Mathematical
Inequalities
& Applications
Volume 3, Number 3 (2000), 355–368

SOME COMPANION INEQUALITIES TO JENSEN'S INEQUALITY

M. MATIĆ AND J. PEČARIĆ

(*communicated by J. Sándor*)

Abstract. We prove a pair of general companion inequalities to Jensen's inequality in its discrete and integral form. Slater's inequality as well as the generalization of the counterpart to Jensen's inequality along with some further results are deduced from these general inequalities.

1. Introduction

The function $\varphi : C \to \mathbf{R}$ is said to be convex on a convex subset C of a real linear space X if

$$\varphi(tx + (1-t)y) \leqslant t\varphi(x) + (1-t)\varphi(y) \tag{1.1}$$

holds for all $x, y \in C$ and $0 \leqslant t \leqslant 1$. φ is said to be strictly convex if the inequality in (1.1) is strict whenever $x \neq y$ and $0 < t < 1$. It is well known that the convexity of φ is equivalent to the requirement that for any vectors $x_i \in C$, $i = 1, \dots, n$ $(n \geqslant 2)$ and for any nonnegative real numbers p_i, $i = 1, \dots, n$ with $P_n := \sum_{i=1}^n p_i > 0$ Jensen's inequality

$$\varphi\left(\frac{1}{P_n}\sum_{i=1}^n p_i x_i\right) \leqslant \frac{1}{P_n}\sum_{i=1}^n p_i \varphi(x_i) \tag{1.2}$$

holds. In the case when φ is strictly convex the inequality in (1.2) is strict unless $x_i = c$ for all indices i with $p_i > 0$ and for some vector $c \in C$.

Jensen's inequality has many integral analogues. We shall use the simplest one expressed in the language of Lebesgue integral (see for example [6, p. 61]:

if $(\Omega, \mathcal{A}, \mu)$ is a measure space with $0 < \mu(\Omega) < \infty$ and if $f \in L^1(\mu)$ is such that $a < f(t) < b$ for all $t \in \Omega$, $-\infty \leqslant a < b \leqslant \infty$, then the inequality

$$\varphi\left(\frac{1}{\mu(\Omega)}\int_\Omega f\,d\mu\right) \leqslant \frac{1}{\mu(\Omega)}\int_\Omega \varphi(f)\,d\mu \tag{1.3}$$

is valid for any convex function $\varphi : (a, b) \to \mathbf{R}$. In the case when φ is strictly convex on (a, b) we have equality in (1.3) if and only if f is constant almost everywhere on Ω.

Slater ([7, Theorems 1 and 2]) proved an interesting companion inequality to Jensen's inequality (in its discrete and integral form):

Mathematics subject classification (1991): 26D15, 26A51, 26B25.
Key words and phrases: Convex functions, inequalities, Jensen's inequality.

THEOREM A. *Suppose that the convex function* $\varphi:(a,b)\to\mathbf{R}$ *is monotonic on* (a,b).

For $x_1,\cdots,x_n\in(a,b)$ *and for* $p_1,\cdots,p_n\geqslant 0$ *with* $P_n:=\sum_{i=1}^n p_i>0$, *if* $\sum_{i=1}^n p_i\varphi'_+(x_i)\neq 0$, *then*

$$\frac{1}{P_n}\sum_{i=1}^n p_i\varphi(x_i)\leqslant\varphi\left(\sum_{i=1}^n p_i\varphi'_+(x_i)x_i\Big/\sum_{i=1}^n p_i\varphi'_+(x_i)\right). \tag{1.4}$$

When φ *is strictly convex, inequality in (1.4) becomes equality if and only if* $x_i=c$ *for some* $c\in(a,b)$ *and for all* i *with* $p_i>0$.

Also if $(\Omega,\mathcal{A},\mu)$ *is a measure space with* $0<\mu(\Omega)<\infty$ *and if* $f:\Omega\to(a,b)$ *is such that* $\varphi(f)$, $\varphi'_+(f)$ *and* $\varphi'_+(f)f$ *are all in* $L^1(\mu)$, *then*

$$\frac{1}{\mu(\Omega)}\int_\Omega\varphi(f)d\mu\leqslant\varphi\left(\int_\Omega\varphi'_+(f)f\,d\mu\Big/\int_\Omega\varphi'_+(f)d\mu\right) \tag{1.5}$$

holds whenever $\int_\Omega\varphi'_+(f)d\mu\neq 0$. *In the case when* φ *is strictly convex, we have equality in (1.5) if and only if* f *is constant almost everywhere on* Ω.

Here φ'_+ denotes the right derivative of φ and similarly φ'_- denotes the left derivative of φ. Both inequalities (1.4) and (1.5) remain valid if any occurrence of $\varphi'_+(x)$ is replaced by any value from the interval $[\varphi'_-(x),\varphi'_+(x)]$.Pečarić in [3] noted that (1.4) remains true if we drop the assumption about monotonicity on φ, provided

$$\sum_{i=1}^n p_i\varphi'_+(x_i)x_i\Big/\sum_{i=1}^n p_i\varphi'_+(x_i)\in(a,b).$$

In the same paper a generalization of (1.4) to the case when φ is convex function defined on the open convex subset C in $\mathbf{R}^m$ was proved. Further generalization of (1.4) to the case when φ is convex function defined on the open convex subset C in arbitrary normed real linear space X was proved by Pečarić and Andrica in [4].

Another companion inequality to Jensen's inequality is a converse proved by Dragomir and Goh in [1]:

THEOREM B. *Let* $\varphi:C\to\mathbf{R}$ *be a differentiable convex function defined on an open and convex subset* C *of* $\mathbf{R}^m$. *If* $x_i\in C$, $i=1,\cdots,n$ $(n\geqslant 2)$ *are arbitrary vectors and* p_i $i=1,\cdots,n$ *nonnegative real numbers with* $P_n:=\sum_{i=1}^n p_i>0$, *then the inequalities*

$$\begin{aligned}0&\leqslant\frac{1}{P_n}\sum_{i=1}^n p_i\varphi(x_i)-\varphi\left(\frac{1}{P_n}\sum_{i=1}^n p_ix_i\right)\\&\leqslant\frac{1}{P_n}\sum_{i=1}^n p_i\langle\nabla\varphi(x_i),x_i\rangle-\left\langle\frac{1}{P_n}\sum_{i=1}^n p_i\nabla\varphi(x_i),\frac{1}{P_n}\sum_{i=1}^n p_ix_i\right\rangle\end{aligned} \tag{1.6}$$

hold. In the case when φ *is strictly convex, we have equalities in both inequalities in (1.6) if and only if there is some* $c\in C$ *such that* $x_i=c$ *holds for all* i *with* $p_i>0$. *(Here* $\langle\cdot,\cdot\rangle$ *denotes the usual inner product in* $\mathbf{R}^m$.*)*

The result above was stated in [1] with $P_n = 1$ and without the conclusion about equalities in the case of strict convexity.

The paper is organized as follows.

In section 2 we prove a couple of general companion inequalities related to Jensen's discrete inequality (1.2). We show that the Jensen's inequality (1.2), as well as Slater's inequality (1.4) along with its earlier mentioned generalizations due to Pečarić, can be obtained from these general inequalities as a special cases. Moreover, one of these general inequalities provides a generalization of the result stated in Theorem B to the case when φ is arbitrary convex function defined on an open and convex subset C in normed real linear space. In section 2, we also state the analogous results related to Jensen's integral inequality (1.3).

In section 3 we consider the cases when our general inequalities are the best possible, both in the discrete case as well as in the integral case. Also, we prove that inequality (1.6) specialized to the case $m = 1$ under additional assumptions on the function φ, is tighter than Slater's inequality (1.4). Under the same assumptions on φ this is true for the integral analogue of (1.4) when it is compared with Slater's integral inequality (1.5).

In section 4 we apply the results obtained in sections 2 and 3, by suitable choice of the function φ, to obtain the classical inequalities between the harmonic, geometric and arithmetic means, as well as the inequalities of Wang and Wang, and Ky Fan, (see for example [2, pp. 25–28]), and certain refinements of these inequalities.

2. Companion Inequalities to Jensen's inequality

Let X be a real linear space and let X^* be the algebraic dual space of X, that is, the real space of all linear functionals $x^* : X \to \mathbf{R}$. If $\varphi : C \to \mathbf{R}$ is any real–valued function defined on a subset C in X, then for any fixed point $y \in C$ we can define the abstract subdifferential $\partial\varphi(y)$ of φ at y as

$$\partial\varphi(y) := \{a^*(y;\cdot) \in X^* : \varphi(x) \geqslant \varphi(y) + a^*(y;x-y), \forall x \in C\}.$$

Of course it may happen that $\partial\varphi(y) = \emptyset$ for some vector $y \in C$. However, when C is an open convex subset in a normed real linear space X and φ is convex function defined on C, we have

$$\partial\varphi(y) \neq \emptyset, \forall y \in C$$

(see for example [5,p. 108 Theorem B]). Also, when φ is strictly convex, the inequality

$$\varphi(x) \geqslant \varphi(y) + a^*(y;x-y),\ \forall x,y \in C \tag{2.1}$$

is strict unless $x = y$.

THEOREM 2.1. *Let $\varphi : C \to \mathbf{R}$ be a convex function defined on an open convex subset C in a normed real linear space X. For the given vectors $x_i \in C$, $i = 1,\cdots,n$, and nonegative real numbers p_i such that $P_n := \sum_{i=1}^n p_i > 0$ let*

$$\bar{x} := \frac{1}{P_n}\sum_{i=1}^n p_i x_i, \quad \bar{y} := \frac{1}{P_n}\sum_{i=1}^n p_i \varphi(x_i).$$

If $c, d \in C$ *are arbitrarily chosen vectors, then we have*

$$\varphi(c) + a^*(c; \bar{x} - c) \leqslant \bar{y} \leqslant \varphi(d) + \frac{1}{P_n} \sum_{i=1}^{n} p_i a^*(x_i; x_i - d). \tag{2.2}$$

Also, when φ *is strictly convex, we have equality in the left inequality in (2.2) if and only if* $x_i = c$ *holds for all indices* i *with* $p_i > 0$*, while equality holds in the right inequality in (2.2) if and only if* $x_i = d$ *holds for all indices* i *with* $p_i > 0$.

Proof. For a fixed index i we can take $x = x_i$ and $y = c$ in (2.1) to obtain

$$\varphi(c) + a^*(c; x_i - c) \leqslant \varphi(x_i). \tag{2.3}$$

Multiplying (2.3) by $p_i \geqslant 0$ and summing over $i = 1, \cdots, n$, and using the fact that $a^*(c; \cdot)$ is a linear functional, we get

$$P_n \varphi(c) + a^*(c; \sum_{i=1}^{n} p_i x_i - P_n c) \leqslant \sum_{i=1}^{n} p_i \varphi(x_i).$$

Now dividing this by $P_n > 0$, by the linearity of $a^*(c; \cdot)$ we get the first inequality in (2.2). Also, when φ is strictly convex the equality in (2.3) holds if and only if $x_i = c$. So, it is easy to see that our assertion on equality in the first inequality in (2.2) is true.

To obtain the second inequality in (2.2) we first put $y = x_i$ and $x = d$ in (2.1), and then, using the linearity of $a^*(x_i; \cdot)$, rewrite it in the form

$$\varphi(x_i) \leqslant \varphi(d) + a^*(x_i; x_i - d). \tag{2.4}$$

Multiplication by $p_i \geqslant 0$ and summation over $i = 1, \cdots, n$ yields

$$\sum_{i=1}^{n} p_i \varphi(x_i) \leqslant P_n \varphi(d) + \sum_{i=1}^{n} p_i a^*(x_i; x_i - d).$$

Dividing this by $P_n > 0$ we get the second inequality in (2.2). Since in the case when φ is strictly convex we have equality in (2.4) if and only if $x_i = d$, our assertion on equality in the second inequality in (2.2) is obviously true. □

Now we give a simple corollary to Theorem 2.1.

COROLLARY 2.2. *Under the assumptions of Theorem 2.1 we have*

$$0 \leqslant \bar{y} - \varphi(\bar{x}) \leqslant \frac{1}{P_n} \sum_{i=1}^{n} p_i a^*(x_i; x_i - \bar{x}). \tag{2.5}$$

If φ *is strictly convex, then we have equalities throughout in (2.5) if and only if there is some* $c \in C$ *such that* $x_i = c$ *holds for all* i *with* $p_i > 0$.

Proof. Simply take $c = d = \bar{x}$ and apply Theorem 2.1. □

REMARKS. (a) If $\varphi : C \to \mathbf{R}$ is a differentiable convex function defined on an open convex subset C in $\mathbf{R}^m$, and if $y \in C$, then $\partial\varphi(y)$ consists of a single element $a^*(y; \cdot)$

given by $a^*(y;z) = \langle \nabla\varphi(y), z\rangle$ ($z \in \mathbf{R}^m$). So, Corollary 2.2 is a generalization of Theorem B discussed in the introduction.

(b) If there is a vector $\bar{\bar{x}}$ such that

$$\sum_{i=1}^{n} p_i a^*(x_i; x_i - \bar{\bar{x}}) \leqslant 0$$

holds, then from the right inequality in (2.2) with $d = \bar{\bar{x}}$ it follows that we have

$$\bar{y} \leqslant \varphi(\bar{\bar{x}}).$$

This result was proved by Pečarić and Andrica [4]. When C is an open convex subset in $\mathbf{R}^m$, we can take $a^*(y;\cdot) \in \partial\varphi(y)$ ($y \in C$) to be defined by

$$a^*(y;z) = \sum_{j=1}^{m} \partial_{j+}\varphi(y)\zeta_j, \; z = (\zeta_1, \cdots, \zeta_m) \in \mathbf{R}^m$$

where $\partial_{j+}\varphi(y)$ denotes right partial derivative of φ over j–th variable at y. Pečarić [3] proved that in this case the vector $\bar{\bar{x}} = (\bar{\bar{\xi}}_1, \cdots, \bar{\bar{\xi}}_m)$ can be given explicitely by

$$\bar{\bar{\xi}}_j := \frac{\sum_{i=1}^{n} p_i \partial_{j+}\varphi(x_i)\xi_{ij}}{\sum_{i=1}^{n} p_i \partial_{j+}\varphi(x_i)}, \; j = 1, \cdots, m,$$

where ξ_{ij} denotes the j–th coordinate of x_i.

In the simplest case when $\varphi : (a,b) \to \mathbf{R}$ is a convex function defined on an open interval (a,b) in $\mathbf{R}$, for any $y \in (a,b)$ we have that $a^*(y;\cdot) \in \partial\varphi(y)$ is given by $a^*(y;z) = \alpha z$ ($z \in \mathbf{R}$), where α is any value from the interval $[\varphi'_-(y), \varphi'_+(y)]$. For convenience we shall always take $\alpha = \varphi'_+(y)$ so that, in this case, Theorem 2.1 states that for any $c, d \in (a,b)$ we have

$$\varphi(c) + \varphi'_+(c)(\bar{x} - c) \leqslant \bar{y} \leqslant \varphi(d) + \frac{1}{P_n}\sum_{i=1}^{n} p_i \varphi'_+(x_i)(x_i - d). \tag{2.6}$$

The proof of this fact depends on the key inequality (2.1), that is (in this case)

$$\varphi(x) \geqslant \varphi(y) + \varphi'_+(y)(x - y), \; \forall x, y \in (a,b) \tag{2.7}$$

with strict inequality when $x \neq y$ in the case when φ is strictly convex. Using inequality (2.7) we can easily prove the integral version of Theorem 2.1 for Lebesgue integral. Here we state this result without proof.

THEOREM 2.3. *Let* $(\Omega, \mathcal{A}, \mu)$ *be a measure space with* $0 < \mu(\Omega) < \infty$ *and let* $\varphi : (a,b) \to \mathbf{R}$ *be a convex function defined on an open interval* (a,b) *in* $\mathbf{R}$. *If* $f : \Omega \to (a,b)$ *is such that* f, $\varphi(f)$, $\varphi'_+(f)$ *and* $f\varphi'_+(f)$ *are all in* $L^1(\mu)$, *then for any* $c, d \in (a,b)$ *we have*

$$\varphi(c) + \varphi'_+(c)(\bar{x} - c) \leqslant \bar{y} \leqslant \varphi(d) + \frac{1}{\mu(\Omega)}\int_{\Omega}(f - d)\varphi'_+(f)d\mu, \tag{2.8}$$

where

$$\bar{x} := \frac{1}{\mu(\Omega)} \int_{\Omega} f \, d\mu \, , \; \bar{y} := \frac{1}{\mu(\Omega)} \int_{\Omega} \varphi(f) d\mu .$$

Further, when φ is strictly convex, we have equality in the left inequality in (2.8) if and only if $f(t) = c$ almost everywhere on Ω, while equality holds in the right inequality in (2.8) if and only if $f(t) = d$ almost everywhere on Ω.

REMARKS. (a) First note that we may take $(\Omega, \mathcal{A}, \mu)$ to be a discrete measure space with $\Omega = \{1, 2, \cdots, n\}$ and $\mu(\{i\}) = p_i$ $(i = 1, \cdots, n)$ and then apply Theorem 2.3 to the function f defined by $f(i) = x_i$ $(i = 1, \cdots, n)$. We get the inequalities (2.6), which means that we can interpret these inequalities as a special case of the inequalities (2.8).

(b) As a corollary to Theorem 2.3 we can obtain an integral analogue of Theorem B. Namely, we can simply set $c = d = \bar{x}$ in (2.8) and rewrite the obtained inequalities in the form

$$0 \leqslant \bar{y} - \varphi(\bar{x}) \leqslant \frac{1}{\mu(\Omega)} \int_{\Omega} (f - \bar{x}) \varphi'_{+}(f) d\mu .$$

(c) When $\int_{\Omega} \varphi'_{+}(f) d\mu \neq 0$, we can define a value $\bar{\bar{x}}$ by

$$\bar{\bar{x}} := \int_{\Omega} f \varphi'_{+}(f) d\mu \Big/ \int_{\Omega} \varphi'_{+}(f) d\mu .$$

If $\bar{\bar{x}} \in (a, b)$ holds, then the second inequality in (2.8) with $d = \bar{\bar{x}}$ reduces to the inequality $\bar{y} \leqslant \varphi(\bar{\bar{x}})$, which is in fact the Slater's inequality (1.5).

3. Some further results

In this section we first address the question of optimizing the inequalities (2.2) given in Theorem 2.1. That is, we want to find out if there is a choice of c and of d for which the inequalities (2.2) are sharp. Our next theorem gives the answer to this question.

THEOREM 3.1. *Let the assumptions of Theorem 2.1 be satisfied. If $\bar{x}$ and $\bar{y}$ are defined as in Theorem 2.1, then we have*

$$\varphi(c) + a^*(c; \bar{x} - c) \leqslant \varphi(\bar{x}) \leqslant \bar{y}, \quad \forall c \in C. \tag{3.1}$$

When φ is strictly convex, the first inequality in (3.1) is strict unless $c = \bar{x}$. Furthermore, if there exists a vector $\bar{d} \in C$ such that the corresponding functional $a^*(\bar{d}; \cdot) \in \partial\varphi(\bar{d})$ satisfies

$$a^*(\bar{d}; \cdot) = \frac{1}{P_n} \sum_{i=1}^{n} p_i a^*(x_i; \cdot), \tag{3.2}$$

then

$$\begin{aligned}\bar{y} &\leqslant \varphi(\bar{d}) + \frac{1}{P_n}\sum_{i=1}^{n} p_i a^*(x_i; x_i - \bar{d}) \\ &\leqslant \varphi(d) + \frac{1}{P_n}\sum_{i=1}^{n} p_i a^*(x_i; x_i - d), \quad \forall d \in C. \end{aligned} \tag{3.3}$$

When φ is strictly convex, the second inequality in (3.3) is strict unless $d = \bar{d}$.

Proof. By setting $c = \bar{x}$ in the first inequality in (2.2), we get $\varphi(\bar{x}) \leqslant \bar{y}$. Also, setting $x = \bar{x}$ and $y = c$ in (2.1), gives the first inequality in (3.1). For the strictly convex φ, this inequality is strict unless $c = \bar{x}$. If (3.2) holds for some vector $\bar{d} \in C$, then for arbitrary $d \in C$ we can apply (2.1) with $x = d$ and $y = \bar{d}$ to obtain

$$\begin{aligned}\varphi(d) &\geqslant \varphi(\bar{d}) + a^*(\bar{d}; d - \bar{d}) \\ &= \varphi(\bar{d}) + \frac{1}{P_n}\sum_{i=1}^{n} p_i a^*(x_i; d - \bar{d}) \\ &= \varphi(\bar{d}) + \frac{1}{P_n}\sum_{i=1}^{n} p_i a^*(x_i; x_i - \bar{d} - (x_i - d)) \\ &= \varphi(\bar{d}) + \frac{1}{P_n}\sum_{i=1}^{n} p_i a^*(x_i; x_i - \bar{d}) - \frac{1}{P_n}\sum_{i=1}^{n} p_i a^*(x_i; x_i - d).\end{aligned}$$

This, in turn, implies the second inequality in (3.3). In the strictly convex case this inequality is strict unless $d = \bar{d}$. The first inequality in (3.3) follows by the second inequality in (2.2) with $d = \bar{d}$. □

COROLLARY 3.2. *Suppose that $\varphi : (a,b) \to \mathbf{R}$ is a convex function defined on an open interval (a,b) in $\mathbf{R}$ and that the first derivative $\varphi'(x)$ exists at every point x in (a,b). For $x_i \in (a,b)$ and nonnegative real numbers p_i, $i = 1,\cdots,n$, with $P_n = \sum_{i=1}^{n} p_i > 0$, let*

$$\bar{x} := \frac{1}{P_n}\sum_{i=1}^{n} p_i x_i, \quad \bar{y} := \frac{1}{P_n}\sum_{i=1}^{n} p_i \varphi(x_i).$$

Then, we have

$$\varphi(c) + \varphi'(c)(\bar{x} - c) \leqslant \varphi(\bar{x}) \leqslant \bar{y}, \quad \forall c \in (a,b). \tag{3.4}$$

If φ is strictly convex, then the first inequality in (3.4) is strict unless $c = \bar{x}$. Furthermore, there exists at least one $\bar{d} \in (a,b)$ such that

$$\varphi'(\bar{d}) = \frac{1}{P_n}\sum_{i=1}^{n} p_i \varphi'(x_i), \tag{3.5}$$

and

$$\bar{y} \leqslant \varphi(\bar{d}) + \frac{1}{P_n}\sum_{i=1}^{n} p_i\varphi'(x_i)(x_i - \bar{d})$$
$$\leqslant \varphi(d) + \frac{1}{P_n}\sum_{i=1}^{n} p_i\varphi'(x_i)(x_i - d), \quad \forall d \in (a,b). \tag{3.6}$$

When φ is strictly convex, there exists exactly one $\bar{d}$ for which (3.5) and (3.6) hold, and the second inequality in (3.6) is strict unless $d = \bar{d}$.

Proof. Since $\varphi'(y)$ exists for arbitrary $y \in (a,b)$, the subdifferential $\partial\varphi(y)$ consists of exactly one element $a^*(y;\cdot)$ given by

$$a^*(y;z) = \varphi'(y)z, \ z \in \mathbf{R}.$$

Therefore, the condition (3.2) is in this case equivalent to the requirement that

$$\varphi'(\bar{d})z = \frac{1}{P_n}\sum_{i=1}^{n} p_i\varphi'(x_i)z, \quad \forall z \in \mathbf{R},$$

which is, in turn, equivalent to (3.5). The derivative φ' is nondecreasing and continuous on (a,b) (see for example [5, pp. 3–7]), which implies that $I = \varphi'((a,b))$ is an interval in $\mathbf{R}$ (not necessarily open). So $(1/P_n)\sum_{i=1}^{n} p_i\varphi'(x_i)$ must be in I since it is a convex combination of $\varphi'(x_i) \in I$ ($i = 1,\cdots,n$). This shows that there must exist at least one $\bar{d} \in (a,b)$ which satisfies (3.5). When φ is strictly convex, the derivative φ' is strictly increasing and exactly one $\bar{d} \in (a,b)$ satisfies (3.5). Now we apply Theorem 3.1 to get desired conclusions. □

REMARK. When φ is strictly convex, the value $\bar{d}$ is uniquely determined by (3.5) which can be rewriten as

$$\bar{d} = M_{\varphi'}(x_1,\cdots,x_n;p_1,\cdots,p_n) := (\varphi')^{-1}\left(\frac{1}{P_n}\sum_{i=1}^{n} p_i\varphi'(x_i)\right).$$

Thus, $\bar{d}$ is the quasiarithmetic mean with respect to the strictly increasing and continuous function φ' of $(x_1,\cdots,x_n)$ with weights $(p_1,\cdots,p_n)$.

The result proved in Corollary 3.2 can be regarded as the special case of the following more general result.

THEOREM 3.3. *Let $(\Omega,\mathcal{A},\mu)$ be a measure space with $0 < \mu(\Omega) < \infty$ and let $\varphi : (a,b) \to \mathbf{R}$ be a convex function such that $\varphi'(x)$ exists at every point $x \in (a,b)$. If $f : \Omega \to (a,b)$ is such that f, $\varphi(f)$, $\varphi'(f)$ and $f\varphi'(f)$ are all in $L^1(\mu)$, and if*

$$\bar{x} := \frac{1}{\mu(\Omega)}\int_\Omega f\,d\mu\ , \ \bar{y} := \frac{1}{\mu(\Omega)}\int_\Omega \varphi(f)d\mu,$$

then

$$\varphi(c) + \varphi'(c)(\bar{x} - c) \leqslant \varphi(\bar{x}) \leqslant \bar{y}, \quad \forall c \in (a,b). \tag{3.7}$$

When φ is strictly convex, the first inequality in (3.7) is strict unless $c = \bar{x}$. Also, there exists at least one $\bar{d} \in (a,b)$ such that

$$\varphi'(\bar{d}) = \frac{1}{\mu(\Omega)} \int_\Omega \varphi'(f)d\mu \tag{3.8}$$

holds, and

$$\begin{aligned} \bar{y} &\leqslant \varphi(\bar{d}) + \frac{1}{\mu(\Omega)} \int_\Omega (f - \bar{d})\varphi'(f)d\mu \\ &\leqslant \varphi(d) + \frac{1}{\mu(\Omega)} \int_\Omega (f - d)\varphi'(f)d\mu, \quad \forall d \in (a,b). \end{aligned} \tag{3.9}$$

When φ is strictly convex, there exists exactly one $\bar{d}$ for which (3.8) and (3.9) hold and the second inequality in (3.9) is strict unless $d = \bar{d}$.

Proof. The argument for the first inequality in (3.7) is the same as the one given for the first inequality in (3.4). The second inequality in (3.7) and the first one in (3.9) follow from the inequalities (2.8) if we set $c = \bar{x}$ and $d = \bar{d}$. Also, we have

$$\frac{1}{\mu(\Omega)} \int_\Omega \varphi'(f)d\mu \in \varphi'((a,b))$$

so that the argument for the existence of $\bar{d}$ and for its uniqueness (in the case when φ is strictly convex) is the same as the one given in the proof of Corollary 3.2. To prove the second inequality in (3.9), it is enough to minimize the function $\psi : (a,b) \to \mathbf{R}$ defined as

$$\psi(d) := \varphi(d) + \frac{1}{\mu(\Omega)} \int_\Omega (f - d)\varphi'(f)d\mu, \quad d \in (a,b).$$

The first derivative of ψ is

$$\psi'(d) = \varphi'(d) - \frac{1}{\mu(\Omega)} \int_\Omega \varphi'(f)d\mu, \quad d \in (a,b).$$

Since the function φ' is nondecreasing and continuous on (a,b), it is obvious that ψ' is nondecreasing and continuous on (a,b), too. So, if $\bar{d}$ satisfies (3.8), then the function ψ attains its minimum at $\bar{d}$. □

For $d = \bar{x}$, Theorem 3.3 gives

$$\varphi(\bar{x}) \leqslant \bar{y} \leqslant D_J := \varphi(\bar{x}) + \frac{1}{\mu(\Omega)} \int_\Omega (f - \bar{x})\varphi'(f)d\mu. \tag{3.10}$$

Also, if $\int_\Omega \varphi'(f)d\mu \neq 0$, then we can define

$$\bar{\bar{x}} := \int_\Omega f\varphi'(f)d\mu \Big/ \int_\Omega \varphi'(f)d\mu,$$

and the Slater's inequality

$$\bar{y} \leqslant D_S := \varphi(\bar{\bar{x}}) \tag{3.11}$$

is valid whenever $\bar{\bar{x}} \in (a,b)$ holds (this happens for example in the case when φ is monotone). We now prove that, under the additional assumptions on φ, the second inequality in (3.10) is sharper than inequality (3.11).

THEOREM 3.4. *Let the assumptions of Theorem 3.3 hold. Suppose that* $\int_\Omega \varphi'(f)d\mu \neq 0$ *holds. If either* φ *is nondecreasing and* φ' *is concave on* (a,b), *or* φ *is nonincreasing and* φ' *is convex on* (a,b), *then*

$$\bar{y} \leqslant D_J \leqslant D_S.$$

Proof. The relation (3.10) implies

$$(\bar{\bar{x}} - \bar{x}) \int_\Omega \varphi'(f)d\mu = \int_\Omega (f - \bar{x})\varphi'(f)d\mu \geqslant 0. \tag{3.12}$$

Also, the difference $D_S - D_J$ can be writen as

$$D_S - D_J = \varphi(\bar{\bar{x}}) - \varphi(\bar{x}) - (\bar{\bar{x}} - \bar{x})\frac{1}{\mu(\Omega)} \int_\Omega \varphi'(f)d\mu. \tag{3.13}$$

If φ is nondecreasing, then we have $\int_\Omega \varphi'(f)d\mu > 0$, and (3.12) implies $\bar{\bar{x}} - \bar{x} \geqslant 0$. Also, Jensen's inequality applied to the concave function φ' yields

$$\frac{1}{\mu(\Omega)} \int_\Omega \varphi'(f)d\mu \leqslant \varphi'\left(\frac{1}{\mu(\Omega)} \int_\Omega f\,d\mu\right) = \varphi'(\bar{x}),$$

so that from (3.13) we get

$$D_S - D_J \geqslant \varphi(\bar{\bar{x}}) - \varphi(\bar{x}) - \varphi'(\bar{x})(\bar{\bar{x}} - \bar{x}) \geqslant 0. \tag{3.14}$$

Similarly, if φ is nonincreasing, then we have $\int_\Omega \varphi'(f)d\mu < 0$, and (3.12) implies $\bar{\bar{x}} - \bar{x} \leqslant 0$. Now, Jensen's inequality applied to the convex function φ' yields

$$\frac{1}{\mu(\Omega)} \int_\Omega \varphi'(f)d\mu \geqslant \varphi'\left(\frac{1}{\mu(\Omega)} \int_\Omega f\,d\mu\right) = \varphi'(\bar{x}),$$

and from (3.13) we get (3.14) again. □

REMARK. If we specialize $(\Omega, \mathcal{A}, \mu)$ to be a discrete measure space with $\Omega = \{1, \cdots, n\}$ and $\mu(\{i\}) = p_i \geqslant 0$ $(i = 1 \cdots, n)$, and then apply Theorem 3.3 to a function f defined by $f(i) = x_i \in (a,b)$, $i = 1, \cdots, n$, we get the conclusions stated in Corollary 3.2. In that case $\bar{x}, \bar{y}$ and $\bar{\bar{x}}$ are given by

$$\bar{x} = \frac{1}{P_n}\sum_{i=1}^n p_i x_i,\ \bar{y} = \frac{1}{P_n}\sum_{i=1}^n p_i \varphi(x_i) \text{ and } \bar{\bar{x}} = \sum_{i=1}^n p_i \varphi'(x_i)x_i \Big/ \sum_{i=1}^n p_i \varphi'(x_i),$$

respectively, provided that $\sum_{i=1}^n p_i \varphi'(x_i) \neq 0$. If, additionally, either φ is nondecreasing and φ' is concave on (a,b), or φ is nonincreasing and φ' is convex on (a,b), then by Theorem 3.4 we have $\bar{y} \leqslant D_J \leqslant D_S$, where

$$D_J = \varphi(\bar{x}) + \frac{1}{P_n}\sum_{i=1} p_i \varphi'(x_i)(x_i - \bar{x}) \text{ and } D_S = \varphi(\bar{\bar{x}}).$$

4. Applications to some classical inequalities

Let $x_1, \cdots, x_n$ and $p_1, \cdots, x_n$ be positive real numbers and let $P_n := \sum_{i=1}^{n} p_i$. Let H_n, G_n and A_n denote the weighted harmonic, geometric and arithmetic mean, respectively, defined by

$$H_n := P_n \left(\sum_{i=1}^{n} p_i/x_i \right)^{-1}, \; G_n := \left(\prod_{i=1}^{n} x_i^{p_i} \right)^{1/P_n} \text{ and } A_n := \frac{1}{P_n} \sum_{i=1}^{n} p_i x_i. \tag{4.1}$$

It is well–known that $H_n \leqslant G_n \leqslant A_n$. We call these inequalities shortly HGA. Here we shall deduce HGA as a consequence of our results from sections 2 and 3.

THEOREM 4.1. *Suppose that $x_1, \cdots, x_n$ and $p_1, \cdots, p_n$ are positive real numbers and let H_n, G_n and A_n be defined by (4.1). For any positive real numbers c and d we have*

$$d \exp\left\{1 - \frac{d}{H_n}\right\} \leqslant H_n \leqslant G_n \leqslant A_n \leqslant c \exp\left\{\frac{A_n}{c} - 1\right\}. \tag{4.2}$$

The equality in the first inequality in (4.2) holds if and only if $d = H_n$, while the equality in the last inequality in (4.2) holds if and only if $c = A_n$. Also, the second and the third inequality in (4.2) are strict unless there is an $K > 0$ such that $x_i = K$ for all $i = 1, \cdots, n$.

Proof. The function $\varphi(x) = -\ln x$ is strictly convex on the interval $(0, \infty)$ and we have $\varphi'(x) = -1/x$, for $x > 0$. Also, we have

$$\bar{x} = \frac{1}{P_n} \sum_{i=1}^{n} p_i x_i = A_n \text{ and } \bar{y} = \frac{1}{P_n} \sum_{i=1}^{n} p_i \varphi(x_i) = -\ln G_n.$$

Furthermore,

$$\frac{1}{P_n} \sum_{i=1}^{n} p_i \varphi'(x_i) x_i = -1 \text{ and } \frac{1}{P_n} \sum_{i=1}^{n} p_i \varphi'(x_i) = -\frac{1}{H_n},$$

so that $\bar{\bar{x}} = H_n$. Also, the equation (3.5) reduces to $-1/\bar{d} = -1/H_n$ which yields $\bar{d} = H_n$. Now we can apply Corollary 3.2 to get the inequalities

$$-\ln c - \frac{1}{c}(A_n - c) \leqslant -\ln A_n \leqslant -\ln G_n \leqslant -\ln H_n \leqslant -\ln d - 1 + \frac{d}{H_n},$$

which are obviously equivalent to the inequalities (4.2). The assertions on the cases when equalities hold are true by Theorem 2.1 and Corollary 3.2. □

REMARK. It should be noted that we can not apply Theorem 3.4 to the function $\varphi(x) = -\ln x$ since it is strictly decreasing and its derivative $\varphi'(x) = -1/x$ is strictly concave on $(0, \infty)$. In fact, we have $\bar{\bar{x}} = \bar{d} = H_n$ so that Corollary 3.2 implies

$$D_S = -\ln H_n \leqslant D_J = -\ln A_n - 1 + \frac{A_n}{H_n}.$$

It is easy to see that this inequality is strict unless $H_n = A_n$.

Suppose that $x_1, \cdots, x_n$ and $p_1, \cdots, p_n$ are positive real numbers such that $x_i \in (0, 1/2]$ for all $i = 1, \cdots, n$. Then we have $1 - x_i \in [1/2, 1)$ for all $i = 1, \cdots, n$, and we can consider the means H_n, G_n and A_n defined by (4.1), as well as the means H_n', G_n' and A_n' defined by

$$H_n' := P_n \left(\sum_{i=1}^{n} p_i/(1 - x_i) \right)^{-1}, \; G_n' := \left(\prod_{i=1}^{n} (1 - x_i)^{p_i} \right)^{1/P_n}$$
$$\text{and } A_n' := \frac{1}{P_n} \sum_{i=1}^{n} p_i (1 - x_i) = 1 - A_n. \tag{4.3}$$

It is well–known that the inequalities

$$H_n/H_n' \leqslant G_n/G_n' \leqslant A_n/A_n' \tag{4.4}$$

are valid. The first inequality in (4.4) is known in the literature as the inequality of Wang and Wang while the second one is known as the inequality of Ky Fan (see for example [2, pp. 25–28]). Again we shall use our results from the previous sections to deduce the inequalities (4.4).

THEOREM 4.2. *Suppose that $x_1, \cdots, x_n$ and $p_1, \cdots, p_n$ are positive real numbers such that $x_i \in (0, 1/2]$ for all $i = 1, \cdots, n$. Then the value $\bar{d}$ given by*

$$\bar{d} = \frac{1}{2} - \sqrt{\frac{1}{4} - \frac{H_n' H_n}{H_n' + H_n}} \tag{4.5}$$

is a well defined number in the interval $(0, 1/2]$ and for any two real numbers c and d from the interval $(0, 1/2]$ we have

$$\frac{d}{1-d} \exp\left\{ \frac{1}{H_n'} - d \frac{H_n' + H_n}{H_n' H_n} \right\} \leqslant \frac{\bar{d}}{1-\bar{d}} \exp\left\{ \frac{1}{H_n'} - \bar{d} \frac{H_n' + H_n}{H_n' H_n} \right\}$$
$$\leqslant \frac{G_n}{G_n'} \leqslant \frac{A_n}{A_n'} \leqslant \frac{c}{1-c} \exp\left\{ \frac{A_n - c}{c(1-c)} \right\}. \tag{4.6}$$

Also, we have

$$\frac{H_n}{H_n'} \leqslant \frac{\bar{d}}{1-\bar{d}} \exp\left\{ \frac{1}{H_n'} - \bar{d} \frac{H_n' + H_n}{H_n' H_n} \right\} \leqslant \frac{G_n}{G_n'}. \tag{4.7}$$

In the first inequality in (4.6) the equality holds if and only if $d = \bar{d}$, while in the last inequality in (4.6) the equality holds if and only if $c = A_n$. Also, the equalities hold throughout in (4.4) if and only if there exists $K > 0$ such that $x_i = K$ for all $i = 1, \cdots, n$.

Proof. Consider the function $\varphi(x) = \ln(1-x) - \ln x$ which is defined on $(0, 1)$. It is strictly convex on the interval $(0, 1/2]$ and has a derivative $\varphi'(x) = -1/(1-x) - 1/x$. It is easy to see that we can apply our results from the previous sections to this function in spite of the fact that the interval on which this function is convex is not open. We have

$$\bar{x} = \frac{1}{P_n}\sum_{i=1}^{n} p_i x_i = A_n \text{ and } \bar{y} = \frac{1}{P_n}\sum_{i=1}^{n} p_i \varphi(x_i) = \ln(G'_n/G_n).$$

Furthermore,

$$\frac{1}{P_n}\sum_{i=1}^{n} p_i \varphi'(x_i) x_i = -\frac{1}{H'_n} \text{ and } \frac{1}{P_n}\sum_{i=1}^{n} p_i \varphi'(x_i) = -\frac{1}{H_n} - \frac{1}{H'_n},$$

which implies $\bar{\bar{x}} = H_n/(H'_n + H_n)$. Also, the equation (3.5) reduces to

$$-\frac{1}{1-\bar{d}} - \frac{1}{\bar{d}} = -\frac{1}{H'_n} - \frac{1}{H_n},$$

and this is equivalent to the equation

$$\bar{d}^2 - \bar{d} + H'_n H_n/(H'_n + H_n) = 0.$$

This last equation has exactly one solution $\bar{d}$ in interval $(0, 1/2]$ and this solution is given by (4.5). Now we can apply Corollary 3.2 to obtain the inequalities

$$\ln\frac{1-c}{c} - \frac{1}{c(1-c)}(A_n - c) \leqslant \ln\frac{1-A_n}{A_n} = \ln\frac{A'_n}{A_n} \leqslant \ln\frac{G'_n}{G_n} \tag{4.8}$$

and

$$\begin{aligned}\ln\frac{G'_n}{G_n} &\leqslant \ln\frac{1-\bar{d}}{\bar{d}} - \frac{1}{H'_n} - \bar{d}\left(-\frac{1}{H_n} - \frac{1}{H'_n}\right)\\ &\leqslant \ln\frac{1-d}{d} - \frac{1}{H'_n} - d\left(-\frac{1}{H_n} - \frac{1}{H'_n}\right).\end{aligned} \tag{4.9}$$

From (4.8) and (4.9) we get the inequalities (4.6), while the inequality (4.7) follows by the first inequality in (4.6) if we set $d = H_n/(H'_n + H_n)$. The assertions on the cases when equalities hold are true by Theorem 2.1 and Corollary 3.2. □

REFERENCES

[1] S. S. DRAGOMIR AND C. J. GOH, *A counterpart of Jensen's discrete inequality for differentiable convex mappings and applications in information theory*, Math. Comput. Modelling **24 (2)** (1996), 1–11.

[2] D. S. MITRINOVIĆ, J. E. PEČARIĆ AND A. M. FINK, *Classical and new inequalities in analysis*, Kluwer Academ. Publ., 1993.

[3] J. E. PEČARIĆ, *A multidimensional generalization of Slater's inequality*, J. Approx. Theory **44 (3)** (1985), 292–294.

[4] J. E. PEČARIĆ AND D. ANDRICA, *Abstract Jessen's inequality for convex functions and applications*, Mathematica (Cluj) **29 (52)** (1987), 61–65.

[5] A. W. ROBERTS AND D. E. VARBERG, *Convex Functions*, Academic Press, New York–London, 1972.

[6] W. RUDIN, *Real and complex analysis*, McGraw–Hill, London, 1970.

[7] M. L. SLATER, *A companion inequality to Jensen's inequality*, J. Approx. Theory **32** (1981), 160–166.

(Received March 12, 1999)

Marko Matić
Mathematics Department, FESB
R. Boškovića b.b, Split
Croatia
e-mail: mmatic@fesb.hr

Josip Pečarić
Faculty of Textile Technology
University of Zagreb
Pierottijeva 6, Zagreb
Croatia
e-mail: pecaric@hazu.hr