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CRITERIA FOR BOUNDEDNESS OF FUZZY DIFFERENTIAL EQUATIONS

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Abstract. The first part of this paper deals with the existence of maximum internal of existence for fuzzy differential equation. Then, using the idea of perturbing Lyapunov functions, is discussed the boundedness of solutions for fuzzy differential equation.

1. Introduction

A differential and integral calculus for fuzzy-valued mappings has been developed in papers [1, 2, 3, 10] and the investigation of fuzzy differential equations has been initiated in [4, 5, 7, 10, 6, 8].

In this paper, we first study the existence of maximum interval of solution of fuzzy differential equations and then develop the concepts of boundedness which corresponds to Lyapunov theory for fuzzy differential systems. Then, using comparison result in [7], we discuss UUB and utilizing the method of perturbing Lyapunov method [9], we obtain the equibounded criteria.

2. Preliminaries

Let $P_k(R^n)$ denote the family of all nonempty compact, convex subsets of R^n. If $\alpha, \beta \in R$ and $A, B \in P_k(R^n)$, then

$$\alpha(A+B) = \alpha A + \beta B, \quad \alpha(\beta A) = (\alpha\beta)A, \quad 1A = A$$

and if $\alpha, \beta \geqslant 0$, then $(\alpha+\beta)A = \alpha A + \beta B$. Let $I = [t_0, t_0 + a], t_0 \geqslant 0$ and $a > 0$. For any set $A \subset R^n$, we denote by clA, A^c and ∂A, the closure, the complement and the boundary. Denote

$$E^n = \{u : R^n \to [0,1] \ \ |u \text{ satisfies } (i) - (iv) \text{ below}\},$$

(i) u is normal, that is, there exists an $x_0 \in R^n$ such that $u(x_0) = 1$;
(ii) u is fuzzy convex, that is, for $x, y \in R^n$ and $0 \leqslant \lambda \leqslant 1$,

$$u(\lambda x + (1-\lambda)y) \geqslant \min[u(x), u(y)];$$

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(iii) u is upper semicontinuous;
(iv) $[u]^0 = cl[x \in R^n : u(x) > 0]$ is compact.

For $0 < \alpha \leqslant 1$, we denote $[u]^\alpha = [x \in R^n : u(x) \geqslant \alpha]$. Then from (i) to (iv), it follows that the α-level sets $[u]^\alpha \in P_k(R^n)$ for $0 \leqslant \alpha \leqslant 1$.

For later purposes, we define $\widehat{0} \in E^n$ as $\widehat{0}(x) = 1$ if $x = 0$ and $\widehat{0}(x) = 0$ if $x \neq 0$.

Let $d_H(A,B)$ be the Hausdorff distance between the sets $A, B \in P_k(R^n)$, that is,

$$d_H(A,B) = inf\{\varepsilon | A \subset N(B,\varepsilon), B \subset N(A,\varepsilon)\},$$

where $N(A,\varepsilon) = \{x \in R^n |\ ||x-y|| < \varepsilon \text{ for some } y \in A\}$. Then we define

$$d[u,v] = \sup_{0 \leqslant \alpha \leqslant 1} d_H([u]^\alpha, [v]^\alpha),$$

It is well known that d is a metric in E^n and that (E^n, d) is a complete metric space. We list the following properties of $d[u,v]$ (see [5]).

$$\begin{gathered} d[u+w, v+w] = d[u,v] \quad \text{and} \quad d[u,v] = d[v,u], \\ d[\lambda u, \lambda v] = |\lambda| d[u,v], \\ d[u,v] \leqslant d[u,w] + d[w,v], \end{gathered}$$

for all $u, v, w \in E^n$ and $\lambda \in R$.

Let $I = [t_0, t_0 + a]$ with $a > 0$ and $x, y \in E^n$. If there exists a $z \in E^n$ such that $x = y + z$, then z is called the H-difference of x and y and is denoted by $x - y$.

DEFINITION 2.1. A mapping $F : I \to E^n$ is differentiable at $t \in I$ if there exists a $F'(t) \in E^n$ such that the limits

$$\lim_{h \to 0^+} \frac{F(t+h) - F(t)}{h} \quad \text{and} \quad \lim_{h \to 0^+} \frac{F(t) - F(t-h)}{h}$$

exist and equal to $F'(t)$.

Here the limits are taken in the metric space (E^n, d). At the end points of I we consider only the one-sided derivatives.

DEFINITION 2.2. [5] Let $F : I \to E^n$ and denote $F_\alpha(t) = [F(t)]^\alpha$. The integral of F over I, denoted $\int_I F(t)dt$ or $\int_{t_0}^{t_0+a} F(t)dt$, is defined levelwise by the equation

$$\begin{aligned} [\textstyle\int_I F(t)dt]^\alpha &= \textstyle\int_I F_\alpha(t)dt \\ &= \{\textstyle\int_I f(t)dt | f : I \to R^n \text{ is a measurable selection for } F_\alpha\}. \end{aligned}$$

It is well known that, if $F : I \to E^n$ is continuous, then it is integrable and

$$\int_{c_1}^{c_2} F(t)dt = \int_{c_1}^{c_3} F(t)dt + \int_{c_3}^{c_2} F(t)dt.$$

Where $c_1, c_2, c_3 \in I$. Also, the following properties of the integral are valid (see [2, 3, 4, 5]). If $F, G : I \to E^n$ are integrable, $\lambda \in R$, then the following hold:

$$\int (F+G) = \int F + \int G;$$

$$\int \lambda F = \lambda \int F, \quad \lambda \in R;$$

$$d[F,G] \text{ is integrable;}$$

$$d[\int_{c_1}^{c_2} F, \int_{c_1}^{c_2} G] \leqslant \int_{c_1}^{c_2} d[F,G].$$

If $F : I \to E^n$ be continuous, then, the function $G(t) = \int_a^t F(s)ds, \quad t \in I$, is differentiable and $G'(t) = F(t)$. Furthermore,

$$F(t) - F(t_0) = \int_{t_0}^{t} F'(s)ds.$$

Now, if F is continuously differentiable on I, then we have the following mean value theorem([5]):

$$d[F(b), F(a)] \leqslant (b-a)\sup\{d[F'(t), \widehat{0}] : t \in I\}.$$

As a consequence, we have that

$$d[G(b), G(a)] \leqslant (b-a)\sup\{d[F(t), \widehat{0}] : t \in I\}.$$

(see [1, 2, 3, 4, 5] for details).

In this paper, let

$$K = \{a(t) : a(t) \in C(R_+, R_+) \text{ is strict increasing and } a(0) = 0\}.$$

3. Continuation of solutions and the maximum interval of existence

In this section, we will discuss the question of whether the solution $u = u(t)$ of the equation $u' = f(t,u)$ which are uniquely defined in the neighborhood of an initial point, can be extended.

Consider equation

$$u' = f(t,u), \quad u(t_0) = u_0, \tag{1}$$

where $u \in E^n$, $f(t,u) \in C(I \times S(\rho), E^n)$ and $S(\rho) = [u \in E^n : d[u, \widehat{0}] < \rho]$. Let us begin with the following lemma(see [5]):

LEMMA 3.1. *A mapping $u : I \to E^n$ is a solution to the problem (1) if and only if it is continuous and satisfies the integral equation*

$$u(t) = u_0 + \int_{t_0}^{t} f(s, u(s))ds, \tag{2}$$

for all $t \in I$.

THEOREM 3.1. *Assume that* $f(t,u) \in C(I \times S(\rho), E^n)$ *and there exists a constant* $L > 0$ *such that*

$$d[f(t,u_1), f(t,u_2)] \leqslant Ld[u_1,u_2], \quad t \in I, \quad u_1,u_2 \in S(\rho).$$

Suppose that $u(t,t_0,u_0) = u(t)$ *is a solution of equation* (1) *on the interval* $[t_0,\beta)$. *Then* $\lim_{t\to\beta-0} u(t)$ *exists, and furthermore if* $\beta \neq t_0+a$ *and* $(\beta, u(\beta-0)) \in I \times S(\rho)$, *the solution* $u(t)$ *can be continued to the right.*

Proof. By Lemma 2.1, $u(t)$ satisfies

$$u(t) = u_0 + \int_{t_0}^{t} f(s,u(s))ds, \quad t \in [t_0,\beta).$$

If $b_n = u(\beta - 1/n)$, then for n sufficiently large and $m > n$ we have

$$\begin{aligned} d[b_m,b_n] &= d[u(\beta-1/m), u(\beta-1/n)] \\ &\leqslant (1/n-1/m)\sup\{d[u',\widehat{0}] : t \in I\} \\ &= (1/n-1/m)\sup\{d[f(t,u(s)),\widehat{0}] : t \in I\} \leqslant M(1/n-1/m), \end{aligned}$$

by mean value theorem,where M is a bound for $f(t,u)$ in $I \times S(\rho)$. This implies that $\{b_n\}$ is a Cauchy sequence, from which it follows that $\lim_{t\to\beta_0} u(t)$ exists.

Suppose that the point $(\beta, u(\beta-0)) \in I \times S(\rho)$, then the function $\overline{u}(t)$ defined by

$$\overline{u}(t) = u(t), \quad t \in [t_0,\beta), \quad \overline{u}(\beta) = u(\beta-0)$$

is a solution of equation (1), defined on $[t_0,\beta]$. This follows from the relation

$$\overline{u}(t) = u_0 + \int_{t_0}^{t} f(s,\overline{u}(s))ds, \quad t \in [t_0,\beta],$$

which implies that the left-hand derivative $u'(\beta)$ exists and equals $f(\beta,\overline{u}(\beta))$, which is finite.

Since $S(\rho)$ is open and $(\beta,u(\beta-0)) \in I \times S(\rho)$, there exists a solution $u_1(t)$ of equation (1) passing through the point $(\beta,u(\beta-0))$ and defined on some interval $\beta \leqslant t \leqslant \beta+\alpha$, for some $\alpha > 0$. Now define the function

$$\begin{aligned} v(t) &= \overline{u}(t), \quad t \in [t_0,\beta], \\ v(t) &= u_1(t), \quad t \in [\beta,\beta+\alpha), \end{aligned}$$

and we assert that $v(t)$ is a solution for $t \in [t_0,\beta+\alpha)$. It is easy to see that we need only to show the existence and continuity of $v'(t)$ at $t = \beta$.

In fact, from Lemma 2.1, we have

$$v(t) = \overline{u}(\beta) + \int_{\beta}^{t} f(s,v(s))ds, \quad t \in [\beta,\beta+\alpha),$$

and

$$\overline{u}(\beta) = u_0 + \int_{t_0}^{\beta} f(s, v(s))ds$$

since $\overline{u}(t)$ is a solution. This gives

$$v(t) = u_0 + \int_{t_0}^{t} f(s, v(s))ds, \quad t \in [t_0, \beta + \alpha),$$

and since f and $v(t)$ are continuous, this implies that $v'(t) = f(t, v(t)), t \in [t_0, \beta + \alpha)$. Therefore $v(t)$ is a continuation(to the right) of $u(t)$, and this completes the proof.

From this Theorem, we can obtain the following conclusion:

THEOREM 3.2. *Suppose the hypotheses of Theorem 3.1 are satisfied for the differential equation* (1). *Then, given initial values* (t_0, u_0), *there exists a solution* $u(t, t_0, u_0)$, *defined on* $[t_0, \beta)$, $\beta \in I$, *such that, if* $v(t)$ *is any other solution and* $v(t_0) = u_0$, *then its interval of definition is contained in* $[t_0, \beta)$.

The interval $[t_0, \beta)$ is called (right side) maximum interval of existence of solution $u(t)$.

THEOREM 3.3. *Suppose the hypotheses of Theorem 3.1 are satisfied for the differential equation* (1) *in the domain* $I \times S(\rho)$. *Let* $u(t)$ *be a solution of* (1) *and let* $[t_0, \beta)$ *be its maximum interval of existence. If* $\beta \neq t_0 + a$ *and* B *is any closed bounded set contained in* $I \times S(\rho)$, *then there exists an* $\epsilon > 0$ *such that the point* $(t, u(t))$ *does not belong to* B *if* $t > \beta - \epsilon$.

COROLLARY 3.1. *If the conditions of Theorem 3.3 are hold except that* $a = \infty$ *and* $s(\rho)$ *is replaced by* E^n. *If* $u(t, t_0, u_0)$ *is a solution with maximum interval of existence* $[t_0, \beta)$ *and* $\beta < \infty$, *then*

$$\lim_{t \to \beta - 0} d[u(t, t_0, u_0), \widehat{0}] = \infty.$$

4. Boundness criteria

Consider the fuzzy differential equation

$$u' = f(t, u), \quad u(t_0) = u_0, \tag{3}$$

where $u \in E^n$, $f(t, u) \in C(R_+ \times E^n, E^n)$. In this paper, we always suppose that equation (3) has at least one solution.

DEFINITION 4.1. We say that the solution of (3) of equation (3) is

i) equi-bounded, if given $B_1 > 0$, and $t_0 \in R_+$, there exists a $B_2(t_0, B_1)$ such that

$$d[u_0, \widehat{0}] < B_1 \quad \text{implies} \quad d[u(t), \widehat{0}] < B_2, \quad t \geqslant t_0;$$

ii) uniformly bounded(UB), if B_2 in *i*) is independent of t_0;

iii) uniformly ultimately bounded(UUB) for bound B, if for each $B_3 > 0$, there exists $T > 0$, such that $d[u_0, \widehat{0}] \leqslant B_3, t_0 \in R_+, t \geqslant t_0 + T$ imply that $d[u(t,t_0,u_0), \widehat{0}] < B$.

To investigate the boundedness of the solutions of equation (3), the following comparison result in term of a Lyapunov function is very important which has been proved in [7].

THEOREM 4.1. *Assume that*

$$V \in C[R_+ \times S(\rho), R_+], |V(t,u_1) - V(t,u_2)| \leqslant Ld[u_1,u_2], L > 0 \text{ and}$$

$$D^+V(t,u) \equiv \limsup_{h \to 0^+} \frac{1}{h}[V(t+h, u+hf(t,u)) - V(t,u)] \leqslant g(t, V(t,u)),$$

where $g \in C[R_+^2, R]$. *Then, if* $u(t)$ *is any solution of* (3) *existing on* $[t_0, \infty)$ *such that* $V(t_0,u_0) \leqslant w_0$, *we have*

$$V(t,u(t)) \leqslant r(t,t_0,w_0), t \geqslant t_0,$$

where $r(t,t_0,w_0)$ *is the maximal solution of the scalar differential equation*

$$w' = g(t,w), \quad w(t_0) = w_0 \geqslant 0,$$

existing on $[t_0, \infty)$.

COROLLARY 4.1. *The function* $g(t,w) \equiv 0$ *is admissible in Theorem 3.1 to yield the estimate*

$$V(t,u(t)) \leqslant V(t_0,u_0), \quad t \geqslant t_0.$$

We shall first prove a result similar to Lyapunou-type theorem.

THEOREM 4.2. *Suppose that there exist* $\rho > 0$, $V(t,u) \in C[R_+ \times S^c(\rho), R_+]$, *such that*

$$|V(t,u_1) - V(t,u_2)| \leqslant Ld[u_1,u_2], \quad L > 0,$$
$$b(d[u,\widehat{0}]) \leqslant V(t,u) \leqslant a(d[u,\widehat{0}]),$$

where $a(.), b(.) \in K$. *Then*

(A_1) $\lim_{h \to 0^+} \frac{1}{h}[V(t+h, u+hf(t,u)) - V(t,u)] \leqslant 0$
implies that the equation (3) *is* (UB) *and*

(A_2) $\lim_{h \to 0^+} \frac{1}{h}[V(t+h, u+hf(t,u)) - V(t,u)] \leqslant -c, \quad c > 0$
implies that the equation (3) *is* (UUB).

Proof. At first, we show that all the solutions of equation (3) are exists on $[t_0, \infty)$. If it is not true, there exists a solution $u(t,t_0,u_0)$ with the maximum existent interval $[t_0, \eta), \eta < \infty$. This means that

$$\limsup_{t \to \eta} d[u(t,t_0,u_0), \widehat{0}] = \infty \tag{4}$$

by the Corollary of Theorem 2.3. Thus, the set

$$S_{t_0} = \{t | d[u(t,t_0,u_0), \widehat{0}] \geqslant \rho\}$$

is not empty. Let $t_1 = inf\, S_{t_0}$, then $d[u(t_1,t_0,u_0),\widehat{0}] = \rho$. The Corollary of Theorem 3.1 implies that

$$V(t,u) \leqslant V(t_1,u(t_1,t_0,u_0)), \quad t \in S_{t_0}.$$

By hypothesis (A_1), we get

$$b(d[u,\widehat{0}]) \leqslant V(t,u) \leqslant V(t_1,u(t_1)) \leqslant a(d[u(t_1),\widehat{0}])$$

and so

$$d[u(t),\widehat{0}] \leqslant b^{-1}(a(d[u(t_1),\widehat{0}])).$$

This contradicts (4). Thus, all solutions of equation (3) exist on $[t_0,\infty)$.

Next, we prove the boundness. Suppose that (A_1) is hold. For any $B_1 > 0$, without loss generality, $B_1 > \rho$, let $B_2 = b^{-1}(a(B_1))$. We claim that,

$$d[u(t_0),\widehat{0}] < B_1 \text{ implies } d[u(t,t_0,u_0),\widehat{0}] \leqslant B_2.$$

In fact, if the set

$$S = \{t | d[u(t,t_0,u_0),\widehat{0}] \geqslant \rho\}$$

is not empty, without loss generality, let $t_0 \in S$, we have, as proved before, that

$$V(t,u(t)) \leqslant V(t_0,u_0), \quad t \in S.$$

By hypothesis, we obtain that

$$\begin{aligned} b(d[u(t),\widehat{0}]) &\leqslant V(t,u) \leqslant V(t_0,u_0) \\ &\leqslant a(d[u_0,\widehat{0}]) \leqslant a(B_1). \end{aligned}$$

Thus,

$$d[u,\widehat{0}] \leqslant b^{-1}(a(B_1)), \quad t \in S.$$

It is clear that

$$d[u,\widehat{0}] \leqslant B_1 \leqslant b^{-1}(a(B_1)), \quad t \in R_+ - S.$$

Hence, solutions of (3) is UB.

Now, we prove (A_2) implies UUB of equation (3) about the bound $B = b^{-1}(a(2\rho))$.

For any $B_2 > \rho$, take $T = [a(B_2) - b(\rho)]/c$, we claim that,

$$d[u_0,\widehat{0}] < B_2 \text{ implies, } d[u(t,t_0,u_0),\widehat{0}] < B, \quad t \geqslant t_0 + T.$$

It is easy to see that, from the proof of the first part, we need only prove that there exists a point $t_1 \in [t_0,t_0+T]$ such that $d[u(t_1,t_0,u_0),\widehat{0}] < 2\rho$, then

$$d[u(t,t_0,u_0),\widehat{0}] < b^{-1}(a(2\rho)) = B, \quad t \geqslant t_1.$$

If it is not true, then there exists a $t_0 \in R_+$ and u_0, such that

$$d[u_0,\widehat{0}] < B_2 \quad \text{and} \quad d[u(t,t_0,u_0),\widehat{0}] \geqslant 2\rho, \;\; t \in [t_0,t_0+T] \tag{5}$$

hold. Let $m(t) = V(t, u(t))$, then

$$\begin{aligned} m(t+h) - m(t) &= V(t+h, u(t+h)) - V(t, u(t)) \\ &= V(t+h, u(t+h)) - V(t+h, u(t) + hf(t, u(t))) \\ &+ V(t+h, u(t) + hf(t, u(t))) - V(t, u(t)) \\ &\leqslant Ld[u(t+h, u(t) + hf(t, u(t))] \\ &+ V(t+h, u(t) + hf(t, u(t))) - V(t, u(t)). \end{aligned}$$

Let $u(t+h) = u(t) + z(t)$, where $z(t)$ is the H-difference for small $h > 0$ which is assumed to exist. Hence using the properties of $d[u, v]$, we see that

$$\begin{aligned} d[u(t+h),\ u(t) + hf(t, u(t))] &= d[u(t) + z(t), u(t) + hf(t, u(t))] \\ &= d[z(t), hf(t, u(t))] = d[u(t+h) - u(t), hf(t, u(t))], \end{aligned}$$

and

$$\begin{aligned} D^+m(t) &= \limsup_{h\to 0^+} \tfrac{1}{h}[m(t+h) - m(t)] \\ &\leqslant D^+V(t, u(t)) + L \limsup_{h\to 0^+} \tfrac{1}{h}[d[u(t+h), u(t) + hf(t, u(t))]] \\ &\leqslant -c + L \limsup_{h\to 0^+} \tfrac{1}{h}[d[u(t+h) - u(t), hf(t, u(t))] \\ &\leqslant -c + L \limsup_{h\to 0^+} d[\tfrac{u(t+h)-u(t)}{h}, f(t, u(t))] = -c. \end{aligned}$$

We therefore have the scalar differential inequality

$$D^+m(t) \leqslant -c, \quad m(t_0) = V(t_0, u(t_0)) \leqslant a(B_2).$$

Solving this equation, we obtain

$$m(t) \leqslant m(t_0) - c(t - t_0) \leqslant a(B_2) - c(t - t_0), \quad t \in [t_0, t_0 + T]$$

and so

$$\begin{aligned} b(d[u(t_0+T), \widehat{0}]) &\leqslant V(t_0+T, u(t_0+T)) = m(t_0+T) \\ &\leqslant a(B_2) - cT = b(\rho), \end{aligned}$$

that is

$$d[u(t_0+T), \widehat{0}] \leqslant \rho.$$

This contradicts (5) and the proof is complete.

THEOREM 4.3. *Suppose that there exists* $\rho > 0$, $V(t, u) \in C[R_+ \times S^c(\rho), R_+]$, $b(d[u, \widehat{0}]) \leqslant V(t, u) \leqslant a(d[u, \widehat{0}])$, *such that*

$$|V(t, u_1) - V(t, u_2)| \leqslant Ld[u_1, u_2], \quad L > 0;$$

$$\limsup_{h\to 0^+} \frac{1}{h}[V(t+h,u(t)+hf(t,u)) - V(t,u)] \leqslant g(t,V(t,u)),$$

where $g \in C(R_+^2, R)$ *and* $a(.),\ b(.) \in K$. *Then, the bound concept of the scale equation*

$$w' = g(t,w), \quad w(t_0) = w_0 \geqslant 0, \tag{6}$$

implies corresponding bound concept of equation (3).

Proof. It is enough to prove the case of UUB because the proof of the rests are the same.

Suppose (6) is UUB. Then there exists a $B > \rho$ (without loss generality, let $b(B) > \rho$), such that, for any $B_1 > 0$, there exists a $T > 0$ and satisfies that

$$0 \leqslant w(t_0) = w_0 < B_1 \quad \text{implies} \quad \gamma(t,t_0,w_0) < b(B) \quad \text{for } t \geqslant t_0 + T,$$

where $\gamma(t,t_0,w_0)$ is maximum solution of (6) through (t_0,w_0). Using Theorem 3.1, we obtain

$$V(t,u) \leqslant \gamma(t,t_0,w_0), \quad \text{if} \quad u(t) \in s^c(\rho),$$

where $w_0 = \max\{d[u(t_0),\widehat{0}],\rho\}$ and thus

$$b(d[u,\widehat{0}]) \leqslant V(t,u) \leqslant \gamma(t,t_0,w_0), \quad u \in S^c(\rho),$$

then

$$d[u,\widehat{0}] \leqslant b^{-1}[\gamma(t,t_0,w_0)] \leqslant b^{-1}[b(B)] = B, \quad u \in S^c(\rho), \quad t \geqslant t_0 + T.$$

It is clear that if $u \in S(\rho)$ for some $t \geqslant t_0 + T$, then $d[u,\widehat{0}] \leqslant B$ since $B > \rho$.

Summing up the analysis above, we obtain that

- For $B > \rho$, for any $B_1 > 0$, there exists a $T > 0$, such that

$$d[u_0,\widehat{0}] < B_1 \quad \text{implies} \quad u(t,t_0,u_0) \in S(B), \quad t \geqslant t_0 + T.$$

This means that equation (3) is UUB. The proof completes.

COROLLARY 4.2. *The function* $g(t,u) = -c(u)$, $c \in K$ *is admissible in Theorem 4.3.*

It is known that the method of perturbing Lyapunov functions introduced in [9] is a useful and important tool in the study of nonuniform properties of solutions because when the Lyapunov function found does not satisfy all the desired conditions, it is fruitful to perturb that Lyapunov function rather than discard it. Next result is obtained by the method of perturbing Lyapunov functions .

THEOREM 4.4. *Assume that*

(i) $\rho > 0, V_1 \in C[R_+ \times S(\rho), R_+]$, V_1 *is bounded for* $(t,u) \in R_+ \times \partial S(\rho)$, *and*

$$|V_1(t,u_1) - V(t,u_2)| \leqslant L_1 d[u_1,u_2], \quad l_1 > 0,$$

$$\begin{aligned} D^+V_1(t,u) &= \limsup_{h\to 0^+} \tfrac{1}{h}[V_1(t+h,u+hf(t,u)) - V_1(t,u)] \\ &\leqslant g_1(t,V_1) \quad (t,u) \in R^+ \times S^c(\rho) \end{aligned} \tag{7}$$

where $g_1 \in C[R_+^2, R]$;

(ii) $V_2 \in C[R_+ \times S^c(\rho), R_+]$,

$$b(d[u,\widehat{0}]) \leqslant V_2(t,u) \leqslant a(d[u,\widehat{0}]), \quad a(.), b(.) \in K, \tag{8}$$

$$D^+V_1 + D^+V_2 \leqslant g_2(t, V_1(t,u) + V_2(t,u)), \quad g_2 \in C[R_+^2, R]; \tag{9}$$

(iii) *the scalar differential equations*

$$w_1' = g_1(t, w_1), \quad w_1(t_0) = w_{10} \geqslant 0, \tag{10}$$

and

$$w_2' = g_2(t, w_2), \quad w_2(t_0) = w_{20} \geqslant 0, \tag{11}$$

are equibounded and uniformly bounded, respectively. Then the system (3) *is equibounded.*

Proof. Let $B_1 > \rho$ and $t_0 \in R_+$ be given. Let $\alpha_1 = \alpha_1(t_0, B_1) = \max(\alpha_0, \alpha^*)$, where $\alpha_0 = \max[V_1(t_0, u_0) : u_0 \in cl\{S(B_1) \bigcap S^c(\rho)\}]$ and $\alpha^* \geqslant V_1(t,u)$ for $(t,u) \in R^+ \times \partial S(\rho)$. Since equation (10) is equibounded, given $\alpha_1 > 0$, and $t_0 \in R_+$, there exists a $\beta_0 = \beta_0(t_0, \alpha_1)$, such that

$$w_1(t, t_0, w_{10}) < \beta_0, \quad t \geqslant t_0, \tag{12}$$

provided $w_{10} < \alpha_1$, where $w_1(t, t_0, w_{10})$ is any solution of (10). Let $\alpha_2 = a(B_1) + \beta_0$, then uniform boundedness of equation (11) yields that

$$w_2(t, t_0, w_{20}) < \beta_1(\alpha_2), \quad t \geqslant t_0, \tag{13}$$

provided $w_{20} < \alpha_2$, where $w_2(t, t_0, w_{20})$ is any solution of (11). Choose B_2 satisfies

$$b(B_2) > \beta_1(\alpha_2). \tag{14}$$

We now claim that $u_0 \in S(B_1)$ implies that $u(t, t_0, u_0) \in S(B_2)$ for $t \geqslant t_0$, where $u(t, t_0, u_0)$ is any solution of (3).

If it is not true, there exists a solution $u(t, t_0, u_0)$ of (3) with $u_0 \in S(B_1)$, such that for some $t^* > t_0$, $d[u(t^*, t_0, u_0), \widehat{0}] = B_2$. Since $B_1 > \rho$, there are two possibilities to consider:

1) $u(t, t_0, u_0) \in S^c(\rho)$ for $t \in [t_0, t^*]$;

2) there exists a $\bar{t} \geqslant t_0$ such that $u(\bar{t}, t_0, u_0) \in \partial S(\rho)$ and $u(t, t_0, u_0) \in S^c(\rho)$ for $t \in [\bar{t}, t^*]$.

If 1) holds, we can find $t_1 > t_0$,such that

$$\begin{cases} u(t_1, t_0, u_0) \in \partial S(B_1) & \\ u(t^*, t_0, u_0) \in \partial S(B_2) & \text{and} \\ u(t, t_0, u_0) \in S^c(B_1), & t \in [t_1, t^*]. \end{cases} \tag{15}$$

Setting $m(t) = V_1(t, u(t, t_0, u_0)) + V_2(t, u(t, t_0, u_0))$ for $t \in [t_1, t^*]$, then using the similar argument the proof of Theorem 4.1, we can obtain the differential inequality

$$D^+m(t) \leqslant g_2(t, m(t)), \quad t \in [t_1, t^*],$$

and so

$$m(t) \leqslant \gamma_2(t, t_1, m(t_1)), \quad t \in [t_1, t^*],$$

where $\gamma_2(t, t_1, v_0)$ is the maximal solution of (11) with $\gamma_2(t_1, t_1, v_0) = v_0$. Thus

$$\begin{aligned} V_1(t^*, u(t^*, t_0, u_0)) \;+\;& V_2(t^*, u(t^*, t_0, u_0)) \\ \leqslant\;& \gamma_2(t^*, t_1, V_1(t_1, u(t_1, t_0, u_0)) + V_2(t_1, u(t_1, t_0, u_0))). \end{aligned} \tag{16}$$

Similarly, we also have

$$V_1(t_1, u(t_1, t_0, u_0)) \leqslant \gamma_1(t_1, t_0, V_1(t_0, u_0)), \tag{17}$$

where $\gamma_1(t, t_0, u_0)$ is the maximal solution of (10). Set $w_{10} = V_1(t_0, u_0) < \alpha_1$, then

$$V_1(t_1, u(t_1, t_0, u_0)) \leqslant \gamma_1(t_1, t_0, V_1(t_0, u_0)) \leqslant \beta_0$$

since (12). Furthermore, $V_2(t_1, u(t_1, t_0, u_0)) \leqslant a(B_1)$ because of (8) and (15), consequently, we have

$$\begin{aligned} w_{20} &= V_1(t_1, u(t_1, t_0, u_0)) + V_2(t_1, u(t_1, t_0, u_0)) \\ &\leqslant \beta_0 + a(B_1) = \alpha_2. \end{aligned} \tag{18}$$

Combining (8), (13), (14), (15), and (18), we obtain

$$b(B_2) \leqslant m(t^*) \leqslant \gamma(t^*) \leqslant \beta_1(\alpha_2) < b(B_2), \tag{19}$$

which is a contradiction.

If case 2) holds, we also arrive at the inequality (16), where $t_1 > \bar{t}$ satisfies (15). We now have, in place of (17), the relation

$$V_1(t_1, u(t_1, t_0, u_0)) \leqslant \gamma_1(t_1, \bar{t}, V_1(\bar{t}, u(\bar{t}, t_0, x_0))).$$

Since $u(\bar{t}, t_0, u_0) \in \partial S(\rho)$ and $V_1(\bar{t}, u(\bar{t}, t_0, x_0)) \leqslant \alpha^* \leqslant \alpha_1$, arguing as before, we get the contradiction (19). This proves that

- for any $B_1 > \rho$, $t_0 > 0$ are given, there exists a B_2 such that $u_0 \in S(B_1)$ implies $u(t, t_0, u_0) \in S(B_2), t \geqslant t_0$.

For $B_1 < \rho$, we set $B_2(t_0, B_1) = B_2(t_0, \rho)$ and hence the proof is complete.

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REFERENCES

[1] D. DUBOIS AND H. PRADE, *Towards fuzzy differential calculus, Part I*, Fuzzy Sets and Systems, **8** (1982), 1–17.

[2] D. DUBOIS AND H. PRADE, *Towards fuzzy differential calculus, Part II*, Fuzzy Sets and Systems, **8** (1982), 105–116.

[3] D. DUBOIS AND H. PRADE, *Towards fuzzy differential calculus, Part III*, Fuzzy Sets and Systems, **8** (1982), 225–234.

[4] M. FRIEDMAN, M. MING AND A. KANDEL, *On the validity of the Peano's theorem for fuzzy differential equations,* Fuzzy Sets and Systems, **86** (1997), 331–334.

[5] O. KALEVA, *Fuzzy differential equations*, Fuzzy Sets and Systems, **24** (1987), 301–317.
[6] O. KALEVA, *The Cauchy problem for Fuzzy differential equations*, Fuzzy Sets and Systems, **35** (1990), 389–396.
[7] V. LAKSHMIKANTHAM AND S. G. LEELA, *Stability Theory of Fuzzy Differential Equations via Differential Inequalities* (to appear).
[8] V. LAKSHMIKANTHAM AND R. N. MOHAPATRA, *Basic properties of solutions of fuzzy differential equations* (to appear).
[9] V. LAKSHMIKANTHAM AND S. G. LEELA, *On perturbing Lyapunov functions*, Math. System Theory, **10** (1976), 85–90.
[10] M. L. PURI AND D. A. RALESCU, *Differentials for fuzzy functions*, J. Math. Appl., **91** (1983), 552–558.

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