambda}(|X|), \tag{1.1}$$

where $|X| = (X^*X)^{\frac{1}{2}}$. If $|\vec{\lambda}(X)| \prec_{(\log)} \vec{\lambda}(Y)$ where $\vec{\lambda}(Y)$ is nonnegative, then $|\mathrm{Tr}(X)| \leqslant \mathrm{Tr}(Y)$ holds; in fact,

$$|\mathrm{Tr}(X)| \leqslant \sum_{i=1}^{n} |\lambda_i(X)| \leqslant \sum_{i=1}^{n} \lambda_i(Y) = \mathrm{Tr}(Y).$$

When A and B are $n\times n$ positive semidefinite matrices, the log majorization due to Araki [3] is written as

$$\vec{\lambda}((A^{\frac{1}{2}}BA^{\frac{1}{2}})^r) \prec_{(\log)} \vec{\lambda}(A^{\frac{r}{2}}B^rA^{\frac{r}{2}}) \quad \text{for} \quad r \geqslant 1,$$

or equivalently,

$$\vec{\lambda}((A^{\frac{s}{2}}B^sA^{\frac{s}{2}})^{\frac{1}{s}}) \prec_{(\log)} \vec{\lambda}((A^{\frac{t}{2}}B^tA^{\frac{t}{2}})^{\frac{1}{t}}) \quad \text{for} \quad 0 < s \leqslant t. \tag{1.2}$$

Here we use the convention $A^0 = I$. This is regarded as strengthening the famous Golden-Thompson inequality ([4, p. 261])

$$\mathrm{Tr}(e^{H+K}) \leqslant \mathrm{Tr}(e^He^K) \tag{1.3}$$

for Hermitian matrices H,K. Indeed, as $s \to +0$ in (1.2) with $A = e^H$ and $B = e^K$, the Lie-Trotter formula ([4, p. 254]) yields

$$\vec{\lambda}(e^{H+K}) \prec_{(\log)} \vec{\lambda}((e^{\frac{t}{2}H}e^{tK}e^{\frac{t}{2}H})^{\frac{1}{t}}) \quad \text{for} \quad t > 0, \tag{1.4}$$

and for $t = 1$ this implies (1.3). (See [2, 9] for more about the strengthened Golden-Thompson inequality.)

According to the log majorization given in [2, Theorem 4.1], when A,B are positive semidefinite, we have

$$\vec{\lambda}(|A^{p_1}B^{q_1}A^{p_2}B^{q_2}\cdots A^{p_K}B^{q_K}|) \prec_{(\log)} \vec{\lambda}(|AB|) \tag{1.5}$$

for every $p_i, q_i \geqslant 0$ ($1 \leqslant i \leqslant K$) such that $\sum_{i=1}^K p_i = \sum_{i=1}^K q_i = 1$ and

$$\sum_{i=1}^{j} p_i \leqslant \sum_{i=1}^{j} q_i \ \ (1 \leqslant j \leqslant K-1), \qquad \sum_{i=1}^{j-1} q_i \leqslant \sum_{i=1}^{j} p_i \ \ (2 \leqslant j \leqslant K-1). \tag{$*$}$$

Although the majorization (1.5) implies in particular

$$\mathrm{Tr}(|A^{p_1}B^{q_1}A^{p_2}B^{q_2}\cdots A^{p_K}B^{q_K}|) \leqslant \mathrm{Tr}(|AB|),$$

the trace inequality

$$|\mathrm{Tr}(A^{p_1}B^{q_1}A^{p_2}B^{q_2}\cdots A^{p_K}B^{q_K})| \leqslant \mathrm{Tr}(AB) \tag{1.6}$$

is not a consequence of (1.5).

Our primary motivation has been to consider whether the trace inequality (1.6) is true for every $p_i, q_i \geqslant 0$ with $\sum_{i=1}^K p_i = \sum_{i=1}^K q_i = 1$ or not. It may be also interesting to find cases where the opposite trace inequality

$$\mathrm{Tr}((A^{\frac{1}{K}}B^{\frac{1}{K}})^K) \leqslant |\mathrm{Tr}(A^{p_1}B^{q_1}A^{p_2}B^{q_2}\cdots A^{p_K}B^{q_K})| \tag{1.7}$$

holds, which is regarded as a far extended version of the Golden-Thompson inequality. These problems seem rather subtle; in fact, we have no counterexamples to them for the moment. In this paper we obtain (1.6) under stronger assumptions on the exponents (given below) and also (1.7) for some particular cases.

In Section 2, we prove a variant of the majorization (1.5) where the right-hand side $\vec{\lambda}(|AB|) = \vec{\lambda}((AB^2A)^{\frac{1}{2}})$ is replaced by $\vec{\lambda}(AB) = \vec{\lambda}(A^{\frac{1}{2}}BA^{\frac{1}{2}})$ and the assumptions (*) on $p_i, q_i \geqslant 0$ with $\sum_{i=1}^K p_i = \sum_{i=1}^K q_i = 1$ are strengthened to

$$0 \leqslant \sum_{i=1}^{j} q_i - \sum_{i=1}^{j} p_i \leqslant \frac{1}{2} \quad (1 \leqslant j \leqslant K-1), \tag{1.8}$$

$$0 \leqslant \sum_{i=1}^{j} p_i - \sum_{i=1}^{j-1} q_i \leqslant \frac{1}{2} \quad (1 \leqslant j \leqslant K). \tag{1.9}$$

Here we use the convention $\sum_{i=1}^{j-1} q_i = 0$ for $j = 1$. As a consequence we have the trace inequality (1.6) for such $p_i, q_i \geqslant 0$. In Section 3, we obtain similar majorization results for products of integer powers of Hermitian matrices.

In Section 4, we consider $\mathrm{Tr}(f_1(A)g_1(B)f_2(A)g_2(B))$ where f_i, g_i are nonnegative increasing functions on $[0,\infty)$, and compare it with other forms such as $\mathrm{Tr}(f_1(A)f_2(A)$ $g_1(B)g_2(B))$. Trace inequalities among such products of matrix functions are obtained for some restricted cases where positive semidefinite matrices A, B have at most two or three different eigenvalues.

2. Case of positive semidefinite matrices

Throughout this section let A, B be $n \times n$ positive semidefinite matrices. We discuss log majorizations and trace inequalities of the form (1.6) for multiple products of powers of A, B. As in [2] the technique of anti-symmetric tensors plays a key role in the proof of the next theorem.

THEOREM 2.1. *Let $p_i, q_i \geqslant 0$ $(1 \leqslant i \leqslant K)$ be such that $\sum_{i=1}^K p_i = \sum_{i=1}^K q_i = 1$. If p_i, q_i satisfy the assumptions (1.8) and (1.9) stated above, then*

$$\vec{\lambda}(|A^{p_1}B^{q_1}A^{p_2}B^{q_2}\cdots A^{p_K}B^{q_K}|) \prec_{(\log)} \vec{\lambda}(AB). \tag{2.1}$$

In particular,

$$|\mathrm{Tr}(A^{p_1}B^{q_1}A^{p_2}B^{q_2}\cdots A^{p_K}B^{q_K})| \leqslant \mathrm{Tr}(AB). \tag{2.2}$$

Proof. We first show that

$$\lambda_1(|A^{p_1}B^{q_1}A^{p_2}B^{q_2}\cdots A^{p_K}B^{q_K}|) \leqslant \lambda_1(AB). \tag{2.3}$$

To do so, we may assume that A,B are invertible. By homogeneity it suffices to prove that $\lambda_1(AB) \leqslant 1$ implies $\lambda_1(|A^{p_1}B^{q_1}A^{p_2}B^{q_2}\cdots A^{p_K}B^{q_K}|) \leqslant 1$. Assume $\lambda_1(A^{\frac{1}{2}}BA^{\frac{1}{2}}) = \lambda_1(AB) \leqslant 1$, so $A^{\frac{1}{2}}BA^{\frac{1}{2}} \leqslant I$. Hence $B \leqslant A^{-1}$ and $A \leqslant B^{-1}$. Here for two Hermitian matrices X,Y the inequality $X \leqslant Y$ means that $Y - X$ is positive semidefinite. Since $0 \leqslant 2p_1 \leqslant 1$ and $0 \leqslant 2(q_1 - p_1) \leqslant 1$, the well-known Löwner's theorem (see [4] p. 149) yields

$$B^{q_1}A^{2p_1}B^{q_1} \leqslant B^{2(q_1-p_1)}$$

and hence

$$A^{p_2}B^{q_1}A^{2p_1}B^{q_1}A^{p_2} \leqslant A^{p_2}B^{2(q_1-p_1)}A^{p_2} \leqslant A^{2(p_1+p_2-q_1)}.$$

Repeating this argument yields

$$B^{q_K}A^{p_K}\cdots B^{q_1}A^{2p_1}B^{q_1}\cdots A^{p_K}B^{q_K} \leqslant B^{2(q_1+\cdots+q_K-p_1-\cdots-p_K)} = I,$$

and (2.3) has been proved.

Now (2.1) is a consequence of (2.3) via anti-symmetric tensor powers. Replace A,B in (2.3) by their k-fold anti-symmetric tensor powers $\wedge^k A, \wedge^k B$ for $1 \leqslant k \leqslant n$ (see [4] for details on anti-symmetric tensors of matrices). Since

$$|(\wedge^k A)^{p_1}(\wedge^k B)^{q_1}\cdots(\wedge^k A)^{p_K}(\wedge^k B)^{q_K}| = \wedge^k(|A^{p_1}B^{q_1}\cdots A^{p_K}B^{q_K}|),$$

$$(\wedge^k A)(\wedge^k B) = \wedge^k(AB),$$

we have

$$\begin{aligned}\prod_{i=1}^{k}\lambda_i(|A^{p_1}B^{q_1}\cdots A^{p_K}B^{q_K}|) &= \lambda_1(\wedge^k(|A^{p_1}B^{q_1}\cdots A^{p_K}B^{q_K}|))\\ &\leqslant \lambda_1(\wedge^k(AB)) = \prod_{i=1}^{k}\lambda_i(AB)\end{aligned}$$

for $1 \leqslant k \leqslant n$. Furthermore, for $k = n$,

$$\begin{aligned}\prod_{i=1}^{n}\lambda_i(|A^{p_1}B^{q_1}\cdots A^{p_K}B^{q_K}|) &= |\det(A^{p_1}B^{q_1}\cdots A^{p_K}B^{q_K})|\\ &= \det(A)\det(B) = \prod_{i=1}^{n}\lambda_i(AB),\end{aligned}$$

so (2.1) has been proved.

Finally, note that (2.1) gives

$$\mathrm{Tr}(|A^{p_1}B^{q_1}A^{p_2}B^{q_2}\cdots A^{p_K}B^{q_K}|) \leqslant \mathrm{Tr}(AB),$$

which is stronger than (2.2). □

Note that the assumptions of the above theorem are satisfied when $0 \leqslant p_i = q_i \leqslant \frac{1}{2}$ $(1 \leqslant i \leqslant K)$ with $\sum_{i=1}^{K} p_i = 1$.

It is evident that the equality in (2.2) is attained when $AB = BA$. Conversely, it is well-known that the equality case in many inequalities of Golden-Thompson type implies the commutativity of matrices. For instance, according to [8], the equality $\mathrm{Tr}((A^{1/m}B^{1/m})^m) = \mathrm{Tr}(AB)$ for some integer $m \geqslant 2$ implies that $AB = BA$.

For example, let

$$A = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}.$$

We compute

$$\mathrm{Tr}(AB) = 3, \quad \mathrm{Tr}(|AB|) = \sqrt{10},$$

$$\mathrm{Tr}(|A^pBA^{1-p}|) = \sqrt{4^p + 4^{1-p} + 5} \quad \text{for} \quad 0 \leqslant p \leqslant 1.$$

Hence $\mathrm{Tr}(|A^pBA^{1-p}|) \leqslant \mathrm{Tr}(|AB|)$ holds for all $0 \leqslant p \leqslant 1$ as is assured by (1.5). But $\mathrm{Tr}(|A^pBA^{1-p}|) \leqslant \mathrm{Tr}(AB)$ is equivalent to $4^p + 4^{1-p} \leqslant 4$, which is valid just for $p = \frac{1}{2}$. This says that both assumptions (1.8) and (1.9) are essential for the majorization (2.1) to hold. However, it may be possible that the weaker trace inequality (2.2) is true without assumptions on the exponents.

Restricted to the case $K = 2$ we have:

COROLLARY 2.2. *(i) If* $|p - \frac{1}{2}| + |q - \frac{1}{2}| \leqslant \frac{1}{2}$, *then* $|\vec{\lambda}(A^pB^qA^{1-p}B^{1-q})| \prec_{(\log)} \vec{\lambda}(AB)$ *and hence* $|\mathrm{Tr}(A^pB^qA^{1-p}B^{1-q})| \leqslant \mathrm{Tr}(AB)$.

(ii) $\mathrm{Tr}((A^{\frac{1}{2}}B^{\frac{1}{2}})^2) \leqslant \mathrm{Tr}(A^{\frac{1}{2}}B^qA^{\frac{1}{2}}B^{1-q}) \leqslant \mathrm{Tr}(AB)$ *for* $0 \leqslant q \leqslant 1$.

Proof. (i) Note that the eigenvalues of $A^pB^qA^{1-p}B^{1-q}$ are the same as those of the cyclic permutations $B^qA^{1-p}B^{1-q}A^p$, $A^{1-p}B^{1-q}A^pB^q$ and $B^{1-q}A^pB^qA^{1-p}$. So the assertion is an immediate consequence of Theorem 2.1 and the majorization (1.1).

(ii) Let $B = \sum_{i=1}^{n} \mu_i Q_i$ with orthogonal projections Q_i. Then using arithmetic-geometric mean inequality we have

$$\begin{aligned}
&\mathrm{Tr}(A^{\frac{1}{2}}B^qA^{\frac{1}{2}}B^{1-q}) \\
&\quad = \sum_{i=1}^{n} \mu_i \mathrm{Tr}(A^{\frac{1}{2}}Q_iA^{\frac{1}{2}}Q_i) + \sum_{i<j} (\mu_i^q\mu_j^{1-q} + \mu_i^{1-q}\mu_j^q)\mathrm{Tr}(A^{\frac{1}{2}}Q_iA^{\frac{1}{2}}Q_j) \\
&\quad \geqslant \sum_{i=1}^{n} \mu_i \mathrm{Tr}(A^{\frac{1}{2}}Q_iA^{\frac{1}{2}}Q_i) + \sum_{i<j} 2\mu_i^{\frac{1}{2}}\mu_j^{\frac{1}{2}}\mathrm{Tr}(A^{\frac{1}{2}}Q_iA^{\frac{1}{2}}Q_j) \\
&\quad = \mathrm{Tr}((A^{\frac{1}{2}}B^{\frac{1}{2}})^2).
\end{aligned}$$

The second inequality is a particular case of (i). □

Note that the second inequality of the above (ii) can be shown by using (1.5) instead of the new majorization (2.2). However, (2.2) is really useful to prove (i); we have no other way to obtain the trace inequality in (i) for general positive semidefinite A, B.

It seems that the behavior of the function $(p,q) \mapsto |\mathrm{Tr}(A^pB^qA^{1-p}B^{1-q})|$ on $[0,1]\times[0,1]$ is rather complicated for general $n\times n$ positive semidefinite matrices A,B. Indeed, when A,B are 3×3, it is not known whether the trace inequality

$$|\mathrm{Tr}(A^pB^qA^{1-p}B^{1-q})| \leqslant \mathrm{Tr}(AB)$$

is valid for all $0 \leqslant p,q \leqslant 1$ without additional condition such as (1.8) and (1.9). But in Section 4 the above inequality will be shown to hold when A,B are 2×2 matrices.

When H,K are Hermitian matrices, since $\mathrm{Tr}(e^{H+K}) \leqslant \mathrm{Tr}((e^{\frac{1}{2}H}e^{\frac{1}{2}K})^2)$ by (1.4), the Golden-Thompson inequality (1.3) can be refined due to Corollary 2.2 (ii) as follows:

$$\mathrm{Tr}(e^{H+K}) \leqslant \mathrm{Tr}(e^{\frac{1}{2}H}e^{qK}e^{\frac{1}{2}H}e^{(1-q)K}) \leqslant \mathrm{Tr}(e^He^K) \quad \text{for} \quad 0 \leqslant q \leqslant 1.$$

REMARK 2.3. The following results generalize Corollary 2.2 (ii). These are essentially included in [5], so we omit the proofs.

(i) If g_1,g_2 are nonnegative increasing functions on $[0,\infty)$, then

$$\mathrm{Tr}(Ag_1(B)Ag_2(B)) \leqslant \mathrm{Tr}(A^2g_1(B)g_2(B)).$$

(ii) If g_1 is nonnegative increasing and g_2 is nonnegative decreasing on $[0,\infty)$, then

$$\mathrm{Tr}(Ag_1(B)Ag_2(B)) \geqslant \mathrm{Tr}(A^2g_1(B)g_2(B)).$$

(iii) For every nonnegative functions g_1,g_2 on $[0,\infty)$,

$$\mathrm{Tr}((A\sqrt{g_1g_2}(B))^2) \leqslant \mathrm{Tr}(Ag_1(B)Ag_2(B)).$$

COROLLARY 2.4. *Let* l_i,m_i $(1 \leqslant i \leqslant K \leqslant 4)$ *be nonnegative integers such that* $\sum_{i=1}^K l_i = \sum_{i=1}^K m_i = 4$. *Then*

$$|\vec{\lambda}(A^{l_1}B^{m_1}A^{l_2}B^{m_2}\cdots A^{l_K}B^{m_K})| \prec_{(\log)} \vec{\lambda}(A^4B^4).$$

In particular,

$$|\mathrm{Tr}(A^{l_1}B^{m_1}A^{l_2}B^{m_2}\cdots A^{l_K}B^{m_K})| \leqslant \mathrm{Tr}(A^4B^4).$$

Proof. According to the cyclic permutation of the product $A^{l_1}B^{m_1}\cdots A^{l_K}B^{m_K}$ and the change of roles of A,B, we have to consider only the following six cases:

$$AB^3A^3B, \qquad A^2B^2AB^2A, \qquad A^2B^2A^2B^2,$$

$$A^2B^2ABAB, \quad ABAB^2ABA, \quad ABABABAB.$$

Put $C = A^4$ and $D = B^4$. It suffices to prove that

$$\vec{\lambda}(|C^{p_1}D^{q_1}C^{p_2}D^{q_2}\cdots C^{p_K}D^{q_K}|) \prec_{(\log)} \vec{\lambda}(CD)$$

for $p_i = l_i/4$ and $q_i = m_i/4$ corresponding to the above six cases. But the assumptions (1.8) and (1.9) on p_i,q_i in Theorem 2.1 can be easily checked for each of these cases. □

The assumptions in Theorem 2.1 are not satisfied for any cyclic permutation of the product $A^{\frac{3}{5}}B^{\frac{1}{5}}A^{\frac{1}{5}}B^{\frac{3}{5}}A^{\frac{1}{5}}B^{\frac{1}{5}}$, so the above restriction $K \leqslant 4$ can not be avoided as long as we apply Theorem 2.1.

3. Case of Hermitian matrices

When we treat multiple products of integer powers of A, B, the proofs of [2, Theorem 4.1] and Theorem 2.1 work well even if A, B are Hermitian. So the next theorems are obtained in the same way as previous.

THEOREM 3.1. *Let A, B be Hermitian matrices and let l_i, m_i $(1 \leqslant i \leqslant K)$ be nonnegative integers such that $\sum_{i=1}^{K} l_i = \sum_{i=1}^{K} m_i = N$ and*

$$\sum_{i=1}^{j} l_i \leqslant \sum_{i=1}^{j} m_i \ \ (1 \leqslant j \leqslant K-1), \quad \sum_{i=1}^{j-1} m_i \leqslant \sum_{i=1}^{j} l_i \ \ (2 \leqslant j \leqslant K-1).$$

Then

$$\vec{\lambda}(|A^{l_1}B^{m_1}A^{l_2}B^{m_2}\cdots A^{l_K}B^{m_K}|) \prec_{(\log)} \vec{\lambda}(|A^N B^N|).$$

THEOREM 3.2. *Let A, B be Hermitian matrices and let l_i, m_i $(1 \leqslant i \leqslant K)$ be nonnegative integers such that $\sum_{i=1}^{K} l_i = \sum_{i=1}^{K} m_i = 2N$, an even integer, and*

$$0 \leqslant \sum_{i=1}^{j} m_i - \sum_{i=1}^{j} l_i \leqslant N \quad (1 \leqslant j \leqslant K-1),$$

$$0 \leqslant \sum_{i=1}^{j} l_i - \sum_{i=1}^{j-1} m_i \leqslant N \quad (1 \leqslant j \leqslant K).$$

Then

$$\vec{\lambda}(|A^{l_1}B^{m_1}A^{l_2}B^{m_2}\cdots A^{l_K}B^{m_K}|) \prec_{(\log)} \vec{\lambda}(A^{2N}B^{2N}).$$

COROLLARY 3.3. *If A, B are Hermitian matrices, then*

$$\mathrm{Tr}((AB)^{2N}) \leqslant \mathrm{Tr}(|(AB)^{2N}|) \leqslant \mathrm{Tr}(A^{2N}B^{2N})$$

for every positive integer N.

Proof. When $l_i = m_i = 1$ $(1 \leqslant i \leqslant 2N)$, Theorem 3.2 yields $\vec{\lambda}(|(AB)^{2N}|) \prec_{(\log)} \vec{\lambda}(A^{2N}B^{2N})$, so we get the second inequality. For the first inequality, by (1.1) we need only to check that $\mathrm{Tr}((AB)^{2N})$ is real. But it is easy to see that $\mathrm{Tr}((AB)^m)$ is real for every positive integer m. In fact, choose $c > 0$ such that $\tilde{A} = A + cI$ is positive definite, then $\mathrm{Tr}(AB) = \mathrm{Tr}(\tilde{A}B) - c\mathrm{Tr}(B)$ is real (this argument is from [7]). Since

$$\mathrm{Tr}((AB)^{2m}) = \mathrm{Tr}(A \cdot (BA)^{m-1}BAB(AB)^{m-1}),$$

$$\mathrm{Tr}((AB)^{2m+1}) = \mathrm{Tr}(A \cdot (BA)^{m}B(AB)^{m})$$

with Hermitian $(BA)^{m-1}BAB(AB)^{m-1}$ and $(BA)^m B(AB)^m$, we have the assertion. □

The above corollary extends [6, Theorem 1] where $\mathrm{Tr}((AB)^{2^k}) \leqslant \mathrm{Tr}(A^{2^k}B^{2^k})$ for every positive integer k was shown.

4. Trace inequalities under restriction on number of eigenvalues

In this section let A, B be $n \times n$ positive semidefinite matrices as in Section 2. We discuss trace inequalities for matrix functions of the form $f_1(A)g_1(B)f_2(A)g_2(B)$, where f_i, g_i are nonnegative increasing functions on $[0, \infty)$. The particular case $f_1 = f_2$ was treated in Remark 2.3. It would be impossible to get certain trace inequalities without further restrictions on f_i, g_i and/or A, B. Below we put rather strict restrictions on the number of different eigenvalues of A, B in compensation for f_i, g_i being general.

THEOREM 4.1. *Let A, B be positive semidefinite matrices and assume that both A, B have at most two different eigenvalues. If f_i, g_i $(1 \leqslant i \leqslant K)$ be nonnegative increasing functions on $[0, \infty)$, then*

$$\begin{aligned} 0 &\leqslant \operatorname{Tr}(f_1(A)g_1(B)f_2(A)g_2(B)\cdots f_K(A)g_K(B)) \\ &\leqslant \operatorname{Tr}(f_1(A)f_2(A)\cdots f_K(A)g_1(B)g_2(B)\cdots g_K(B)). \end{aligned}$$

Proof. By homogeneity and continuity we may assume that

$$f_i(A) = I + a_iP, \quad g_i(B) = I + b_iQ \quad (1 \leqslant i \leqslant K),$$

where P, Q are orthogonal projections and $a_i, b_i \geqslant 0$. Then we can write

$$\begin{aligned} &\operatorname{Tr}(f_1(A)g_1(B)f_2(A)g_2(B)\cdots f_K(A)g_K(B)) \\ &= \operatorname{Tr}(I) + \sum_{1\leqslant i_1<\cdots<i_k\leqslant K} a_{i_1}\cdots a_{i_k}\operatorname{Tr}(P) + \sum_{1\leqslant j_1<\cdots<j_l\leqslant K} b_{j_1}\cdots b_{j_l}\operatorname{Tr}(Q) \\ &\qquad + \sum_{\substack{1\leqslant i_1<\cdots<i_k\leqslant K \\ 1\leqslant j_1<\cdots<j_l\leqslant K}} a_{i_1}\cdots a_{i_k}b_{j_1}\cdots b_{j_l}\operatorname{Tr}((PQ)^m) \\ &\geqslant 0, \end{aligned}$$

where $1 \leqslant m \leqslant K$ is determined depending on $i_1, \ldots, i_k; j_1, \ldots, j_l$ in the following way: when $i_1 \leqslant j_1$, m is the number of $r \in \{1, \ldots, k\}$ such that $i_r \leqslant j_s < i_{r+1}$ (with $i_{k+1} = K + 1$) for some $s \in \{1, \ldots, l\}$; when $j_1 < i_1$, m is the number of $r \in \{1, \ldots, k\}$ such that $i_{r-1} \leqslant j_s < i_r$ (with $i_0 = 1$) for some $s \in \{1, \ldots, l\}$. On the other hand,

$$\begin{aligned} &\operatorname{Tr}(f_1(A)f_2(A)\cdots f_K(A)g_1(B)g_2(B)\cdots g_K(B)) \\ &= \operatorname{Tr}(I) + \sum_{1\leqslant i_1<\cdots<i_k\leqslant K} a_{i_1}\cdots a_{i_k}\operatorname{Tr}(P) + \sum_{1\leqslant j_1<\cdots<j_l\leqslant K} b_{j_1}\cdots b_{j_l}\operatorname{Tr}(Q) \\ &\qquad + \sum_{\substack{1\leqslant i_1<\cdots<i_k\leqslant K \\ 1\leqslant j_1<\cdots<j_l\leqslant K}} a_{i_1}\cdots a_{i_k}b_{j_1}\cdots b_{j_l}\operatorname{Tr}(PQ). \end{aligned}$$

Since $\operatorname{Tr}((PQ)^m) \leqslant \operatorname{Tr}(PQ)$ for $m \geqslant 1$, the theorem is proved. $\square$

THEOREM 4.2. *Let A,B be positive semidefinite matrices and assume that both A,B have at most two different eigenvalues. If f_1,f_2,g_1,g_2 are nonnegative increasing functions on $[0,\infty)$, then*

$$\mathrm{Tr}((\sqrt{f_1f_2}(A)\sqrt{g_1g_2}(B))^2) \leqslant \mathrm{Tr}(f_1(A)g_1(B)f_2(A)g_2(B)).$$

Proof. We may assume that

$$f_i(A) = I + a_iP, \quad g_i(B) = I + b_iQ \quad (i = 1,2),$$

where P,Q and a_i,b_i are as in the previous proof. Then

$$\sqrt{f_1f_2}(A) = I + aP \quad \text{with} \quad a = \sqrt{(1+a_1)(1+a_2)} - 1,$$

$$\sqrt{g_1g_2}(B) = I + bQ \quad \text{with} \quad b = \sqrt{(1+b_1)(1+b_2)} - 1.$$

So we compute

$$\begin{aligned}
&\mathrm{Tr}((\sqrt{f_1f_2}(A)\sqrt{g_1g_2}(B))^2)\\
&\quad = \mathrm{Tr}((I+aP)(I+bQ)(I+aP)(I+bQ))\\
&\quad = \mathrm{Tr}(I) + (2a+a^2)\mathrm{Tr}(P) + (2b+b^2)\mathrm{Tr}(Q)\\
&\qquad + (4ab+2a^2b+2ab^2)\mathrm{Tr}(PQ) + a^2b^2\mathrm{Tr}((PQ)^2)
\end{aligned}$$

and

$$\begin{aligned}
&\mathrm{Tr}(f_1(A)g_1(B)f_2(A)g_2(B))\\
&\quad = \mathrm{Tr}((I+a_1P)(I+b_1Q)(I+a_2P)(I+b_2Q))\\
&\quad = \mathrm{Tr}(I) + (a_1+a_2+a_1a_2)\mathrm{Tr}(P) + (b_1+b_2+b_1b_2)\mathrm{Tr}(Q)\\
&\qquad + [(a_1+a_2)(b_1+b_2) + a_1a_2(b_1+b_2) + (a_1+a_2)b_1b_2]\mathrm{Tr}(PQ)\\
&\qquad + a_1a_2b_1b_2\mathrm{Tr}((PQ)^2).
\end{aligned}$$

Direct computations give

$$2a + a^2 = a_1 + a_2 + a_1a_2,$$

$$2b + b^2 = b_1 + b_2 + b_1b_2,$$

and

$$\begin{aligned}
&4ab + 2a^2b + 2ab^2 + a^2b^2\\
&\quad = (1+a)^2(1+b)^2 - (1+a)^2 - (1+b)^2 + 1\\
&\quad = (a_1+a_2)(b_1+b_2) + a_1a_2(b_1+b_2) + (a_1+a_2)b_1b_2 + a_1a_2b_1b_2.
\end{aligned}$$

Furthermore, since

$$(1+a_1)(1+a_2) - (1+\sqrt{a_1a_2})^2 = a_1 + a_2 - 2\sqrt{a_1a_2} \geqslant 0,$$

we get $\sqrt{a_1a_2} \leqslant \sqrt{(1+a_1)(1+a_2)} - 1$, so $a_1a_2 \leqslant a^2$. Similarly $b_1b_2 \leqslant b^2$ and hence

$$a_1a_2b_1b_2 \leqslant a^2b^2.$$

Combining the above estimates we obtain

$$\begin{aligned}&\mathrm{Tr}(f_1(A)g_1(B)f_2(A)g_2(B)) - \mathrm{Tr}((\sqrt{f_1f_2}(A)\sqrt{g_1g_2}(B))^2)\\&\quad = (a^2b^2 - a_1a_2b_1b_2)[\mathrm{Tr}(PQ) - \mathrm{Tr}((PQ)^2)] \geqslant 0,\end{aligned}$$

as desired. □

By Theorems 4.1 and 4.2 we have:

COROLLARY 4.3. *Let A,B be positive semidefinite matrices and assume that both A,B have at most two different eigenvalues. Then:*

(i) $0 \leqslant \mathrm{Tr}(A^{p_1}B^{q_1}A^{p_2}B^{q_2}\cdots A^{p_K}B^{p_K}) \leqslant \mathrm{Tr}(AB)$ *for* $p_i,q_i \geqslant 0$ *with* $\sum_{i=1}^K p_i = \sum_{i=1}^K q_i = 1$.

(ii) $\mathrm{Tr}((A^{1/2}B^{1/2})^2) \leqslant \mathrm{Tr}(A^pB^qA^{1-p}B^{1-q}) \leqslant \mathrm{Tr}(AB)$ *for* $0 \leqslant p,q \leqslant 1$.

COROLLARY 4.4. *Let A,B be 2×2 positive semidefinite matrices. Then:*

(i) $0 \leqslant \mathrm{Tr}(A^{p_1}B^{q_1}A^{p_2}B^{q_2}\cdots A^{p_K}B^{p_K}) \leqslant \mathrm{Tr}(AB)$ *for* $p_i,q_i \geqslant 0$ *with* $\sum_{i=1}^K p_i = \sum_{i=1}^K q_i = 1$.

(ii) $\mathrm{Tr}((A^{1/2}B^{1/2})^2) \leqslant \mathrm{Tr}(A^pB^qA^{1-p}B^{1-q}) \leqslant \mathrm{Tr}(AB)$ *for* $0 \leqslant p,q \leqslant 1$.

THEOREM 4.5. *Let A,B be positive semidefinite matrices. Assume that A has at most two different eigenvalues and B has at most three different eigenvalues. If f_1,f_2,g_1,g_2 are nonnegative increasing functions on $[0,\infty)$, then*

$$\mathrm{Tr}(f_1(A)g_1(B)f_2(A)g_2(B)) \leqslant \mathrm{Tr}(f_1(A)f_2(A)g_1(B)g_2(B)). \tag{4.1}$$

In particular,

$$\mathrm{Tr}(A^pB^qA^{1-p}B^{1-q}) \leqslant \mathrm{Tr}(AB) \quad \textit{for} \quad 0 \leqslant p,q \leqslant 1.$$

Proof. We may assume that

$$f_1(A) = I + a_1P, \quad f_2(A) = I + a_2P,$$

$$g_1(B) = I + b_1^+Q_+ - b_1^-Q_-, \quad g_2(B) = I + b_2^+Q_+ - b_2^-Q_-,$$

where P,Q_+,Q_- are orthogonal projections with $Q_+ \perp Q_-$ and $a_1,a_2,b_1^+,b_2^+ \geqslant 0$, $0 \leqslant b_1^-,b_2^- \leqslant 1$. Then we estimate

$$\begin{aligned}&\mathrm{Tr}(f_1(A)f_2(A)g_1(B)g_2(B)) - \mathrm{Tr}(f_1(A)g_1(B)f_2(A)g_2(B))\\&\quad = a_1a_2b_1^+b_2^+(\mathrm{Tr}(PQ_+) - \mathrm{Tr}((PQ_+)^2) + a_1a_2b_1^-b_2^-(\mathrm{Tr}(PQ_-) - \mathrm{Tr}((PQ_-)^2)\\&\qquad\quad + a_1a_2(b_1^+b_2^- + b_1^-b_2^+)\mathrm{Tr}(PQ_+PQ_-)\\&\quad \geqslant 0\end{aligned}$$

as in the proof of Theorem 4.2. □

Note that there are 3×3 positive semidefinite matrices A,B and nonnegative increasing functions f_i,g_i ($i = 1,2$) such that

$$\mathrm{Tr}(f_1(A)g_1(B)f_2(A)g_2(B)) > \mathrm{Tr}(f_1(A)f_2(A)g_1(B)g_2(B)).$$

For example, let

$$A = \begin{bmatrix} \frac{3}{2} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{2} \end{bmatrix} = I + \frac{1}{2}P - \frac{1}{2}Q$$

with orthogonal projections

$$P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad Q = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Choose a unitary matrix

$$U = \begin{bmatrix} \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{6}} & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{6}} \end{bmatrix}$$

and put $B = UAU^*$. For $0 < a < 1$ take increasing functions f_1, f_2 such that

$$f_1\left(\frac{1}{2}\right) = 1 - a^2, \quad f_1(1) = 1, \quad f_1\left(\frac{3}{2}\right) = 1 + a,$$
$$f_2\left(\frac{1}{2}\right) = 1 - a, \quad f_2(1) = 1, \quad f_2\left(\frac{3}{2}\right) = 1 + a^2.$$

Then $f_1(A) = I + aP - a^2Q$, $f_2(A) = I + a^2P - aQ$ and $f_1(B) = Uf_1(A)U^* = I + aR - a^2S$, $f_2(B) = Uf_2(A)U^* = I + a^2R - aS$ with orthogonal projections $R = UPU^*$ and $S = UQU^*$. We compute

$$\begin{aligned}\varphi(a) &= \mathrm{Tr}(f_1(A)f_2(A)f_1(B)f_2(B)) - \mathrm{Tr}(f_1(A)f_1(B)f_2(A)f_2(B)) \\ &= -\frac{1}{18}a^8 + \frac{1}{4}a^7 + \frac{11}{18}a^6 + \frac{1}{4}a^5 - \frac{1}{18}a^4 \\ &= -\frac{1}{36}a^4(a+1)^2(2a^2 - 13a + 2).\end{aligned}$$

Hence we get $\varphi(a) < 0$ if $0 < a < \frac{13-3\sqrt{17}}{4}$. In this way, we notice that the assumption in Theorem 4.5 is rather optimal for (4.1) to hold for all nonnegative increasing f_i, g_i.

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