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ON GENERALIZATIONS OF OSTROWSKI INEQUALITY VIA SOME EULER–TYPE IDENTITIES

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Abstract. Some generalizations of Ostrowski inequality are given, by using some Euler-type identities.

1. Introduction

The following Ostrowski inequality is well known [1]:

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(t)dt \right| \leqslant \left[\frac{1}{4} + \frac{\left(x-\frac{a+b}{2}\right)^2}{(b-a)^2} \right](b-a)M, \quad x \in [a,b], \tag{1.1}$$

where $f : [a,b] \to \mathbf{R}$ is a differentiable function such that $|f'(x)| \leqslant M$, for every $x \in [a,b]$. It has been generalized over the last years in a number of ways.

For every function $f : [a,b] \to \mathbf{R}$ with n $(\geqslant 1)$ continuous derivatives and for every $x \in [a,b]$ the following formula is valid [2, p.17]:

$$f(x) = \frac{1}{b-a}\int_a^b f(t)dt + T_{n-1}(x) + R_n(x) \tag{1.2}$$

where

$$T_m(x) = \sum_{k=1}^{m} \frac{(b-a)^{k-1}}{k!} B_k\left(\frac{x-a}{b-a}\right)\left[f^{(k-1)}(b) - f^{(k-1)}(a)\right] \tag{1.3}$$

with convention $T_0(x) = 0$, and

$$R_n(x) = -\frac{(b-a)^{n-1}}{n!}\int_a^b \left[B_n^*\left(\frac{x-t}{b-a}\right) - B_n\left(\frac{x-a}{b-a}\right)\right] f^{(n)}(t)dt.$$

Here, $B_k(x)$, $k \geqslant 0$, are the Bernoulli polynomials, B_k, $k \geqslant 0$, the Bernoulli numbers, and $B_k^*(x)$, $k \geqslant 0$, are periodic functions of period one, related to the Bernoulli polynomials as

$$B_k^*(x) = B_k(x), \; 0 \leqslant x < 1$$

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and

$$B_k^*(x+1) = B_k^*(x),\ x \in \mathbf{R}.$$

To make reading easier let us recall here some properties of the Bernoulli polynomials, used below [3, 23.1.]. First

$$B_0(x) = 1, \quad B_1(x) = x - \frac{1}{2}, \quad B_2(x) = x^2 - x + \frac{1}{6},$$

and

$$B_n'(x) = nB_{n-1}(x),\ n \in \mathbf{N}.$$

In particular this means that $B_0^* = 1$, B_1^* is a discontinuous function with a jump of -1 at each integer, and B_k^*, $k \geqslant 2$, is a continuous function. If $n = 2r$ then $B_{2r}(x)$ has its maximal and minimal values at the points 0, 1 and $\frac{1}{2}$. Note that

$$B_{2r}(0) = B_{2r}(1) = B_{2r},$$

and

$$B_{2r}\left(\frac{1}{2}\right) = -(1 - 2^{1-2r})B_{2r}.$$

For $n = 2r+1$, and $r \geqslant 1$, we have

$$0 < (-1)^{r+1}B_{2r+1}(x) < \frac{2(2r+1)!}{(2\pi)^{2r+1}(1-2^{-2r})},\ 0 < x < \frac{1}{2}$$

and

$$B_{2r+1}(x) = -B_{2r+1}(1-x).$$

In this paper we shall give two modified versions of the identity (1.2), and using them we shall prove some generalizations of the Ostrowski inequality.

2. Some integral identities

LEMMA 1. *Let* $a, b \in \mathbf{R}$, $a < b$, $x \in [a,b]$ *and* $\varphi : \mathbf{R} \to \mathbf{R}$ *be defined by*

$$\varphi(t) = B_1^*\left(\frac{x-t}{b-a}\right).$$

Then for every continuous function $F : [a,b] \to \mathbf{R}$ *we have*

$$\int_{[a,b]} F(t)d\varphi(t) = -\frac{1}{b-a}\int_a^b F(t)dt + F(x),\ \textit{for}\ a < x < b$$

and

$$\int_{[a,b]} F(t)d\varphi(t) = -\frac{1}{b-a}\int_a^b F(t)dt + F(a),\ \textit{for}\ x = a\ \textit{or}\ x = b,$$

with Riemann-Stieltjes integrals on the left hand sides.

Proof. If $a < x < b$ the function φ is differentiable on $[a,b]\backslash\{x\}$ and its derivative is equal to $\frac{-1}{b-a}$, since $B_1(t) = t - \frac{1}{2}$. Further, it has a jump of $\varphi(x+0) - \varphi(x-0) = 1$ at x, which gives the first formula. For $x = a$ or $x = b$ the function φ is differentiable on (a,b) and its derivative is equal to $\frac{-1}{b-a}$. Further, it has jump of $\varphi(a+0) - \varphi(a) = 1$ at the point a, while $\varphi(b) - \varphi(b-0) = 0$, which gives the second formula. □

THEOREM 1. *Let $f : [a,b] \to \mathbf{R}$ be such that $f^{(n-1)}$ is continuous function of bounded variation on $[a,b]$ for some $n \geqslant 1$. Then for every $x \in [a,b]$*

$$f(x) = \frac{1}{b-a}\int_a^b f(t)dt + T_n(x) + R_n^1(x), \tag{2.1}$$

where $T_n(x)$ is defined by (1.3) and

$$R_n^1(x) = -\frac{(b-a)^{n-1}}{n!}\int_{[a,b]} B_n^*\left(\frac{x-t}{b-a}\right) df^{(n-1)}(t).$$

Proof. Consider the integrals

$$I_k(x) := \frac{(b-a)^{k-1}}{k!}\int_{[a,b]} B_k^*\left(\frac{x-t}{b-a}\right) df^{(k-1)}(t) \quad (1 \leqslant k \leqslant n).$$

By partial integration we have

$$\begin{aligned} I_k(x) &= \frac{(b-a)^{k-1}}{k!} B_k^*\left(\frac{x-t}{b-a}\right) f^{(k-1)}(t)\Big|_a^b - \\ &\quad \frac{(b-a)^{k-1}}{k!}\int_{[a,b]} f^{(k-1)}(t) dB_k^*\left(\frac{x-t}{b-a}\right). \end{aligned} \tag{2.2}$$

For every $k \geqslant 1$ and every $x \in [a,b)$ we have

$$B_k^*\left(\frac{x-b}{b-a}\right) = B_k^*\left(\frac{x-a}{b-a} - 1\right) = B_k^*\left(\frac{x-a}{b-a}\right) = B_k\left(\frac{x-a}{b-a}\right). \tag{2.3}$$

Also, for $k \geqslant 2$ the above formula is valid for every $x \in [a,b]$. The identity (2.2) for $k = 1$ becomes

$$I_1(x) = B_1^*\left(\frac{x-t}{b-a}\right) f(t)\Big|_a^b - \int_{[a,b]} f(t) dB_1^*\left(\frac{x-t}{b-a}\right).$$

If $x \in [a,b)$, then using Lemma 1 and (2.3) we get

$$\begin{aligned} I_1(x) &= B_1\left(\frac{x-a}{b-a}\right)[f(b) - f(a)] + \frac{1}{b-a}\int_a^b f(t)dt - f(x) \\ &= T_1(x) + \frac{1}{b-a}\int_a^b f(t)dt - f(x). \end{aligned}$$

If $x = b$, then using Lemma 1 we get

$$\begin{aligned} I_1(b) &= B_1^*(0)f(b) - B_1^*(1)f(a) + \frac{1}{b-a}\int_a^b f(t)dt - f(a) \\ &= -\frac{1}{2}f(b) + \frac{1}{2}f(a) + \frac{1}{b-a}\int_a^b f(t)dt - f(a) \\ &= \frac{1}{2}[f(b) - f(a)] + \frac{1}{b-a}\int_a^b f(t)dt - f(b) \\ &= T_1(b) + \frac{1}{b-a}\int_a^b f(t)dt - f(b). \end{aligned}$$

So, for every $x \in [a,b]$ we have

$$I_1(x) = T_1(x) + \frac{1}{b-a}\int_a^b f(t)dt - f(x), \tag{2.4}$$

which is just the identity (2.1) for $n = 1$, since $I_1(x) = -R_1^1(x)$. Further, for every $k \geqslant 2$

$$\frac{d}{dt}B_k^*\left(\frac{x-t}{b-a}\right) = -\frac{k}{b-a}B_{k-1}^*\left(\frac{x-t}{b-a}\right),$$

except for t from the discrete set $x + (b-a)\mathbf{Z} \subset \mathbf{R}$, since the Bernoulli polynomials satisfy $\frac{d}{dt}B_k(t) = kB_{k-1}(t)$. Using the above formula and the fact that $B_k^*\left(\frac{x-t}{b-a}\right)$ is continuous for $k \geqslant 2$, we get

$$\begin{aligned} &-\frac{(b-a)^{k-1}}{k!}\int_{[a,b]} f^{(k-1)}(t)dB_k^*\left(\frac{x-t}{b-a}\right) \\ &\quad = \frac{(b-a)^{k-2}}{(k-1)!}\int_a^b B_{k-1}^*\left(\frac{x-t}{b-a}\right)f^{(k-1)}(t)dt \\ &\quad = \frac{(b-a)^{k-2}}{(k-1)!}\int_{[a,b]} B_{k-1}^*\left(\frac{x-t}{b-a}\right)df^{(k-2)}(t) \\ &\quad = I_{k-1}(x). \end{aligned}$$

Using this formula and (2.3), from (2.2) we get the identity

$$I_k(x) = \frac{(b-a)^{k-1}}{k!}B_k\left(\frac{x-a}{b-a}\right)\left[f^{(k-1)}(b) - f^{(k-1)}(a)\right] + I_{k-1}(x),$$

which holds for $k = 2, \cdots, n$ and for every $x \in [a,b]$. So, for $n \geqslant 2$ and for every $x \in [a,b]$ we get

$$I_n(x) = \sum_{k=2}^{n}\frac{(b-a)^{k-1}}{k!}B_k\left(\frac{x-a}{b-a}\right)\left[f^{(k-1)}(b) - f^{(k-1)}(a)\right] + I_1(x),$$

which, in combination with (2.4) yields

$$I_n(x) = T_n(x) + \frac{1}{b-a}\int_a^b f(t)dt - f(x).$$

This proves our assertion, since $I_n(x) = -R_n^1(x)$.

□

THEOREM 2. *Let $f : [a,b] \to \mathbf{R}$ be such that $f^{(n-1)}$ is continuous function of bounded variation on $[a,b]$ for some $n \geqslant 1$. Then for every $x \in [a,b]$*

$$f(x) = \frac{1}{b-a}\int_a^b f(t)dt + T_{n-1}(x) + R_n^2(x), \tag{2.5}$$

where $T_{n-1}(x)$ is defined by (1.3) and

$$R_n^2(x) = -\frac{(b-a)^{n-1}}{n!}\int_{[a,b]}\left[B_n^*\left(\frac{x-t}{b-a}\right) - B_n\left(\frac{x-a}{b-a}\right)\right] df^{(n-1)}(t).$$

Proof. Note that

$$\begin{aligned} R_n^2(x) &= R_n^1(x) + \frac{(b-a)^{n-1}}{n!}B_n\left(\frac{x-a}{b-a}\right)\int_{[a,b]} df^{(n-1)}(t) \\ &= R_n^1(x) + \frac{(b-a)^{n-1}}{n!}B_n\left(\frac{x-a}{b-a}\right)\left[f^{(n-1)}(b) - f^{(n-1)}(a)\right] \\ &= R_n^1(x) + T_n(x) - T_{n-1}(x), \end{aligned}$$

that is

$$R_n^1(x) = -T_n(x) + T_{n-1}(x) + R_n^2(x),$$

and apply the identity (2.1) to obtain the identity (2.5).

□

3. Generalizations of the Ostrowski inequality

THEOREM 3. *Let $f : [a,b] \to \mathbf{R}$ be such that $f^{(n-1)}$ is an L-Lipschitzian function on $[a,b]$ for some $n \geqslant 1$. Then*

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(t)dt - T_{n-1}(x)\right| \leqslant \frac{(b-a)^n}{n!}L\int_0^1\left|B_n(t) - B_n\left(\frac{x-a}{b-a}\right)\right| dt,$$

for every $x \in [a,b]$.

Proof. For integrable function $F : [a,b] \to \mathbf{R}$ we have

$$\left|\int_{[a,b]} F(t)df^{(n-1)}(t)\right| \leqslant L\int_a^b |F(t)|\, dt,$$

since $f^{(n-1)}$ is L-Lipschitzian function.

Let us apply this estimation to the formula (2.5). We have

$$\begin{aligned}\left|f(x)-\frac{1}{b-a}\int_a^b f(t)dt-T_{n-1}(x)\right| \\ &=\left|R_n^2(x)\right| \\ &\leqslant \frac{(b-a)^{n-1}}{n!}L\int_a^b\left|B_n^*\left(\frac{x-t}{b-a}\right)-B_n\left(\frac{x-a}{b-a}\right)\right|dt.\end{aligned}$$

Because the function B_n^* has period 1, we have

$$\int_0^1 |B_n^*(y+t)-z|\,dt=\int_0^1 |B_n^*(t)-z|\,dt=\int_0^1 |B_n(t)-z|\,dt,$$

for every $y,z\in\mathbf{R}$. Therefore we have

$$\int_a^b\left|B_n^*\left(\frac{x-t}{b-a}\right)-B_n\left(\frac{x-a}{b-a}\right)\right|dt=(b-a)\int_0^1\left|B_n(t)-B_n\left(\frac{x-a}{b-a}\right)\right|dt,$$

which proves our assertion.

□

REMARK 1. Because of

$$\int_0^1\left|B_1(t)-B_1\left(\frac{x-a}{b-a}\right)\right|dt=\int_0^1\left|t-\frac{x-a}{b-a}\right|dt=\frac{1}{4}+\frac{(x-\frac{a+b}{2})^2}{(b-a)^2},$$

the inequality from the theorem above for $n=1$ reduces to

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(t)dt\right|\leqslant\left[\frac{1}{4}+\frac{(x-\frac{a+b}{2})^2}{(b-a)^2}\right](b-a)L,$$

that is to the Ostrowski inequality for a function f which is L-Lipschitzian on $[a,b]$.

COROLLARY 1. *Let $f:[a,b]\to\mathbf{R}$ be such that f' is an L-Lipschitzian function on $[a,b]$. Then*

$$\begin{aligned}\left|f(x)-\frac{1}{b-a}\int_a^b f(t)dt-\left(x-\frac{a+b}{2}\right)\frac{f(b)-f(a)}{b-a}\right| \\ &\leqslant\frac{1}{2}\left[\frac{8}{3}\delta^3(x)-\delta^2(x)+\frac{1}{12}\right](b-a)^2L \\ &\leqslant\frac{1}{12}(b-a)^2L, \qquad (3.1)\end{aligned}$$

for every $x\in[a,b]$, where

$$\delta(x):=\frac{\left|x-\frac{a+b}{2}\right|}{b-a},\quad x\in[a,b].$$

Proof. For $n = 2$ we have

$$\begin{aligned} T_1(x) &= B_1\left(\frac{x-a}{b-a}\right)[f(b)-f(a)] \\ &= \left(\frac{x-a}{b-a}-\frac{1}{2}\right)[f(b)-f(a)] \\ &= \left(x-\frac{a+b}{2}\right)\frac{f(b)-f(a)}{b-a}. \end{aligned}$$

Also, by simple calculation we get

$$\begin{aligned} &\int_0^1 \left|B_2(t)-B_2\left(\frac{x-a}{b-a}\right)\right| dt \\ &\quad = \int_0^1 \left|t^2-t-\left(\frac{x-a}{b-a}\right)^2+\frac{x-a}{b-a}\right| dt \\ &\quad = \frac{8}{3}\delta^3(x)-\delta^2(x)+\frac{1}{12}, \end{aligned}$$

so that the inequality from Theorem 3 for $n = 2$ reduces to the first inequality in (3.1). Further, consider the function

$$g_1(t) = \frac{8}{3}t^3 - t^2 + \frac{1}{12}, \quad t \in \left[0, \frac{1}{2}\right].$$

We have $g_1(0) = \frac{1}{12}$, $g_1\left(\frac{1}{2}\right) = \frac{1}{6}$ and $g_1(t)$ achieves the minimum at $t = \frac{1}{4}$, $g_1\left(\frac{1}{4}\right) = \frac{1}{16}$, so that

$$\frac{1}{16} \leqslant g_1(t) \leqslant \frac{1}{6}, \quad t \in \left[0, \frac{1}{2}\right].$$

It is obvious that $\delta(x) \in \left[0, \frac{1}{2}\right]$ for all $x \in [a,b]$ and hence

$$\frac{1}{16} \leqslant \frac{8}{3}\delta^3(x)-\delta^2(x)+\frac{1}{12} \leqslant \frac{1}{6}, \quad x \in [a,b],$$

which implies the second inequality in (3.1).

□

REMARK 2. For $x = a$ or $x = b$ we have $\delta(a) = \delta(b) = \frac{1}{2}$, and the first inequality in (3.1) reduces to

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(t)dt\right| \leqslant \frac{1}{12}(b-a)^2 L.$$

For $x = \frac{a+b}{2}$ we have $\delta(\frac{a+b}{2}) = 0$, and the first inequality in (3.1) reduces to

$$\left|f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(t)dt\right| \leqslant \frac{1}{24}(b-a)^2 L.$$

Also, from the proof above we see that the first inequality in (3.1) gives the best estimate when $\delta(x) = \frac{1}{4}$, that is when $x = \frac{3a+b}{4}$ or $x = \frac{a+3b}{4}$. In these two cases we get the inequalities

$$\left| f\left(\frac{3a+b}{4}\right) - \frac{1}{b-a}\int_a^b f(t)dt + \frac{1}{4}\left[f(a)+f(b)\right]\right| \leqslant \frac{1}{32}(b-a)^2 L,$$

and

$$\left| f\left(\frac{a+3b}{4}\right) - \frac{1}{b-a}\int_a^b f(t)dt - \frac{1}{4}\left[f(a)+f(b)\right]\right| \leqslant \frac{1}{32}(b-a)^2 L.$$

Adding these two inequalities with use of the triangle inequality, we obtain the following inequality

$$\left| \frac{1}{2}\left[f\left(\frac{3a+b}{4}\right) + f\left(\frac{a+3b}{4}\right)\right] - \frac{1}{b-a}\int_a^b f(t)dt\right| \leqslant \frac{1}{32}(b-a)^2 L.$$

REMARK 3. If $f : [a,b] \to \mathbf{R}$ has a bounded second derivative on $[a,b]$, then Corollary 1 applies with

$$L = M_2 := \sup_{t\in[a,b]} \left|f''(t)\right|.$$

On the other side, Dragomir and Barnett [6] proved that for every $x \in [a,b]$ the following inequalities are valid:

$$\begin{aligned} &\left| f(x) - \frac{1}{b-a}\int_a^b f(t)dt - \left(x - \frac{a+b}{2}\right)\frac{f(b)-f(a)}{b-a}\right| \\ &\qquad \leqslant \frac{1}{2}\left[\left(\delta^2(x) + \frac{1}{4}\right)^2 + \frac{1}{12}\right](b-a)^2 M_2 \\ &\qquad \leqslant \frac{1}{6}(b-a)^2 M_2. \end{aligned} \tag{3.2}$$

Matić, Pečarić and Ujević [5] partially improved this result. They proved that for every $x \in [a,b]$

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(t)dt - \left(x - \frac{a+b}{2}\right)\frac{f(b)-f(a)}{b-a}\right| \leqslant \frac{1}{12}(b-a)^2 M_2. \tag{3.3}$$

Evidently, the second inequality in (3.1) extends the inequality (3.3) to a wider class of functions. Moreover, the first inequality in (3.1) is much better than the first inequality in (3.2). Namely, if we consider the function $g(t) = g_2(t) - g_1(t)$, where $g_2(t) = \left(t^2 + \frac{1}{4}\right)^2 + \frac{1}{12}$ and $g_1(t)$ is as in the proof of Corollary 1, then

$$g(t) = t^4 - \frac{8}{3}t^3 + \frac{3}{2}t^2 + \frac{1}{16}.$$

We have $g(0) = \frac{1}{16}$, $g\left(\frac{1}{2}\right) = \frac{1}{6}$ and $g(t)$ is strictly increasing on $\left[0, \frac{1}{2}\right]$, which shows that $\frac{1}{16} \leqslant g_2(t) - g_1(t) \leqslant \frac{1}{6}$, for every $t \in \left[0, \frac{1}{2}\right]$. This implies that

$$\frac{8}{3}\delta^3(x) - \delta^2(x) + \frac{1}{12} < \left(\delta^2(x) + \frac{1}{4}\right)^2 + \frac{1}{12}, \text{ for every } x \in [a,b].$$

For example, for $x = a$ or $x = b$ the first inequality in (3.2) yields the trapezoid inequality

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(t)dt\right| \leqslant \frac{1}{6}(b-a)^2 M_2,$$

and for $x = \frac{a+b}{2}$ we get the following midpoint inequality

$$\left|f\left(\frac{a+b}{2}\right) - \frac{1}{b-a}\int_a^b f(t)dt\right| \leqslant \frac{7}{96}(b-a)^2 M_2.$$

The corresponding trapezoid and midpoint inequalities established in Remark 2 are sharper and hold for a wider class of functions.

THEOREM 4. *Let $f : [a,b] \to \mathbf{R}$ be such that $f^{(n-1)}$ is an L-Lipschitzian function on $[a,b]$ for some $n \geqslant 1$. Then*

$$\left|f(x) - \frac{1}{b-a}\int_a^b f(t)dt - T_n(x)\right| \leqslant \frac{(b-a)^n}{n!} L \int_0^1 |B_n(t)|\, dt,$$

for every $x \in [a,b]$.

Proof. By using (2.1), as in the proof of the Theorem 3, we have

$$\left|f(x) - \frac{1}{b-a}\int_a^b f(t)dt - T_n(x)\right| = \left|R_n^1(x)\right| \leqslant \frac{(b-a)^{n-1}}{n!} L \int_a^b \left|B_n^*\left(\frac{x-t}{b-a}\right)\right| dt,$$

and also

$$\int_a^b \left|B_n^*\left(\frac{x-t}{b-a}\right)\right| dt = (b-a)\int_0^1 |B_n(t)|\, dt,$$

which proves our assertion. □

COROLLARY 2. *Let $f : [a,b] \to \mathbf{R}$ be an L-Lipschitzian function on $[a,b]$. Then*

$$\left|f(x) - \frac{1}{b-a}\int_a^b f(t)dt - \left(x - \frac{a+b}{2}\right)\frac{f(b)-f(a)}{b-a}\right| \leqslant \frac{b-a}{4} L, \tag{3.4}$$

for every $x \in [a,b]$.

Proof. Note that

$$\int_0^1 |B_1(t)|\,dt = \int_0^1 \left|t - \frac{1}{2}\right| dt = \frac{1}{4},$$

and apply Theorem 4 with $n = 1$. □

REMARK 4. Matić, Pečarić and Ujević [4] proved that, for every $x \in [a,b]$,

$$\left|f(x) - \frac{1}{b-a}\int_a^b f(t)dt - \left(x - \frac{a+b}{2}\right)\frac{f(b)-f(a)}{b-a}\right| \leqslant \frac{b-a}{4\sqrt{3}}(\Gamma - \gamma), \qquad (3.5)$$

where γ and Γ are constants such that $\gamma \leqslant f'(x) \leqslant \Gamma$ for all $x \in [a,b]$. On the other side, if f has a bounded derivative f' on $[a,b]$, then Corollary 2 applies with

$$L = M_1 := \sup_{t\in[a,b]} |f'(t)|,$$

and we can try to compare the two inequalities (3.4) and (3.5). If for example $\gamma \geqslant 0$ and $\Gamma = M_1$, then $\frac{b-a}{4\sqrt{3}}(\Gamma - \gamma) \leqslant \frac{b-a}{4\sqrt{3}}M_1 < \frac{b-a}{4}M_1$. On the contrary, if for example $\gamma = -M_1$ and $\Gamma = M_1$, then $\frac{b-a}{4\sqrt{3}}(\Gamma - \gamma) = \frac{b-a}{2\sqrt{3}}M_1 > \frac{b-a}{4}M_1$. We conclude that neither of the inequalities (3.4) and (3.5) is better than another one, but (3.4) should be preferred since it holds for a wider class of functions than (3.5) does.

THEOREM 5. *Let $f : [a,b] \to \mathbf{R}$ be such that $f^{(n-1)}$ is continuous function of bounded variation on $[a,b]$ for some $n \geqslant 1$. Then*

$$\left|f(x) - \frac{1}{b-a}\int_a^b f(t)dt - T_{n-1}(x)\right| \leqslant \frac{(b-a)^{n-1}}{n!} \max_{t\in[0,1]} \left|B_n(t) - B_n\left(\frac{x-a}{b-a}\right)\right| \cdot V_a^b(f^{(n-1)}),$$

for every $x \in [a,b]$, where $V_a^b(f^{(n-1)})$ is the total variation of $f^{(n-1)}$ on $[a,b]$. Moreover, for $n = 2r$ we have

$$\left|f(x) - \frac{1}{b-a}\int_a^b f(t)dt - T_{2r-1}(x)\right| \leqslant \frac{(b-a)^{2r-1}}{(2r)!}\left[(1-2^{-2r})|B_{2r}| + \left|2^{-2r}B_{2r} - B_{2r}\left(\frac{x-a}{b-a}\right)\right|\right] \cdot V_a^b(f^{(2r-1)}),$$

while for $n = 2r+1$, $r \geqslant 1$, we have

$$\left|f(x) - \frac{1}{b-a}\int_a^b f(t)dt - T_{2r}(x)\right| \leqslant \frac{(b-a)^{2r}}{(2r+1)!}\left[\frac{2(2r+1)!}{(2\pi)^{2r+1}(1-2^{-2r})} + \left|B_{2r+1}\left(\frac{x-a}{b-a}\right)\right|\right] \cdot V_a^b(f^{(2r)}).$$

Proof. If $F:[a,b]\to\mathbf{R}$ is bounded on $[a,b]$ and the Riemann-Stieltjes integral

$$\int_{[a,b]} F(t)df^{(n-1)}(t)$$

exists, then

$$\left|\int_{[a,b]} F(t)df^{(n-1)}(t)\right| \leqslant \max_{t\in[a,b]}|F(t)|\cdot V_a^b(f^{(n-1)}).$$

Let us apply this estimation to the formula (2.5). We have

$$\begin{aligned}
&\left|f(x)-\frac{1}{b-a}\int_a^b f(t)dt-T_{n-1}(x)\right|\\
&\quad=\left|R_n^2(x)\right|\\
&\quad\leqslant\frac{(b-a)^{n-1}}{n!}\max_{t\in[a,b]}\left|B_n^*\left(\frac{x-t}{b-a}\right)-B_n\left(\frac{x-a}{b-a}\right)\right|\cdot V_a^b(f^{(n-1)})\\
&\quad=\frac{(b-a)^{n-1}}{n!}\max_{t\in[0,1]}\left|B_n(t)-B_n\left(\frac{x-a}{b-a}\right)\right|\cdot V_a^b(f^{(n-1)}),
\end{aligned}$$

which proves the first inequality. If $n=2r$ then $B_{2r}(t)$ has its minimum and maximum at points $0,\ 1$ and $\frac{1}{2}$. Since $B_{2r}(0)=B_{2r}(1)=B_{2r}$, we have

$$\max_{t\in[0,1]}\left|B_{2r}(t)-B_{2r}\left(\frac{x-a}{b-a}\right)\right|=\max\left\{|A|,|B|\right\},$$

where

$$A=B_{2r}-B_{2r}\left(\frac{x-a}{b-a}\right),\ \ B=B_{2r}\left(\frac{1}{2}\right)-B_{2r}\left(\frac{x-a}{b-a}\right).$$

Also, $A\leqslant 0\leqslant B$ or $B\leqslant 0\leqslant A$, so that

$$\max\left\{|A|,|B|\right\}=\frac{1}{2}\left(|A+B|+|A-B|\right).$$

Using the above formula and

$$B_{2r}\left(\frac{1}{2}\right)=-(1-2^{1-2r})B_{2r},$$

we get

$$\max_{t\in[0,1]}\left|B_{2r}(t)-B_{2r}\left(\frac{x-a}{b-a}\right)\right|=(1-2^{-2r})\left|B_{2r}\right|+\left|2^{-2r}B_{2r}-B_{2r}\left(\frac{x-a}{b-a}\right)\right|,$$

which gives the second inequality. For $n=2r+1$, and $r\geqslant 1$, we have

$$0<(-1)^{r+1}B_{2r+1}(x)<\frac{2(2r+1)!}{(2\pi)^{2r+1}(1-2^{-2r})},\ \ 0<x<\frac{1}{2},$$

and

$$B_{2r+1}(x)=-B_{2r+1}(1-x),$$

which implies

$$\max_{t\in[0,1]}\left|B_{2r+1}(t)-B_{2r+1}\left(\frac{x-a}{b-a}\right)\right|\leqslant\frac{2(2r+1)!}{(2\pi)^{2r+1}(1-2^{-2r})}+\left|B_{2r+1}\left(\frac{x-a}{b-a}\right)\right|.$$

This proves the third inequality.

□

REMARK 5. We have

$$\max_{t\in[0,1]}\left|B_1(t)-B_1\left(\frac{x-a}{b-a}\right)\right|=\max_{t\in[0,1]}\left|t-\frac{x-a}{b-a}\right|=\frac{1}{2}+\left|x-\frac{a+b}{2}\right|.$$

and the first inequality of the preceding theorem for $n=1$ becomes

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(t)dt\right|\leqslant\left[\frac{1}{2}+\left|x-\frac{a+b}{2}\right|\right]\cdot V_a^b(f).$$

COROLLARY 3. *Let $f:[a,b]\to\mathbf{R}$ be such that $f^{(n)}\in L_1[a,b]$ for some $n\geqslant 1$. Then*

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(t)dt-T_{n-1}(x)\right|$$
$$\leqslant\frac{(b-a)^{n-1}}{n!}\max_{t\in[0,1]}\left|B_n(t)-B_n\left(\frac{x-a}{b-a}\right)\right|\cdot\|f^{(n)}\|_1,$$

for every $x\in[a,b]$.
Moreover, for $n=2r$ we have

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(t)dt-T_{2r-1}(x)\right|$$
$$\leqslant\frac{(b-a)^{2r-1}}{(2r)!}\left[(1-2^{-2r})\,|B_{2r}|+\left|2^{-2r}B_{2r}-B_{2r}\left(\frac{x-a}{b-a}\right)\right|\right]\cdot\|f^{(2r)}\|_1,$$

while for $n=2r+1$, and $r\geqslant 1$, we have

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(t)dt-T_{2r}(x)\right|$$
$$\leqslant\frac{(b-a)^{2r}}{(2r+1)!}\left[\frac{2(2r+1)!}{(2\pi)^{2r+1}(1-2^{-2r})}+\left|B_{2r+1}\left(\frac{x-a}{b-a}\right)\right|\right]\cdot\|f^{(2r+1)}\|_1.$$

Proof. Note that in this case we have

$$V_a^b(f^{(n-1)})=\int_a^b\left|f^{(n)}(t)\right|dt=\|f^{(n)}\|_1,\tag{3.6}$$

and apply the preceding theorem.

□

THEOREM 6. *Let $f : [a,b] \to \mathbf{R}$ be such that $f^{(n-1)}$ is continuous function of bounded variation on $[a,b]$ for some $n \geqslant 1$. Then*

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(t)dt - T_n(x) \right| \leqslant \frac{(b-a)^{n-1}}{n!} \max_{t\in[0,1]} |B_n(t)| \cdot V_a^b(f^{(n-1)}),$$

for every $x \in [a,b]$.

Moreover, for $n = 2r$ we have

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(t)dt - T_{2r}(x) \right| \leqslant \frac{(b-a)^{2r-1}\,|B_{2r}|}{(2r)!} \cdot V_a^b(f^{(2r-1)}),$$

while for $n = 2r+1$, and $r \geqslant 1$ we have

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(t)dt - T_{2r+1}(x) \right| \leqslant \frac{2(b-a)^{2r}}{(2\pi)^{2r+1}(1-2^{-2r})} \cdot V_a^b(f^{(2r)}).$$

Proof. We use the identity (2.1) and apply the argument similar to that in the proof of the preceding theorem.

□

COROLLARY 4. *Let $f : [a,b] \to \mathbf{R}$ be such that $f^{(n)} \in L_1[a,b]$ for some $n \geqslant 1$. Then*

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(t)dt - T_n(x) \right| \leqslant \frac{(b-a)^{n-1}}{n!} \max_{t\in[0,1]} |B_n(t)| \cdot \|f^{(n)}\|_1,$$

for every $x \in [a,b]$.
Moreover, for $n = 2r$ we have

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(t)dt - T_{2r}(x) \right| \leqslant \frac{(b-a)^{2r-1}\,|B_{2r}|}{(2r)!} \|f^{(2r)}\|_1,$$

while for $n = 2r+1$, and $r \geqslant 1$, we have

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(t)dt - T_{2r+1}(x) \right| \leqslant \frac{2(b-a)^{2r}}{(2\pi)^{2r+1}(1-2^{-2r})} \|f^{(2r+1)}\|_1.$$

Proof. Use the formula (3.6) and apply Theorem 6.

□

THEOREM 7. *Let* $f:[a,b]\to\mathbf{R}$ *be such that* $f^{(n)}\in L_\infty[a,b]$ *for some* $n\geqslant 1$. *Then*

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(t)dt-T_{n-1}(x)\right| \leqslant \frac{(b-a)^n}{n!}\int_0^1\left|B_n(t)-B_n\left(\frac{x-a}{b-a}\right)\right|dt\cdot\|f^{(n)}\|_\infty,$$

for every $x\in[a,b]$.

Proof. Since $f^{(n)}$ exists on $[a,b]$, we have

$$R_n^2(x)=-\frac{(b-a)^{n-1}}{n!}\int_{[a,b]}\left[B_n^*\left(\frac{x-t}{b-a}\right)-B_n\left(\frac{x-a}{b-a}\right)\right]f^{(n)}(t)dt, \tag{3.7}$$

and from (2.5) we get the following estimate

$$\begin{aligned}\left|f(x)-\frac{1}{b-a}\int_a^b f(t)dt-T_{n-1}(x)\right| &\leqslant \frac{(b-a)^{n-1}}{n!}\int_a^b\left|B_n^*\left(\frac{x-t}{b-a}\right)-B_n\left(\frac{x-a}{b-a}\right)\right|dt\cdot\|f^{(n)}\|_\infty\\ &=\frac{(b-a)^n}{n!}\int_0^1\left|B_n(t)-B_n\left(\frac{x-a}{b-a}\right)\right|dt\cdot\|f^{(n)}\|_\infty,\end{aligned}$$

which proves our assertion.

□

THEOREM 8. *Let* $f:[a,b]\to\mathbf{R}$ *be such that* $f^{(n)}\in L_\infty[a,b]$ *for some* $n\geqslant 1$. *Then*

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(t)dt-T_n(x)\right|\leqslant\frac{(b-a)^n}{n!}\int_0^1|B_n(t)|\,dt\cdot\|f^{(n)}\|_\infty,$$

for every $x\in[a,b]$.

Proof. Analogously as in the preceding theorem, use the formula

$$R_n^1(x)=-\frac{(b-a)^{n-1}}{n!}\int_{[a,b]}B_n^*\left(\frac{x-t}{b-a}\right)f^{(n)}(t)dt, \tag{3.8}$$

and the identity (2.1).

□

THEOREM 9. *Let* $f:[a,b]\to\mathbf{R}$ *be such that* $f^{(n)}\in L_p[a,b]$ *for some* $n\geqslant 1$ *and* $1<p<\infty$. *Then*

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(t)dt-T_{n-1}(x)\right| \leqslant\frac{(b-a)^{n-1+1/q}}{n!}\left(\int_0^1\left|B_n(t)-B_n\left(\frac{x-a}{b-a}\right)\right|^q dt\right)^{1/q}\cdot\|f^{(n)}\|_p,$$

for every $x \in [a,b]$, *where* $1/p + 1/q = 1$.

Proof. Use the identity (2.5) and the formula (3.7) and apply the Hölder inequality to obtain

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(t)dt - T_{n-1}(x) \right|$$
$$\leqslant \frac{(b-a)^{n-1}}{n!}\int_a^b \left| B_n^*\left(\frac{x-t}{b-a}\right) - B_n\left(\frac{x-a}{b-a}\right)\right| \left| f^{(n)}(t)\right| dt$$
$$\leqslant \frac{(b-a)^{n-1}}{n!}\left(\int_a^b \left| B_n^*\left(\frac{x-t}{b-a}\right) - B_n\left(\frac{x-a}{b-a}\right)\right|^q dt\right)^{1/q} \cdot \|f^{(n)}\|_p$$
$$= \frac{(b-a)^{n-1+1/q}}{n!}\left(\int_0^1 \left| B_n(t) - B_n\left(\frac{x-a}{b-a}\right)\right|^q dt\right)^{1/q} \cdot \|f^{(n)}\|_p,$$

which proves our assertion.

□

COROLLARY 5. *Let* $f : [a,b] \to \mathbf{R}$ *be such that* $f^{(n)} \in L_2[a,b]$ *for some* $n \geqslant 1$. *Then*

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(t)dt - T_{n-1}(x) \right|$$
$$\leqslant \frac{(b-a)^{n-1/2}}{n!}\left(\frac{(n!)^2}{(2n)!}|B_{2n}| + B_n^2\left(\frac{x-a}{b-a}\right)\right)^{1/2} \cdot \|f^{(n)}\|_2,$$

for every $x \in [a,b]$.

Proof. By Theorem 9 for $p = 2$ we have

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(t)dt - T_{n-1}(x) \right|$$
$$\leqslant \frac{(b-a)^{n-1/2}}{n!}\left(\int_0^1 \left| B_n(t) - B_n\left(\frac{x-a}{b-a}\right)\right|^2 dt\right)^{1/2} \cdot \|f^{(n)}\|_2.$$

Further

$$\int_0^1 \left| B_n(t) - B_n\left(\frac{x-a}{b-a}\right)\right|^2 dt$$
$$= \int_0^1 B_n^2(t)dt - 2B_n\left(\frac{x-a}{b-a}\right)\int_0^1 B_n(t)dt + B_n^2\left(\frac{x-a}{b-a}\right).$$

The Bernoulli polynomials satisfy the relation [3, 23.1.12]:

$$\int_0^1 B_n(t)B_m(t)dt = (-1)^{n-1}\frac{n!m!}{(n+m)!}B_{n+m},$$

for every $n, m \geqslant 1$. Therefore, we have

$$\int_0^1 B_n^2(t)dt = (-1)^{n-1}\frac{n!^2}{(2n)!}B_{2n} = \frac{n!^2}{(2n)!}\left|B_{2n}\right|,$$

and

$$\int_0^1 B_n(t)dt = 0,\ n \geqslant 1.$$

Hence

$$\int_0^1 \left|B_n(t) - B_n\left(\frac{x-a}{b-a}\right)\right|^2 dt = \frac{(n!)^2}{(2n)!}\left|B_{2n}\right| + B_n^2\left(\frac{x-a}{b-a}\right),$$

which proves our assertion.

□

THEOREM 10. *Let $f : [a,b] \to \mathbf{R}$ be such that $f^{(n)} \in L_p[a,b]$ for some $n \geqslant 1$ and $1 < p < \infty$. Then*

$$\left|f(x) - \frac{1}{b-a}\int_a^b f(t)dt - T_n(x)\right| \leqslant \frac{(b-a)^{n-1+1/q}}{n!}\left(\int_0^1 \left|B_n(t)\right|^q dt\right)^{1/q} \|f^{(n)}\|_p,$$

for every $x \in [a,b]$, where $1/p + 1/q = 1$.

Proof. Use the identity (2.1) and the formula (3.8) and apply the same argument as in the preceding theorem.

□

COROLLARY 6. *Let $f : [a,b] \to \mathbf{R}$ be such that $f^{(n)} \in L_2[a,b]$ for some $n \geqslant 1$. Then*

$$\left|f(x) - \frac{1}{b-a}\int_a^b f(t)dt - T_n(x)\right| \leqslant (b-a)^{n-1/2}\left(\frac{|B_{2n}|}{(2n)!}\right)^{1/2} \cdot \|f^{(n)}\|_2,$$

for every $x \in [a,b]$.

Proof. Apply Theorem 10 for $p = 2$ and use the argument similar to that in the proof of Corollary 5.

□

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