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ON THE CAUCHY–BUNIAKOWSKY–SCHWARTZ'S INEQUALITY FOR SEQUENCES IN INNER PRODUCT SPACES

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Abstract. In this paper we consider some mappings naturally connected to Cauchy-Buniakowsky-Schwartz's inequality for sequences of vectors in inner product spaces and point out their main properties. Some applications are also given.

1. Introduction

In the following pages, we will assume that $(H;(\cdot,\cdot))$ is an inner product on the real or complex number field $\mathbb{K}$. The following inequality is a variant of the well-known Cauchy-Buniakowsky-Schwartz inequality:

$$\sum_{i\in I} p_i |\alpha_i|^2 \sum_{i\in I} p_i \|x_i\|^2 \geqslant \left\| \sum_{i\in I} p_i \alpha_i x_i \right\|^2 \tag{1.1}$$

where $p_i \geqslant 0$, $\alpha_i \in \mathbb{K}$, $x_i \in H$ for all $i \in I$, where I is a finite part of the natural number set $\mathbb{N}$. If $p_i > 0$ for all $i \in I$, then the inequality holds in (1.1) iff there exists a vector $x_0 \in H$ such that $x_i = \bar{\alpha}_i x_0$ for all $i \in I$.

Indeed, a simple calculation shows that:

$$\begin{aligned} 0 &\leqslant \frac{1}{2} \sum_{(i,j)\in I^2} p_i p_j \|\bar{\alpha}_i x_j - \bar{\alpha}_j x_i\|^2 \\ &= \frac{1}{2} \sum_{(i,j)\in I^2} p_i p_j \left[|\alpha_i|^2 \|x_j\|^2 - 2\operatorname{Re}(\bar{\alpha}_i x_j, \bar{\alpha}_j x_i) + |\alpha_j|^2 \|x_i\|^2 \right] \\ &= \sum_{(i,j)\in I^2} p_i p_j |\alpha_i|^2 \|x_j\|^2 - \sum_{(i,j)\in I^2} p_i p_j \operatorname{Re}(\bar{\alpha}_j x_j, \bar{\alpha}_i x_i) \\ &= \sum_{i\in I} p_i |\alpha_i|^2 \sum_{i\in I} p_i \|x_i\|^2 - \left\| \sum_{i\in I} p_i \alpha_i x_i \right\|^2 \end{aligned}$$

and thus the inequality holds in (1.1) iff $\bar{\alpha}_i x_j = \bar{\alpha}_j x_i$ for all $i, j \in I$. That is, there exists a vector $x_0 \in H$ such that $x_i = \bar{\alpha}_i \cdot x_0$ for all $i \in I$.

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In the following, we shall use the following notations:

$$\begin{aligned}
&\mathfrak{P}_f(\mathbb{N}) := \{I \subset \mathbb{N} \,|I \text{ is finite}\};\\
&\mathfrak{J}(\mathbb{R}) := \left\{p = (p_i)_{i\in\mathbb{N}} \,|p_i \in \mathbb{R} \text{ for all } i \in \mathbb{N}\right\};\\
&\mathfrak{J}_+(\mathbb{R}) := \left\{p = (p_i)_{i\in\mathbb{N}} \,|p_i \geqslant 0 \text{ for all } i \in \mathbb{N}\right\};\\
&\mathfrak{J}(\mathbb{K}) := \left\{\alpha = (\alpha_i)_{i\in\mathbb{N}} \,|\alpha_i \in \mathbb{K} \text{ for all } i \in \mathbb{N}\right\}
\end{aligned}$$

and given by:

$$\mathfrak{J}(H) := \left\{x = (x_i)_{i\in\mathbb{N}} \,|x_i \in H \text{ for all } i \in \mathbb{N}\right\}.$$

By the use of these notations, we can define the following mapping associated with the Cauchy-Buniakowsky-Schwartz inequality (1.1) :

$$\mu : \mathfrak{P}_f(\mathbb{N}) \times \mathfrak{J}_+(\mathbb{R}) \times \mathfrak{J}(\mathbb{K}) \times \mathcal{H}(X) \longrightarrow \mathbb{R},$$

given by:

$$\mu(I,p,\alpha,x) := \left[\sum_{i\in I} p_i |\alpha_i|^2 \sum_{i\in I} p_i \|x_i\|^2 - \left\|\sum_{i\in I} p_i \alpha_i x_i\right\|^2\right]^{\frac{1}{2}}.$$

The main aim of this paper is to point out the fundamental properties of this mapping. Some natural applications are also made.

2. The superadditivity of the mapping $\mu(I,\cdot,\alpha,x)$

We will start with the following result:

THEOREM 1. *Let* $I \in \mathfrak{P}_f(\mathbb{N})$, $\alpha \in \mathfrak{J}(\mathbb{K})$ *and* $x \in \mathfrak{J}(H)$. *Then*

(i) *For all* $p,q \in \mathfrak{J}_+(\mathbb{R})$ *we have the inequality:*

$$\mu(I,p+q,\alpha,x) \geqslant \mu(I,p,\alpha,x) + \mu(I,q,\alpha,x) \geqslant 0. \tag{2.1}$$

That is, the mapping $\mu(I,\cdot,\alpha,x)$ *is superadditive on* $\mathfrak{J}_+(\mathbb{R})$;

(ii) *For all* $p,q \in \mathfrak{J}_+(\mathbb{R})$ *with* $p \geqslant q$, *we have the inequality:*

$$\mu(I,p,\alpha,x) \geqslant \mu(I,q,\alpha,x) \geqslant 0. \tag{2.2}$$

That is, the mapping $\mu(I,\cdot,\alpha,x)$ *is monotonic nondecreasing on* $\mathfrak{J}_+(\mathbb{R})$.

Proof. (i) We will start with the following identity of Lagrange's type:

$$\mu(I,r,\alpha,x)=\left(\frac{1}{2}\sum_{(i,j)\in I^2}r_ir_j\left\|\bar{\alpha}_ix_j-\bar{\alpha}_jx_i\right\|^2\right)^{\frac{1}{2}}.$$

Then for all $p,q\in\mathfrak{J}_+(\mathbb{R})$ we have:

$$\begin{aligned}
&\mu^2(I,p+q,\alpha,x)\\
&\quad=\frac{1}{2}\sum_{(i,j)\in I^2}(p_i+q_i)(p_j+q_j)\left\|\bar{\alpha}_ix_j-\bar{\alpha}_jx_i\right\|^2\\
&\quad=\frac{1}{2}\sum_{(i,j)\in I^2}p_ip_j\left\|\bar{\alpha}_ix_j-\bar{\alpha}_jx_i\right\|^2+\frac{1}{2}\sum_{(i,j)\in I^2}q_iq_j\left\|\bar{\alpha}_ix_j-\bar{\alpha}_jx_i\right\|^2\\
&\qquad+\frac{1}{2}\sum_{(i,j)\in I^2}p_iq_j\left\|\bar{\alpha}_ix_j-\bar{\alpha}_jx_i\right\|^2+\frac{1}{2}\sum_{(i,j)\in I^2}q_ip_j\left\|\bar{\alpha}_ix_j-\bar{\alpha}_jx_i\right\|^2\\
&\quad=\mu^2(I,p,\alpha,x)+\mu^2(I,q,\alpha,x)+\sum_{(i,j)\in I^2}p_iq_j\left\|\bar{\alpha}_ix_j-\bar{\alpha}_jx_i\right\|^2
\end{aligned}$$

as a simple calculation shows that:

$$\sum_{(i,j)\in I^2}p_iq_j\left\|\bar{\alpha}_ix_j-\bar{\alpha}_jx_i\right\|^2=\sum_{(i,j)\in I^2}p_jq_i\left\|\bar{\alpha}_ix_j-\bar{\alpha}_jx_i\right\|^2.$$

We will prove the following inequality:

$$\sum_{(i,j)\in I^2}p_jq_i\left\|\bar{\alpha}_ix_j-\bar{\alpha}_jx_i\right\|^2\geqslant 2\mu(I,p,\alpha,x)\,\mu(I,q,\alpha,x) \tag{2.3}$$

which is equivalent to:

$$\begin{aligned}
&\left(\sum_{(i,j)\in I^2}p_iq_j\left\|\bar{\alpha}_ix_j-\bar{\alpha}_jx_i\right\|^2\right)^2\\
&\qquad\geqslant\sum_{(i,j)\in I^2}p_ip_j\left\|\bar{\alpha}_ix_j-\bar{\alpha}_jx_i\right\|^2\times\sum_{(i,j)\in I^2}q_iq_j\left\|\bar{\alpha}_ix_j-\bar{\alpha}_jx_i\right\|^2.
\end{aligned}\tag{2.4}$$

As we have:

$$\begin{aligned}
&\sum_{(i,j)\in I^2}p_iq_j\left\|\bar{\alpha}_ix_j-\bar{\alpha}_jx_i\right\|^2\\
&\qquad=\sum_{i\in I}p_i|\alpha_i|^2\sum_{i\in I}q_i\|x_i\|^2+\sum_{i\in I}p_i\|x_i\|^2\sum_{i\in I}q_i|\alpha_i|^2\\
&\qquad\qquad-2\operatorname{Re}\left(\sum_{i\in I}p_i\alpha_ix_i,\sum_{i\in I}q_i\alpha_ix_i\right)
\end{aligned}$$

and

$$\sum_{(i,j)\in I^2} p_i p_j \left\|\bar{\alpha}_i x_j - \bar{\alpha}_j x_i\right\|^2 = 2\left(\sum_{i\in I} p_i \left|\alpha_i\right|^2 \sum_{i\in I} p_i \left\|x_i\right\|^2 - \left\|\sum_{i\in I} p_i \alpha_i x_i\right\|^2\right)$$

and

$$\sum_{(i,j)\in I^2} q_i q_j \left\|\bar{\alpha}_i x_j - \bar{\alpha}_j x_i\right\|^2 = 2\left(\sum_{i\in I} q_i \left|\alpha_i\right|^2 \sum_{i\in I} q_i \left\|x_i\right\|^2 - \left\|\sum_{i\in I} q_i \alpha_i x_i\right\|^2\right),$$

the inequality (2.4) becomes:

$$\begin{aligned} &\left[\sum_{i\in I} p_i \left|\alpha_i\right|^2 \sum_{i\in I} q_i \left\|x_i\right\|^2 + \sum_{i\in I} p_i \left\|x_i\right\|^2 \sum_{i\in I} q_i \left|\alpha_i\right|^2 \right. \\ &\qquad\qquad \left. - 2\operatorname{Re}\left(\sum_{i\in I} p_i \alpha_i x_i, \sum_{i\in I} q_i \alpha_i x_i\right)\right]^2 \\ &\qquad \geqslant 4\left(\sum_{i\in I} p_i \left|\alpha_i\right|^2 \sum_{i\in I} p_i \left\|x_i\right\|^2 - \left\|\sum_{i\in I} p_i \alpha_i x_i\right\|^2\right) \\ &\qquad\qquad \times \left(\sum_{i\in I} q_i \left|\alpha_i\right|^2 \sum_{i\in I} q_i \left\|x_i\right\|^2 - \left\|\sum_{i\in I} q_i \alpha_i x_i\right\|^2\right). \end{aligned} \tag{2.5}$$

Denote

$$a := \left(\sum_{i\in I} p_i \left|\alpha_i\right|^2\right)^{\frac{1}{2}},\ b := \left(\sum_{i\in I} p_i \left\|x_i\right\|^2\right)^{\frac{1}{2}} \text{ and } x := \sum_{i\in I} p_i \alpha_i x_i \in H.$$

$$c := \left(\sum_{i\in I} q_i \left|\alpha_i\right|^2\right)^{\frac{1}{2}},\ d := \left(\sum_{i\in I} q_i \left\|x_i\right\|^2\right)^{\frac{1}{2}} \text{ and } y := \sum_{i\in I} q_i \alpha_i x_i \in H.$$

By the use of these notations, the inequality (2.5) becomes

$$\left(a^2 d^2 + b^2 c^2 - 2\operatorname{Re}(x,y)\right)^2 \geqslant 4\left(a^2 b^2 - \|x\|^2\right)\left(c^2 d^2 - \|y\|^2\right) \geqslant 0. \tag{2.6}$$

Now, let us observe that a simple calculation shows us:

$$(abcd - \|x\|\|y\|)^2 \geqslant \left(a^2 b^2 - \|x\|^2\right)\left(c^2 d^2 - \|y\|^2\right) \geqslant 0. \tag{2.7}$$

Since, by the Cauchy-Buniakowsky-Schwartz inequality (1.1) we have

$$abcd \geqslant \|x\|\|y\|,$$

then it is sufficient to prove that

$$a^2d^2 + b^2c^2 - 2\operatorname{Re}(x,y) \geqslant 2\left(abcd - \|x\|\,\|y\|\right) \geqslant 0. \tag{2.8}$$

However,

$$a^2d^2 + b^2c^2 \geqslant 2abcd, \quad a,b,c,d \geqslant 0$$

and Schwartz's inequality in the inner product space $(H;(\cdot,\cdot))$ tells us that

$$\|x\|\,\|y\| \geqslant \operatorname{Re}(x,y).$$

Therefore, the inequality (2.8) is proved. Now, using the inequalities (2.7) and (2.8) we get (2.6). That is, the inequality (2.3) holds.

Finally, we have

$$\begin{aligned}\mu^2(I,p+q,\alpha,x) &\geqslant \mu^2(I,p,\alpha,x) + \mu^2(I,q,\alpha,x) + 2\mu(I,p,\alpha,x)\,\mu(I,q,\alpha,x)\\ &= \left[\mu(I,p,\alpha,x) + \mu(I,q,\alpha,x)\right]^2\end{aligned}$$

and the inequality (2.1) is proved.

(ii) Let $p,q \in \mathfrak{J}_+(\mathbb{R})$ with $p \geqslant q$. Then

$$\mu(I,p,\alpha,x) = \mu(I,q+(p-q),\alpha,x) \geqslant \mu(I,q,\alpha,x) + \mu(I,p-q,\alpha,x)$$

which indicates that

$$\mu(I,p,\alpha,x) - \mu(I,q,\alpha,x) \geqslant \mu(I,p-q,\alpha,x) \geqslant 0$$

and the inequality is proved.

□

The following corollary is important as it gives an interesting refinement of the Cauchy-Buniakowsky-Schwartz inequality (1.1).

COROLLARY 1. *Let $\alpha \in \mathfrak{J}(\mathbb{K})$, $x \in \mathfrak{J}(H)$ and $\beta = (\beta_i)_{i\in\mathbb{N}} \in \mathfrak{J}(\mathbb{R})$. Then for all $I \in \mathfrak{P}_f(\mathbb{N})$ we have the inequality:*

$$\begin{aligned}&\left(\sum_{i\in I}|\alpha_i|^2\sum_{i\in I}\|x_i\|^2 - \left\|\sum_{i\in I}\alpha_i x_i\right\|^2\right)^{\frac{1}{2}}\\ &\qquad\geqslant \left(\sum_{i\in I}|\alpha_i|^2\sin^2\beta_i\sum_{i\in I}\|x_i\|^2\sin^2\beta_i - \left\|\sum_{i\in I}(\sin\beta_i)^2\alpha_i x_i\right\|^2\right)^{\frac{1}{2}}\\ &\qquad\qquad + \left(\sum_{i\in I}|\alpha_i|^2\cos^2\beta_i\sum_{i\in I}\|x_i\|^2\cos^2\beta_i - \left\|\sum_{i\in I}(\cos\beta_i)^2\alpha_i x_i\right\|^2\right)^{\frac{1}{2}}\\ &\qquad\geqslant 0.\end{aligned}$$

The next bound also holds.

COROLLARY 2. *Consider the set of sequences*

$$S(\mathbb{I}) := \left\{ p = (p_i)_{i\in\mathbb{N}} \,|\, 0 \leqslant p_i \leqslant 1 \text{ for all } i \in \mathbb{N} \right\}.$$

Then one has the bound:

$$\sum_{i\in I} |\alpha_i|^2 \sum_{i\in I} \|x_i\|^2 - \left\| \sum_{i\in I} \alpha_i x_i \right\|^2 = \sup_{p\in S(\mathbb{I})} \left[\sum_{i\in I} p_i |\alpha_i|^2 \sum_{i\in I} p_i \|x_i\|^2 - \left\| \sum_{i\in I} p_i \alpha_i x_i \right\|^2 \right] \geqslant 0.$$

3. The superadditivity of the mapping $\mu(\cdot, p, \alpha, x)$

In this section we will investigate the mapping $\mu(\cdot, p, \alpha, x)$ as an index set function defined on $\mathfrak{P}_f(\mathbb{N})$.

The main result is embodied in the following theorem.

THEOREM 2. *Let* $p \in \mathfrak{J}_+(\mathbb{R})$, $\alpha \in \mathfrak{J}(\mathbb{K})$ *and* $x \in \mathfrak{J}(H)$. *Then*

(i) *For all* $I, J \in \mathfrak{P}_f(\mathbb{N})$ *with* $I \cap J = \emptyset$ *one has the inequality:*

$$\mu(I \cup J, p, \alpha, x) \geqslant \mu(I, p, \alpha, x) + \mu(J, p, \alpha, x) \geqslant 0. \tag{3.1}$$

That is, the mapping $\mu(\cdot, p, \alpha, x)$ *is superadditive as an index set function defined on* $\mathfrak{P}_f(\mathbb{N})$;

(ii) *For all* $I, J \in \mathfrak{P}_f(\mathbb{N})$ *with* $\emptyset \neq I \subseteq J$ *we have*

$$\mu(I, p, \alpha, x) \geqslant \mu(J, p, \alpha, x) \geqslant 0. \tag{3.2}$$

That is, the mapping $\mu(\cdot, p, \alpha, x)$ *is monotonic decreasing on* $\mathfrak{P}_f(\mathbb{N})$.

Proof. (i) We have, for all $I, J \in \mathfrak{P}_f(\mathbb{N})$ with $I \cap J = \emptyset$ that:

$$\begin{aligned}
&\mu^2(I \cup J, p, \alpha, x) \\
&\quad = \frac{1}{2} \sum_{(i,j)\in(I\cup J)^2} p_i p_j \|\bar{\alpha}_i x_j - \bar{\alpha}_j x_i\|^2 \\
&\quad = \frac{1}{2} \sum_{(i,j)\in I^2} p_i p_j \|\bar{\alpha}_i x_j - \bar{\alpha}_j x_i\|^2 + \frac{1}{2} \sum_{(i,j)\in J^2} p_i p_j \|\bar{\alpha}_i x_j - \bar{\alpha}_j x_i\|^2 \\
&\quad = \frac{1}{2} \sum_{(i,j)\in I\times J} p_i p_j \|\bar{\alpha}_i x_j - \bar{\alpha}_j x_i\|^2 + \frac{1}{2} \sum_{(i,j)\in J\times I} p_i p_j \|\bar{\alpha}_i x_j - \bar{\alpha}_j x_i\|^2 \\
&\quad = \mu^2(I, p, \alpha, x) + \mu^2(J, p, \alpha, x) + \sum_{(i,j)\in I\times J} p_i p_j \|\bar{\alpha}_i x_j - \bar{\alpha}_j x_i\|^2,
\end{aligned}$$

because

$$\sum_{(i,j)\in I\times J} p_i p_j \left\|\bar{\alpha}_i x_j - \bar{\alpha}_j x_i\right\|^2 = \sum_{(i,j)\in J\times I} p_i p_j \left\|\bar{\alpha}_i x_j - \bar{\alpha}_j x_i\right\|^2 .$$

We will prove the following inequality

$$\sum_{(i,j)\in I\times J} p_i p_j \left\|\bar{\alpha}_i x_j - \bar{\alpha}_j x_i\right\|^2 \geqslant 2\mu\left(I,p,\alpha,x\right)\mu\left(J,p,\alpha,x\right) \tag{3.3}$$

which is the equivalent to

$$\begin{aligned}
&\left[\sum_{i\in I} p_i\left|\alpha_i\right|^2 \sum_{j\in J} p_j \left\|x_j\right\|^2 + \sum_{j\in J} p_j\left|\alpha_j\right|^2 \sum_{i\in I} p_i \left\|x_i\right\|^2 \right.\\
&\qquad\left. -\,2\operatorname{Re}\left(\sum_{i\in I} p_i\alpha_i x_i, \sum_{j\in J} p_j\alpha_j x_j\right)\right]^2\\
&\geqslant 4\left(\sum_{i\in I} p_i\left|\alpha_i\right|^2 \sum_{i\in I} p_i\left\|x_i\right\|^2 - \left\|\sum_{i\in I} p_i\alpha_i x_i\right\|^2\right)\\
&\qquad\times\left(\sum_{j\in J} p_j\left|\alpha_j\right|^2 \sum_{j\in J} p_j\left\|x_j\right\|^2 - \left\|\sum_{j\in J} p_j\alpha_j x_j\right\|^2\right).
\end{aligned} \tag{3.4}$$

Now, if we denote

$$a := \left(\sum_{i\in I} p_i\left|\alpha_i\right|^2\right)^{\frac{1}{2}},\ b := \left(\sum_{i\in I} p_i\left\|x_i\right\|^2\right)^{\frac{1}{2}} \text{ and } x := \sum_{i\in I} p_i\alpha_i x_i \in H,$$

$$c := \left(\sum_{j\in J} p_j\left|\alpha_j\right|^2\right)^{\frac{1}{2}},\ d := \left(\sum_{j\in J} p_j\left\|x_j\right\|^2\right)^{\frac{1}{2}} \text{ and } y := \sum_{j\in J} p_j\alpha_j x_j \in H,$$

then the above inequality is equivalent to:

$$\left(a^2d^2 + b^2c^2 - 2\operatorname{Re}\left(x,y\right)\right)^2 \geqslant 4\left(a^2b^2 - \|x\|^2\right)\left(c^2d^2 - \|y\|^2\right) \geqslant 0 \tag{3.5}$$

which is exactly the inequality (2.6) that was proved above.

Consequently, we have:

$$\begin{aligned}
\mu^2\left(I\cup J,p,\alpha,x\right) &\geqslant \mu^2\left(I,p,\alpha,x\right) + \mu^2\left(J,p,\alpha,x\right) + 2\mu\left(I,p,\alpha,x\right)\mu\left(J,p,\alpha,x\right)\\
&= \left(\mu\left(I,p,\alpha,x\right) + \mu\left(J,p,\alpha,x\right)\right)^2
\end{aligned}$$

which proves the desired inequality (3.1).

(ii) Suppose that $\emptyset \neq I \subset J$ and $J \neq I$. Then we have successively:

$$\begin{aligned}
\mu\left(I,p,\alpha,x\right) &= \mu\left(I\cup\left(I\backslash J\right),p,\alpha,x\right)\\
&\geqslant \mu\left(J,p,\alpha,x\right) + \mu\left(I\backslash J,p,\alpha,x\right)
\end{aligned}$$

which gives us

$$\mu(I,p,\alpha,x) - \mu(J,p,\alpha,x) \geqslant \mu(I\backslash J,p,\alpha,x) \geqslant 0,$$

and the theorem is proved. □

The following corollaries are interesting.

COROLLARY 3. *Let $p \in \mathfrak{J}_+(\mathbb{R})$, $\alpha \in \mathfrak{J}(\mathbb{K})$ and $x \in \mathfrak{J}(H)$. Then we have the inequality:*

$$\begin{aligned}
&\left(\sum_{i=1}^{2n} p_i |\alpha_i|^2 \sum_{i=1}^{2n} p_i \|x_i\|^2 - \left\|\sum_{i=1}^{2n} p_i\alpha_i x_i\right\|^2\right)^{\frac{1}{2}} \\
&\quad \geqslant \left(\sum_{i=1}^{n} p_{2i-1} |\alpha_{2i-1}|^2 \sum_{i=1}^{n} p_{2i-1} \|x_{2i-1}\|^2 - \left\|\sum_{i=1}^{n} p_{2i-1}\alpha_{2i-1} x_{2i-1}\right\|^2\right)^{\frac{1}{2}} \\
&\qquad + \left(\sum_{i=1}^{n} p_{2i} |\alpha_{2i}|^2 \sum_{i=1}^{n} p_{2i} \|x_{2i}\|^2 - \left\|\sum_{i=1}^{n} p_{2i}\alpha_{2i} x_{2i}\right\|^2\right)^{\frac{1}{2}} \\
&\quad \geqslant 0.
\end{aligned}$$

COROLLARY 4. *With the above assumptions, we have:*

$$\begin{aligned}
&\sum_{i=1}^{n} p_i |\alpha_i|^2 \sum_{i=1}^{n} p_i \|x_i\|^2 - \left\|\sum_{i=1}^{n} p_i\alpha_i x_i\right\|^2 \\
&\qquad = \sup_{I\subseteq I_n}\left[\sum_{i\in I} p_i |\alpha_i|^2 \sum_{i\in I} p_i \|x_i\|^2 - \left\|\sum_{i\in I} p_i\alpha_i x_i\right\|^2\right] \geqslant 0,
\end{aligned}$$

where $I_n := \{1,2,...,n\}$ and

$$\sum_{i=1}^{n} p_i |\alpha_i|^2 \sum_{i=1}^{n} p_i \|x_i\|^2 - \left\|\sum_{i=1}^{n} p_i\alpha_i x_i\right\|^2 \geqslant \max_{1\leqslant i<j\leqslant n}\left\{p_i p_j \|\bar{\alpha}_i x_j - \bar{\alpha}_j x_i\|^2\right\} \geqslant 0.$$

The proof of this corollary follows by the statement (ii) of the above theorem. We shall omit the details.

Note that similar results were proved in the paper [3].

4. Some properties of the mapping $\mu(I,p,\alpha,x,\cdot)$

Now let X be a linear space over the real or complex number field $\mathbb{K}$, and $\mathcal{H}(X)$ the class of all non-negative Hermitian forms on X. That is, a mapping belongs to $\mathcal{H}(X)$ if it satisfies the conditions:

(i) $(x,x) \geqslant 0$ for all $x \in X$;
(ii) $(\alpha x + \beta y, z) = \alpha (x,z) + \beta (y,z)$ for all $x,y,z \in X$ and $\alpha, \beta \in \mathbb{K}$;
(iii) $(y,x) = \overline{(x,y)}$ for all $x,y \in X$

If $(\cdot,\cdot) \in \mathcal{H}(X)$, then the following inequality $(x,x)(y,y) \geqslant |(x,y)|^2$ or $\|x\| \|y\| \geqslant |(x,y)|$ for all $x,y \in X$ where $\|x\| = (x,x)^{\frac{1}{2}}$ is the semi-norm associated with the Hermitian form $(\cdot,\cdot)$ is well-known as Schwartz's inequality for Hermitian forms.

Now, let us observe that $\mathcal{H}(X)$ is a convex cone in the linear space of all mappings defined on X^2 with values in $\mathbb{K}$. That is,

(i) $(\cdot,\cdot)_1, (\cdot,\cdot)_2 \in \mathcal{H}(X)$ implies that $(\cdot,\cdot)_1 + (\cdot,\cdot)_2 \in \mathcal{H}(X)$;
(ii) $\alpha \geqslant 0$ and $(\cdot,\cdot) \in \mathcal{H}(X)$ implies that $\alpha(\cdot,\cdot) \in \mathcal{H}(X)$.

Also, we can introduce on $\mathcal{H}(X)$ the following binary relation: $(\cdot,\cdot)_2 \geqslant (\cdot,\cdot)_1$ iff $\|x\|_2 \geqslant \|x\|_1$ for all $x \in X$.

We observe that:

(b) $(\cdot,\cdot) \geqslant (\cdot,\cdot)$ for all $(\cdot,\cdot) \in \mathcal{H}(X)$;
(bb) $(\cdot,\cdot)_3 \geqslant (\cdot,\cdot)_2$ and $(\cdot,\cdot)_2 \geqslant (\cdot,\cdot)_1$ implies that $(\cdot,\cdot)_3 \geqslant (\cdot,\cdot)_1$;
(bbb) $(\cdot,\cdot)_2 \geqslant (\cdot,\cdot)_1$ and $(\cdot,\cdot)_2 \geqslant (\cdot,\cdot)_1$ implies that $(\cdot,\cdot)_2 = (\cdot,\cdot)_1$.

That is, the relation "$\geqslant$" is an order relation on $\mathcal{H}(X)$.

To prove the relation (bbb) we observe that $(\cdot,\cdot)_2 \geqslant (\cdot,\cdot)_1$ and $(\cdot,\cdot)_1 \geqslant (\cdot,\cdot)_2$ is equivalent to $\|x\|_2 = \|x\|_1$ for all $x \in X$, which implies, by the following relation

$$(x,y)_k = \frac{1}{4}\left[\|x+y\|_k^2 - \|x-y\|_k^2 + i\|x+iy\|_k - i\|x-iy\|_k\right], \ x,y \in X, \ k = \overline{1,2}$$

that $(x,y)_2 = (x,y)_1$ for all $x,y \in X$.

Now, let us consider the mapping

$$\mu : \mathfrak{P}_f(\mathbb{N}) \times \mathfrak{J}_+(\mathbb{R}) \times \mathfrak{J}(\mathbb{K}) \times \mathfrak{J}(H) \times \mathcal{H}(X) \longrightarrow \mathbb{R}$$

given by:

$$\begin{aligned}
\mu(I,p,\alpha,x,(\cdot,\cdot)) &:= \left[\sum_{i\in I} p_i |\alpha_i|^2 \sum_{i\in I} p_i (x_i,x_i) - \left(\sum_{i\in I} p_i\alpha_i x_i, \sum_{i\in I} p_i\alpha_i x_i\right)\right]^{\frac{1}{2}} \\
&= \left[\sum_{i\in I} p_i |\alpha_i|^2 \sum_{i\in I} p_i \|x_i\|^2 - \left\|\sum_{i\in I} p_i\alpha_i x_i\right\|^2\right]^{\frac{1}{2}}.
\end{aligned}$$

The main properties of the mapping $\mu(I,p,\alpha,x,\cdot)$ are embodied in the following theorem.

THEOREM 3. *Let $I \in \mathfrak{P}_f(\mathbb{N})$, $p \in \mathfrak{J}_+(\mathbb{R})$, $\alpha \in \mathfrak{J}(\mathbb{K})$ and $x \in \mathfrak{J}(H)$. Then*

(i) *For all $(\cdot,\cdot)_1, (\cdot,\cdot)_2 \in \mathcal{H}(X)$ we have the inequality*

$$\mu(I,p,\alpha,x,(\cdot,\cdot)_1 + (\cdot,\cdot)_2) \leqslant \mu(I,p,\alpha,x,(\cdot,\cdot)_1) + \mu(I,p,\alpha,x,(\cdot,\cdot)_2). \tag{4.1}$$

That is, the mapping $\mu(I,p,\alpha,x,\cdot)$ is subadditive on $\mathcal{H}(X)$;

(*ii*) *For all* $(\cdot,\cdot) \in \mathcal{H}(X)$ *and* $c \geqslant 0$ *we have:*

$$\mu(I,p,\alpha,x,\alpha(\cdot,\cdot)) = \alpha\mu(I,p,\alpha,x,(\cdot,\cdot)_1).$$

That is, the mapping $\mu(I,p,\alpha,x,\cdot)$ *is positive homogeneous on* $\mathcal{H}(X)$;

(*iii*) *For all* $(\cdot,\cdot)_2 \geqslant (\cdot,\cdot)_1, (\cdot,\cdot)_i \in \mathcal{H}(X)$ $\left(i=\overline{1,2}\right)$, *we have that:*

$$\mu(I,p,\alpha,x,(\cdot,\cdot)_2) \geqslant \mu(I,p,\alpha,x,(\cdot,\cdot)_1) \geqslant 0.$$

That is, the mapping $\mu(I,p,\alpha,x,\cdot)$ *is monotonic nondecreasing.*

Proof. (*i*) Let $(\cdot,\cdot)_i \in \mathcal{H}(X)$ $\left(i=\overline{1,2}\right)$. We have

$$\begin{aligned}
&\mu(I,p,\alpha,x,(\cdot,\cdot)_1+(\cdot,\cdot)_2)\\
&\quad=\left[\sum_{i\in I}p_i|\alpha_i|^2\sum_{i\in I}p_i\left(\|x_i\|_1^2+\|x_i\|_2^2\right)-\left\|\sum_{i\in I}p_i\alpha_ix_i\right\|_1^2-\left\|\sum_{i\in I}p_i\alpha_ix_i\right\|_2^2\right]^{\frac{1}{2}}\\
&\quad=\left[\sum_{i\in I}p_i|\alpha_i|^2\sum_{i\in I}p_i\|x_i\|_1^2-\left\|\sum_{i\in I}p_i\alpha_ix_i\right\|_1^2\right.\\
&\qquad\left.+\sum_{i\in I}p_i|\alpha_i|^2\sum_{i\in I}p_i\|x_i\|_2^2-\left\|\sum_{i\in I}p_i\alpha_ix_i\right\|_2^2\right]^{\frac{1}{2}}\\
&\quad\leqslant\left[\sum_{i\in I}p_i|\alpha_i|^2\sum_{i\in I}p_i\|x_i\|_1^2-\left\|\sum_{i\in I}p_i\alpha_ix_i\right\|_1^2\right]^{\frac{1}{2}}\\
&\qquad+\left[\sum_{i\in I}p_i|\alpha_i|^2\sum_{i\in I}p_i\|x_i\|_2^2-\left\|\sum_{i\in I}p_i\alpha_ix_i\right\|_2^2\right]^{\frac{1}{2}}\\
&\quad=\mu(I,p,\alpha,x,(\cdot,\cdot)_1)+\mu(I,p,\alpha,x,(\cdot,\cdot)_2).
\end{aligned}$$

(*ii*) It is obvious.

(*iii*) Suppose that $(\cdot,\cdot)_2 \geqslant (\cdot,\cdot)_1$, i.e., $\|x\|_2 \geqslant \|x\|_1$ for all $x \in X$. Then, by Lagrange's identity, we have:

$$\begin{aligned}
\mu(I,p,\alpha,x,(\cdot,\cdot)_2) &= \left(\frac{1}{2}\sum_{(i,j)\in I^2}p_ip_j\|\bar{\alpha}_ix_j-\bar{\alpha}_jx_i\|_2^2\right)^{\frac{1}{2}}\\
&\geqslant \left(\frac{1}{2}\sum_{(i,j)\in I^2}p_ip_j\|\bar{\alpha}_ix_j-\bar{\alpha}_jx_i\|_1^2\right)^{\frac{1}{2}}\\
&= \mu(I,p,\alpha,x,(\cdot,\cdot)_1)
\end{aligned}$$

and the theorem is proved.

□

The following corollaries are interesting.

COROLLARY 5. *Let* $A : H \longrightarrow H$ *be a bounded linear operator and*

$$\|A\| = \sup\{\|Ax\|, \|x\| = 1\}.$$

Then, for all $p \in \mathfrak{J}_+(\mathbb{R})$, $\alpha \in \mathfrak{J}(\mathbb{K})$ *and* $x \in \mathfrak{J}(H)$ *we have the inequality:*

$$\|A\|^2\left(\sum_{i\in I} p_i|\alpha_i|^2 \sum_{i\in I} p_i\|x_i\|^2 - \left\|\sum_{i\in I} p_i\alpha_i x_i\right\|^2\right)$$
$$\geqslant \sum_{i\in I} p_i|\alpha_i|^2 \sum_{i\in I} p_i\|Ax_i\|^2 - \left\|\sum_{i\in I} p_i\alpha_i Ax_i\right\|^2 \geqslant 0.$$

Proof. The argument follows by the statement (iii) of the above theorem for the norms: $\|x\|_2 := \|A\|\,\|x\|$ and $\|x\|_1 := \|Ax\|$, $x \in H$.

□

COROLLARY 6. *Let* $A : H \longrightarrow H$ *be a positive linear operator with the property that* $(Ax, x) \geqslant m\|x\|^2$ *for all* $x \in H$. *Then for all* $p \in \mathfrak{J}_+(\mathbb{R})$, $\alpha \in \mathfrak{J}(\mathbb{K})$ *and* $x \in \mathfrak{J}(H)$ *we have the inequality:*

$$\sum_{i\in I} p_i|\alpha_i|^2 \sum_{i\in I} p_i(Ax_i, x_i) - \left(\sum_{i\in I} p_i\alpha_i Ax_i, \sum_{i\in I} p_i\alpha_i x_i\right)$$
$$\geqslant m\left(\sum_{i\in I} p_i|\alpha_i|^2 \sum_{i\in I} p_i\|x_i\|^2 - \left\|\sum_{i\in I} p_i\alpha_i x_i\right\|^2\right) \geqslant 0.$$

Proof. The proof is obvious from the statement (iii) of the above theorem applied to the norms $\|x\|_2 := [(Ax, x)]^{\frac{1}{2}}$ and $\|x\|_1 := m^{\frac{1}{2}}\|x\|$, $x \in H$.

□

Finally, the following corollary also holds.

COROLLARY 7. *Let* $\{l_\alpha\}_{\alpha\in\mathcal{A}}$ *be a family of orthonormal vectors in the linear product space* H *where* p, α, x *are as above. Then we have the following refinement of the Cauchy-Buniakowsky-Schwartz inequality:*

$$\sum_{i\in I} p_i|\alpha_i|^2 \sum_{i\in I} p_i\|x_i\|^2 - \left\|\sum_{i\in I} p_i\alpha_i x_i\right\|^2$$
$$\geqslant \sum_{i\in I} p_i|\alpha_i|^2 \sum_{\substack{i\in I\\ \alpha\in\mathcal{A}}} p_i|(x_i, l_\alpha)|^2 - \sum_{\alpha\in\mathcal{A}}\left|\sum_{i\in I} p_i\alpha_i(x_i, l_\alpha)\right|^2 \geqslant 0.$$

The argument follows by the above theorem, choosing $\|x\|_2 := \|x\|$ and $\|x\|_1 := \left(\sum_{\alpha\in\mathcal{A}} |(x,l_\alpha)|^2\right)^{\frac{1}{2}}$. The fact that $\|\cdot\|_2 \geqslant \|\cdot\|_1$ follows by the well-known Bessel's inequality

$$\|x\|^2 \geqslant \sum_{\alpha\in\mathcal{A}} |(x,l_\alpha)|^2, \; x\in H.$$

5. Some properties of supermultiplicity for the Cauchy-Buniakowsky-Schwartz inequality

Further on, we will study the following mappings associated with the Cauchy-Buniakowsky-Schwartz Inequality:

$$\nu,\varphi : \mathfrak{P}_f(\mathbb{N}) \times \mathfrak{J}_+(\mathbb{R}) \times \mathfrak{J}(\mathbb{K}) \times \mathfrak{J}(H) \longrightarrow \mathbb{R}$$

given by

$$\begin{aligned}\nu(I,p,\alpha,x) &:= \frac{1}{P_I}\mu(I,p,\alpha,x)\\ &= \left(\frac{1}{P_I}\sum_{i\in I} p_i |\alpha_i|^2 \frac{1}{P_I}\sum_{i\in I} p_i \|x_i\|^2 - \left\|\frac{1}{P_I}\sum_{i\in I} p_i\alpha_i x_i\right\|^2\right)^{\frac{1}{2}}\end{aligned}$$

and

$$\begin{aligned}\varphi(I,p,\alpha,x) &:= [\nu(I,p,\alpha,x)]^{P_I}\\ &= \left(\frac{1}{P_I}\sum_{i\in I} p_i |\alpha_i|^2 \frac{1}{P_I}\sum_{i\in I} p_i \|x_i\|^2 - \left\|\frac{1}{P_I}\sum_{i\in I} p_i\alpha_i x_i\right\|^2\right)^{\frac{P_I}{2}},\end{aligned}$$

where $P_I > 0$.

The following property of supermultiplicity holds.

THEOREM 4. *Let $I \in \mathfrak{P}_f(\mathbb{N})$, $\alpha \in \mathfrak{J}(\mathbb{K})$ and $x \in \mathfrak{J}(H)$. Then for all $p,q \in \mathfrak{J}_+(H)$ with $P_I, Q_I > 0$ we have the inequality:*

$$\varphi(I,p+q,\alpha,x) \geqslant \varphi(I,p,\alpha,x)\,\varphi(I,q,\alpha,x) \geqslant 0. \tag{5.1}$$

That is, the mapping $\varphi(I,\cdot,\alpha,x)$ is supermultiplicative on $\mathfrak{J}_+(\mathbb{R})$.

Proof. Using the well-known arithmetic mean-geometric mean inequality for real numbers, i.e., we recall that:

$$\frac{\alpha a + \beta b}{\alpha+\beta} \geqslant a^{\frac{\alpha}{\alpha+\beta}} b^{\frac{\beta}{\alpha+\beta}},$$

where $a,b \geqslant 0$ and $\alpha,\beta \geqslant 0$ with $\alpha+\beta>0,$ we have successively:

$$\begin{aligned} \nu(I,p+q,\alpha,x) &= \frac{1}{P_I+Q_I}\mu(I,p+q,\alpha,x) \\ &\geqslant \frac{\mu(I,p,\alpha,x)+\mu(I,q,\alpha,x)}{P_I+Q_I} \\ &= \frac{P_I\nu(I,p,\alpha,x)+Q_I\nu(I,q,\alpha,x)}{P_I+Q_I} \\ &\geqslant [\nu(I,p,\alpha,x)]^{\frac{P_I}{P_I+Q_I}}[\nu(I,q,\alpha,x)]^{\frac{Q_I}{P_I+Q_I}}, \end{aligned}$$

which gives us:

$$[\nu(I,p+q,\alpha,x)]^{(P_I+Q_I)} \geqslant [\nu(I,p,\alpha,x)]^{P_I}[\nu(I,q,\alpha,x)]^{Q_I}$$

and the inequality (5.1) is proved.

□

The following refinement of the Cauchy-Buniakowsky-Schwartz inequality is equivalent to (5.1) :

$$\begin{aligned} &\frac{1}{P_I+Q_I}\sum_{i\in I}(p_i+q_i)|\alpha_i|^2\frac{1}{P_I+Q_I}\sum_{i\in I}(p_i+q_i)\|x_i\|^2 \\ &\qquad -\left\|\frac{1}{P_I+Q_I}\sum_{i\in I}(p_i+q_i)\alpha_i x_i\right\|^2 \\ &\geqslant \left(\frac{1}{P_I}\sum_{i\in I}p_i|\alpha_i|^2\frac{1}{P_I}\sum_{i\in I}p_i\|x_i\|^2-\left\|\frac{1}{P_I}\sum_{i\in I}p_i\alpha_i x_i\right\|^2\right)^{\frac{P_I}{P_I+Q_I}} \\ &\qquad \times\left(\frac{1}{Q_I}\sum_{i\in I}q_i|\alpha_i|^2\frac{1}{Q_I}\sum_{i\in I}q_i\|x_i\|^2-\left\|\frac{1}{Q_I}\sum_{i\in I}q_i\alpha_i x_i\right\|^2\right)^{\frac{Q_I}{P_I+Q_I}} \\ &\geqslant 0, \end{aligned} \tag{5.2}$$

where $p,q\in\mathfrak{J}_+(\mathbb{R})$ with $P_I,Q_I>0$ and α,x,I are as above.

Finally, by the use Theorem 2, we also have the result:

THEOREM 5. *Let $p\in\mathfrak{J}_+(\mathbb{R})$, $\alpha\in\mathfrak{J}(\mathbb{K})$ and $x\in\mathfrak{J}(H)$. Then, for all $I,J\in\mathfrak{P}_f(\mathbb{N})$, with $I\cap J=\emptyset$ and $P_I,Q_I>0,$ we have the inequality:*

$$\varphi(I\cup J,p,\alpha,x)\geqslant\varphi(I,p,\alpha,x)\,\varphi(J,p,\alpha,x)\geqslant 0. \tag{5.3}$$

That is, the mapping $\varphi(\cdot,p,\alpha,x)$ is supermultiplicative as an index map on $\mathfrak{P}_f(\mathbb{N})$.

Note that the inequality (5.3) is equivalent to the following refinement of the Cauchy-Buniakowsky-Schwartz inequality:

$$
\begin{aligned}
&\frac{1}{P_{I\cup J}}\sum_{i\in I\cup J}p_i\left|\alpha_i\right|^2\sum_{i\in I\cup J}p_i\left\|x_i\right\|^2-\left\|\frac{1}{P_{I\cup J}}\sum_{i\in I\cup J}p_i\alpha_i x_i\right\|^2\\
&\quad\geqslant\left(\frac{1}{P_I}\sum_{i\in I}p_i\left|\alpha_i\right|^2\frac{1}{P_I}\sum_{i\in I}p_i\left\|x_i\right\|^2-\left\|\frac{1}{P_I}\sum_{i\in I}p_i\alpha_i x_i\right\|^2\right)^{\frac{P_I}{P_{I\cup J}}}\\
&\qquad\times\left(\frac{1}{P_J}\sum_{i\in J}p_i\left|\alpha_i\right|^2\frac{1}{P_J}\sum_{i\in J}p_i\left\|x_i\right\|^2-\left\|\frac{1}{P_J}\sum_{i\in J}p_i\alpha_i x_i\right\|^2\right)^{\frac{P_J}{P_{I\cup J}}}\\
&\quad\geqslant 0. \qquad (5.4)
\end{aligned}
$$

For other results of the Cauchy-Buniakowsky-Schwartz's type in inner product spaces, see the papers [1]-[12] and the book [13].

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