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NORM INEQUALITIES FOR SOME SINGULAR INTEGRAL OPERATORS

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Abstract. Let $\mathscr{B}$ be a von Neumann algebra and P a selfdjoint projection. For A and B in $\mathscr{B}$, set $S_{A,B} = AP + BQ$ where $Q = I - P$. The operator $S_{A,B}$ will be called a singular integral operator. When $\mathscr{B} = L^\infty(T)$ where $L^\infty(T)$ is the usual Lebesgue space on the unit circle and P is an analytic projection, in [6] we established formulae for norms of $S_{A,B}$ and $(S_{A,B})^{-1}$. In this paper, if $\mathscr{A} = \{D \in \mathscr{B} : PDP = DP\}$ and $(\mathscr{B}, \mathscr{A}, P)$ has a lifting property, then we will establish formulae of norms of $S_{A,B}$ and $(S_{A,B})^{-1}$. These formulae are operator theoretic and different from the previous ones. There are several examples such that $(\mathscr{B}, \mathscr{A}, P)$ has a lifting property. As result, we give several interesting inequalities.

1. Introduction

Let m denote the normalized Lebesgue measure on the unit circle T. For $1 \leqslant p \leqslant \infty$, $L^p(T)$ denotes the usual Lebesgue space on T and $H^p(T)$ denotes the usual Hardy space on T. The canonical example of a singular integral operator is the operator defined by

$$(S_{a,b}F)(\zeta) = \frac{a(\zeta) + b(\zeta)}{2} F(\zeta) + \frac{a(\zeta) - b(\zeta)}{2} (SF)(\zeta)$$

on $L^2(T)$; here, $a(\zeta)$ and $b(\zeta)$ denote functions in $L^\infty(T)$ and

$$(SF)(\zeta) = \frac{1}{\pi i} \int_T \frac{F(\eta)}{\eta - \zeta} d\eta \quad (a.e.\ \zeta \in T),$$

the integral being a Cauchy principal value (cf.[4]). Then $P = (I + S)/2$ is a selfadjoint projection from $L^2(T)$ to $H^2(T), Q = (I - S)/2$ is a selfadjoint projection from $L^2(T)$ to $e^{-i\theta}\overline{H^2(T)}$, and $P + Q = I$ where I denotes the identity operator. Hence

$$S_{a,b} = aP + bQ.$$

The following inequalities are well known and not difficult to establish.

$$\max\{\|a\|_\infty, \|b\|_\infty\} \leqslant \|S_{a,b}\| \leqslant \|\sqrt{|a|^2 + |b|^2}\|_\infty.$$

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and

$$\inf_{\|F\|_2=1} \|S_{a,b}F\|_2^2 \leqslant ess\inf_T(\min\{|a|^2,|b|^2\}).$$

In the previous paper, the author and T.Yamamoto [6] showed the following theorems.

THEOREM A. *Let* $a,b \in L^\infty(T)$. *Then*

$$\|S_{a,b}\|^2 = \inf_{k\in H^\infty(T)} \left\| \frac{|a|^2+|b|^2}{2} + \sqrt{|a\bar{b}+k|^2 + \left(\frac{|a|^2-|b|^2}{2}\right)^2} \right\|_\infty .$$

THEOREM B. *Let* $a,b \in L^\infty(T)$. *Then*

$$\inf_{F\in L^2(T),\|F\|_2=1} \|S_{a,b}F\|_2^2 = \sup_{k\in H^\infty(T)} \left(ess\inf_T \left(\frac{|a|^2+|b|^2}{2} - \sqrt{|a\bar{b}+k|^2 + \left(\frac{|a|^2-|b|^2}{2}\right)^2} \right)\right).$$

In this paper, we give formulae which are similar to those of Theorems A and B. In fact, we prove them for more general situations.

Let K be a complex Hilbert space and H the closed subspace of K. Let P be a selfadjoint projection from K to H and $Q=I-P$ where I denotes the identity operator on K. $\mathscr{B}$ denotes a von Neumann algebra on K which contains I and $\mathscr{A}$ denotes a (perhaps nonselfadjoint) weakly closed subalgebra of $\mathscr{B}$ which has H as an invariant subspace. For A and B in $\mathscr{B}$, set

$$S_{A,B} = AP + BQ.$$

The operator $S_{A,B}$ is called a singular integral operator. In this paper, we give formulae of norms of $S_{A,B}$ and $(S_{A,B})^{-1}$. The formulae are little bit complicated. However, as result we give the following simple inequalities.

$$\begin{aligned}
&\inf_{D\in\mathscr{A}} \sup_{\substack{F,G\in K\\ \|F\|=\|G\|=1}} \left\{ \frac{\|AF\|^2+\|BG\|^2}{2} + |\langle (B^*A+D)F,G\rangle| \right\} \\
&\qquad \leqslant \|S_{A,B}\|^2 \\
&\qquad \leqslant \inf_{D\in\mathscr{A}} \sup_{\substack{F,G\in K\\ \|F\|=\|G\|=1}} \{\max(\|AF\|^2,\|BG\|^2) + |\langle (B^*A+D)F,G\rangle|\}
\end{aligned}$$

and

$$\begin{aligned}
&\sup_{D\in\mathscr{A}} \inf_{\substack{F,G\in K\\ \|F\|=\|G\|=1}} \left\{ \frac{\|AF\|^2+\|BG\|^2}{2} - |\langle (B^*A+D)F,G\rangle| \right\} \\
&\qquad \geqslant \|S_{A,B}^{-1}\|^2 \\
&\qquad \geqslant \sup_{D\in\mathscr{A}} \inf_{\substack{F,G\in K\\ \|F\|=\|G\|=1}} \{\min(\|AF\|^2,\|BG\|^2) - |\langle (B^*A+D)F,G\rangle|\}.
\end{aligned}$$

In §2, we give a definition of a lifting property of $(\mathscr{B},\mathscr{A},P)$ and several examples which have such a property. In §3, we study the norm of $S_{A,B}$ when $(\mathscr{B},\mathscr{A},P)$ has a lifting property. In §4, we study the norm of $S_{A,B}^{-1}$ when $(\mathscr{B},\mathscr{A},P)$ has a lifting property. In §5, we give several remarks.

2. Lifting theorem

In this section, we recall a special case of a lifting theorem which was proved in [5]. The classical case was proved by Cotlar and Sadosky [1].

Suppose $\mathscr{T} = (T_{ij})$ is a 2×2 operator matrix on $K \oplus K$ where $T_{ij} \in \mathscr{B}$ $(i,j = 1,2)$, $T_{11} \geqslant 0$, $T_{22} \geqslant 0$ and $T_{21}^* = T_{12}$. $[\mathscr{B}]$ denotes the set of such operator matrices $\mathscr{T}$. $[\mathscr{A}]_0$ denotes the subset of $[\mathscr{B}]$ such that $T_{12} \in \mathscr{A}$ and $T_{11} = T_{22} = 0$. Let us denote

$$\mathscr{T}[f_1,f_2] = \sum_{i,j=1}^{2} \langle T_{ij} f_i, f_j \rangle .$$

If $\mathscr{T}$ satisfies $\mathscr{T}[f_1,f_2] \geqslant 0$ for all f_1 in H $(resp. K)$ and f_2 in $H^\perp$ $(resp. K)$, then $\mathscr{T}$ is said to be positive on $H \oplus H^\perp$ $(resp. K \oplus K)$ where $H^\perp$ is the orthogonal complement of H in K. When $\mathscr{T}$ in $[\mathscr{B}]$ and $\mathscr{T}$ is positive on $H \oplus H^\perp$, if we can find $\tilde{\mathscr{T}}$ in $\mathscr{T} + [\mathscr{A}]_0$ which is positive on $K \oplus K$ then we say that $\mathscr{T}$ has a lifting $\tilde{\mathscr{T}}$. If any positive $\mathscr{T}$ in $[\mathscr{B}]$ on $H \oplus H^\perp$ has a lifting $\tilde{\mathscr{T}}$, we say that $(\mathscr{B},\mathscr{A},P)$ has a lifting property where P is a selfadjoint projection of K onto H.

EXAMPLE. We give several examples of $(\mathscr{B},\mathscr{A},P)$ which have a lifting property (cf.[5]).

(1) Let $\mathscr{B} = \mathscr{L}(K)$ be the set of all bounded linear operators on K, P a selfadjoint projection from K to H and

$$\mathscr{A} = \{A \in \mathscr{L}(K) : PAP = AP\}.$$

(2) Let U be a bilateral shift operator on K with $UP = PUP$, where P is a selfadjoint projection from K to H. Suppose $\bigcap_{n=0}^{\infty} U^n(H) = \{0\}$ and $\bigcup_{n=0}^{\infty} U^{*n}(H)$ is dense in K. Let $\mathscr{B} = \{T \in \mathscr{L}(K)\,;\, UT = TU\}$ and $\mathscr{A} = \{A \in \mathscr{B}\,;\, PAP = AP\}$.

(3) Let $\mathscr{B}_1$ be a factor with faithful semifinite normal trace τ and let $\mathscr{E}$ be a complete nest of selfadjoint projections in $\mathscr{B}_1$. Let $L^p = L^p(\mathscr{B},\tau)$ $(1 \leqslant p \leqslant \infty)$, be the usual noncommutative Lebesgue spaces and define the noncommutative Hardy space $H^p = H^p(\mathscr{B}_1,\mathscr{E},\tau)$ to be the closed subspace of L^p of elements A for which $(1-P_1)AP_1 = 0$ for all $P_1 \in \mathscr{E}$. Suppose that $\mathscr{B} = L^\infty, \mathscr{A} = H^\infty, K = L^2$ and $H = H^2$.

(4) Let $\mathscr{A}_1$ be a weak-$*$ Dirichlet algebra of $L^\infty(\mu)$ where μ is a probability measure. The abstract Hardy space $H^p(\mu)$, $1 \leqslant p \leqslant \infty$, associated with $\mathscr{A}_1$ are defined as follows. For $1 \leqslant p \leqslant \infty$, $H^p(\mu)$ is the $L^p(\mu)$-closure of $\mathscr{A}_1$, while $H^\infty(\mu)$ is defined to be the weak-$*$ closure of $\mathscr{A}_1$ in $L^\infty(\mu)$. Suppose $\mathscr{B} = L^\infty(\mu)$, $\mathscr{A} = H^\infty(\mu)$, $K = L^2(\mu)$ and $H = H^2(\mu)$.

In the latter section, when $(\mathscr{B},\mathscr{A},P)$ has a lifting property we study the norms of the sigular integral operator and its inverse.

3. Inequality for norm of $S_{A,B}$

In this section, we assume that $(\mathscr{B},\mathscr{A},P)$ has a lifting property. Assuming that $\|AP\| = \|A\|$ and $\|BQ\| = \|B\|$, we give a formula and inequalities for the norm of $S_{A,B}$. In general,

$$S_{A,B}^* S_{A,B} = (PA^* + QB^*)(AP + BQ) = PA^*AP + QB^*BQ + QB^*AP + PA^*BQ.$$

If B^*A is in $\mathscr{A}$, then $\|S_{A,B}\| = \max(\|A\|, \|B\|)$. Not assuming that B^*A is in $\mathscr{A}$, Proposition 1 gives a formula similar to Theorem A.

PROPOSITION 1. *Let A and B be in $\mathscr{B}$. If* $\max(\|A\|, \|B\|) \leqslant \|S_{A,B}\|$, *then*

$$\|S_{A,B}\|^2 = \inf_{D\in\mathscr{A}} \sup_{\substack{F,G\in K\\ \|F\|=\|G\|=1}} \left\{ \frac{\|AF\|^2 + \|BG\|^2}{2} + \sqrt{|\langle (B^*A + D)F, G\rangle|^2 + \left(\frac{\|AF\|^2 - \|BG\|^2}{2}\right)^2} \right\}.$$

There exists at least a D in $\mathscr{A}$ which gives the infimum.

Proof. Put $\gamma = \|S_{A,B}\|$, then

$$\|Af + Bg\|^2 \leqslant \gamma^2 \|f + g\|^2$$

where $f \in H$ and $g \in H^\perp$. Hence

$$\langle (\gamma^2 - A^*A)f, f\rangle + \langle (\gamma^2 - B^*B)g, g\rangle - 2Re\langle (B^*A - \gamma^2)f, g\rangle \geqslant 0$$

for all $f \in H$ and $g \in H^\perp$. Suppose $\mathscr{T} = [T_{ij}]$, where $T_{11} = \gamma^2 - A^*A,\ T_{22} = \gamma^2 - B^*B$ and $T_{12} = T_{21}^* = B^*A - \gamma^2$. By hypothesis on A and B, $T_{11} \geqslant 0$ and $T_{22} \geqslant 0$. Hence $\mathscr{T}$ is positive on $H \oplus H^\perp$. Since $(\mathscr{B},\mathscr{A},P)$ has a lifting property, there exists D in $\mathscr{A}$ such that $\tilde{\mathscr{T}} = [\tilde{\mathscr{T}}_{ij}]$ is positive on $K \oplus K$ where $\tilde{\mathscr{T}}_{11} = T_{11},\ \tilde{T}_{22} = T_{22}$ and $\tilde{T}_{12} = \tilde{T}_{21}^* = T_{12} + D$. Hence

$$\langle (\gamma^2 - A^*A)F, F\rangle \langle (\gamma^2 - B^*B)G, G\rangle \geqslant |\langle (B^*A + D)F, G\rangle|^2$$

for any $F \in K$ and $G \in K$, and so

$$\gamma^4 \|F\|^2\|G\|^2 - \gamma^2(\|AF\|^2\|G\|^2 + \|F\|^2\|BG\|^2) + \|AF\|^2\|BG\|^2 - |\langle (B^*A + D)F, G\rangle|^2 \geqslant 0$$

for any $F \in K$ and $G \in K$. If $\|F\| = \|G\| = 1$, then

$$\gamma^4 - \gamma^2(\|AF\|^2 + \|BG\|^2) + \|AF\|^2\|BG\|^2 - |\langle (B^*A + D)F, G\rangle|^2 \geqslant 0.$$

Therefore

$$\gamma^2 \leqslant \frac{\|AF\|^2+\|BG\|^2}{2} - \sqrt{|\langle (B^*A+D)F,G\rangle|^2 + \left(\frac{\|AF\|^2-\|BG\|^2}{2}\right)^2}$$

or

$$\gamma^2 \geqslant \frac{\|AF\|^2+\|BG\|^2}{2} + \sqrt{|\langle (B^*A+D)F,G\rangle|^2 + \left(\frac{\|AF\|^2-\|BG\|^2}{2}\right)^2}.$$

The first inequality above is not valid because $\gamma^2 \geqslant \max\{\|AF\|^2,\ \|BG\|^2\}$ for any $F,G \in K$ with $\|F\| = \|G\| = 1$. Thus

$$\gamma^2 \geqslant \sup_{\substack{F,G\in K\\ \|F\|=\|G\|=1}} \left\{\frac{\|AF\|^2+\|BG\|^2}{2} + \sqrt{|\langle (B^*A+D)F,G\rangle|^2 + \left(\frac{\|AF\|^2-\|BG\|^2}{2}\right)^2}\right\}.$$

Conversely if

$$\gamma_0^2 = \inf_{D\in\mathscr{A}} \sup_{\substack{F,G\in K\\ \|F\|=\|G\|=1}} \left\{\frac{\|AF\|^2+\|BG\|^2}{2} + \sqrt{|\langle (B^*A+D)F,G\rangle|^2 + \left(\frac{\|AF\|^2-\|BG\|^2}{2}\right)^2}\right\},$$

for any $\varepsilon > 0$ there exists D in $\mathscr{A}$ such that

$$(\gamma_0+\varepsilon)^4 - (\gamma_0+\varepsilon)^2(\|AF\|^2+\|BG\|^2) + \|AF\|^2\|BG\|^2 - |\langle (B^*A+D)F,G\rangle|^2 \geqslant 0$$

for any $F,G \in K$ with $\|F\| = \|G\| = 1$. Hence

$$(\gamma_0+\varepsilon)^4 - (\gamma_0+\varepsilon)^2(\|Af\|^2+\|Bg\|^2) + \|Af\|^2\|Bg\|^2 - |\langle B^*Af,g\rangle| \geqslant 0$$

for any $f \in H$ and $g \in H^\perp$ with $\|f\| = \|g\| = 1$ because $(1-P)DP = 0$. This implies $\gamma_0+\varepsilon \geqslant \|S_{A,B}\|$ and so $\gamma_0 \geqslant \|S_{A,B}\|$ because ε is arbitrary.

COROLLARY 1. *Let A and B be in $\mathscr{B}$. If $A^*A = |a|^2 I$ and $B^*B = |b|^2 I$ where a and b are complex numbers, then*

$$\|S_{A,B}\|^2 = \frac{|a|^2+|b|^2}{2} + \sqrt{\|B^*A+\mathscr{A}\|^2 + \left(\frac{|a|^2-|b|^2}{2}\right)^2}.$$

Proof. Since $(AP)^*(AP) = PA^*AP = |a|^2P$, $\|AP\| = \|A\|$, Similary $\|BQ\| = \|B\|$.

COROLLARY 2. *Let A be in $\mathscr{B}$ and $B = I$. Then*

$$\|S_{A,I}\|^2 = \inf_{D\in\mathscr{A}} \sup_{\substack{F\in K \\ \|F\|=1}} \left\{ \frac{\|AF\|^2+1}{2} + \sqrt{\|(A+D)F\|^2 + \left(\frac{\|AF\|^2-1}{2}\right)^2} \right\}$$

Proof. It is a corollary of the proof of Proposition 1. In fact, note that

$$(\gamma^2-1)\langle(\gamma^2-A^*A)F,F\rangle \geqslant |\langle(A+D)F,G\rangle|^2$$

for any $G\in K$ with $\|G\|=1$ if and only if

$$(\gamma^2-1)\langle(\gamma^2-A^*A)F,F\rangle \geqslant \|(A+D)F\|^2.$$

THEOREM 2. *Let A and B be in $\mathscr{B}$. If* $\max(\|A\|,\|B\|) \leqslant \|S_{A,B}\|$, *then*

$$\begin{aligned}\inf_{D\in\mathscr{A}} \sup_{\substack{F,G\in K \\ \|F\|=\|G\|=1}} & \left\{ \frac{\|AF\|^2+\|BG\|^2}{2} + |\langle(B^*A+D)F,G\rangle| \right\} \\ &\leqslant \|S_{A,B}\|^2 \\ &\leqslant \inf_{D\in\mathscr{A}} \sup_{\substack{F,G\in K \\ \|F\|=\|G\|=1}} \left\{ \max(\|AF\|^2,\|BG\|^2) + |\langle(B^*A+D)F,G\rangle| \right\}\end{aligned}$$

Proof. This is an immediate consequence of Proposition 1. In fact, it can be shown by the following elementary inequality :

$$\begin{aligned}&\frac{\|AF\|^2+\|BG\|^2}{2} + \sqrt{|\langle(B^*A+D)F,G\rangle|^2} \\ &\qquad\leqslant \frac{\|AF\|^2+\|BG\|^2}{2} + \sqrt{|\langle(B^*A+D)F,G\rangle|^2 + \left(\frac{\|AF\|^2-\|BG\|^2}{2}\right)^2} \\ &\qquad\leqslant \frac{\|AF\|^2+\|BG\|^2}{2} + \sqrt{|\langle(B^*A+D)F,G\rangle|^2} + \sqrt{\left(\frac{\|AF\|^2-\|BG\|^2}{2}\right)^2}\end{aligned}$$

4. Inequality for norm of $(S_{A,B})^{-1}$

In this section, we assume that $(\mathscr{B},\mathscr{A},P)$ has a lifting property. Under some conditions on A and B, we give a formula and inequalities of $\inf\{\|S_{A,B}F\| \; ; \; F\in K, \|F\|=1\}$. As in a formula of $\|S_{A,B}\|$, even if $(\mathscr{B},\mathscr{A},P)$ does not have a lifting property, in general it is easy to see that

$$\inf\{\|S_{A,B}F\| \; ; \; F\in K, \|F\|=1\} = \min(\inf\|AF\|, \inf\|BF\|).$$

when A and B^* are in $\mathscr{A}$. Proposition 3 implies that if B^*A is in $\mathscr{A}$, then the same thing is true when $(\mathscr{B},\mathscr{A},P)$ has a lifting property.

PROPOSITION 3. *Let A and B be in $\mathscr{B}$. If* $\inf\{\|Af\| \,;\, f \in H, \|f\| = 1\} = \inf\{\|AF\| \,;\, F \in K, \|F\| = 1\}$ *and* $\inf\{\|Bg\| \,;\, g \in H^{\perp},\ \|g\| = 1\} = \inf\{\|BG\| \,;\, G \in K, \|G\| = 1\}$*, then*

$$\inf_{F\in K\ \|F\|=1} \|S_{A,B}F\|^2 = \sup_{D\in\mathscr{A}} \inf_{\substack{F,G\in K\\ \|F\|=\|G\|=1}} \left\{ \frac{\|AF\|^2+\|BG\|^2}{2} - \sqrt{|\langle (B^*A+D)F,G\rangle|^2 + \left(\frac{\|AF\|^2-\|BG\|^2}{2}\right)^2} \right\}.$$

There exists at least D in $\mathscr{A}$ which gives the supremum.

Proof. Put $\varepsilon = \inf\{\|S_{A,B}F\| \,;\, F \in K, \|F\| = 1\}$, then

$$\|Af + Bg\|^2 \geqslant \varepsilon^2 \|f+g\|^2$$

where $f \in H$ and $g \in H^{\perp}$. Hence

$$\langle (A^*A-\varepsilon^2)f,f\rangle + \langle (B^*B-\varepsilon^2)g,g\rangle - 2Re\langle (B^*A-\varepsilon^2)f,g\rangle \geqslant 0$$

for all $f \in H$ and $g \in H^{\perp}$. Suppose $\mathscr{T} = [T_{ij}]$ where $T_{11} = A^*A-\varepsilon^2$, $T_{22} = B^*B-\varepsilon^2$ and $T_{12} = T_{21}^* = B^*A-\varepsilon^2$. By hypothesis on A and B, $T_{11} \geqslant 0$ and $T_{22} \geqslant 0$. Hence T is positive on $H \oplus H^{\perp}$. As in the proof of Proposition 1, since $(\mathscr{B},\mathscr{A},P)$ has a lifting property, there exists D in $\mathscr{A}$ such that

$$\langle (A^*A-\varepsilon^2)F,F\rangle \langle (B^*B-\varepsilon^2)G,G\rangle \geqslant |\langle (B^*A+D)F,G\rangle|^2$$

for any $F \in K$ and $G \in K$, and so

$$\varepsilon^4 - \varepsilon^2(\|AF\|^2+\|BG\|^2) + \|AF\|^2\|BG\|^2 - |\langle (B^*A+D)F,G\rangle|^2 \geqslant 0$$

for any $F \in K$ and $G \in K$ with $\|F\| = \|G\| = 1$. Therefore

$$\varepsilon^2 \leqslant \frac{\|AF\|^2+\|BG\|^2}{2} - \sqrt{|\langle (B^*A+D)F,G\rangle|^2 + \left(\frac{\|AF\|^2-\|BG\|^2}{2}\right)^2}$$

or

$$\varepsilon^2 \geqslant \frac{\|AF\|^2+\|BG\|^2}{2} + \sqrt{|\langle (B^*A+D)F,G\rangle|^2 + \left(\frac{\|AF\|^2-\|BG\|^2}{2}\right)^2}$$

By hypotheses on A and $B, A^*A \geqslant \varepsilon^2$ and $B^*B \geqslant \varepsilon^2$, and so

$$\varepsilon^2 \leqslant (\|AF\|^2+\|BG\|^2)/2$$

for any $F,G \in K$ with $\|F\| = \|G\| = 1$. This implies that the second inequality is not valid. Thus

$$\varepsilon^2 \leqslant \inf_{\substack{F,G\in K\\ \|F\|=\|G\|=1}} \left\{ \frac{\|AF\|^2+\|BG\|^2}{2} - \sqrt{|\langle (B^*A+D)F,G\rangle|^2 + \left(\frac{\|AF\|^2-\|BG\|^2}{2}\right)^2} \right\}.$$

The converse can be shown as in the proof of Proposition 1.

COROLLARY 3. *Let A and B be in $\mathscr{B}$. If A and B satisfy the condition in Proposition 3, then*

$$\begin{aligned}
&\inf_{F\in K\ \|F\|=1} \|S_{A,B}F\|^2 \\
&\qquad \geqslant \sup_{D\in\mathscr{A}} \inf_{\substack{F,G\in K \\ \|F\|=\|G\|=1}} \{\min(\|AF\|^2, \|BG\|^2) - |\langle (B^*A+D)F, G\rangle|\} \\
&\qquad \geqslant \min(\inf_{\substack{F\in K \\ \|F\|=1}} \|AF\|^2, \inf_{G\in K} \|BG\|^2) - \|B^*A+\mathscr{A}\|.
\end{aligned}$$

COROLLARY 4. *Let A and B be in $\mathscr{B}$. If $A^*A=|a|^2I$ and $B^*B=|b|^2I$ where a and b are complex numbers, then*

$$\begin{aligned}
&\inf_{F\in K\ \|F\|=1} \|S_{A,B}F\|^2 \\
&\qquad = \frac{|a|^2+|b|^2}{2} - \sqrt{\|B^*A+\mathscr{A}\|^2 + \left(\frac{|a|^2-|b|^2}{2}\right)^2}.
\end{aligned}$$

COROLLARY 5. *Let A be in $\mathscr{B}$ and $B=I$. Then*

$$\begin{aligned}
&\inf \|S_{A,I}F\|^2 \\
&\qquad = \sup_{D\in\mathscr{A}} \inf_{\substack{F\in K \\ \|F\|=1}} \left\{ \frac{\|AF\|^2+1}{2} - \sqrt{\|(A+D)F\|^2 + \left(\frac{\|AF\|^2-1}{2}\right)^2} \right\}.
\end{aligned}$$

Proof. It is a corollary of the proof of Proposition 3 similarly to the proof of Corollary 2.

THEOREM 4. *Let A and B be in $\mathscr{B}$. If* $\inf\{\|Af\|\ ;\ f\in H, \|f\|=1\} = \inf\{\|AF\|\ ;\ F\in K, \|F\|=1\}$ *and* $\inf\{\|Bg\|\ ;\ g\in H^\perp, \|g\|=1\} = \inf\{\|BG\|\ ;\ G\in K, \|G\|=1\}$*, then*

$$\begin{aligned}
&\sup_{D\in\mathscr{A}} \inf_{\substack{F,G\in K \\ \|F\|=\|G\|=1}} \left\{ \frac{\|AF\|^2+\|BG\|^2}{2} - |\langle (B^*A+D)F, G\rangle| \right\} \\
&\qquad \geqslant \inf_{F\in K\ \|F\|=1} \|S_{A,B}F\|^2 \\
&\qquad \geqslant \sup_{D\in\mathscr{A}} \inf_{\substack{F,G\in K \\ \|F\|=\|G\|=1}} \{\min(\|AF\|^2,\ \|BG\|^2) - |\langle (B^*A+D)F, G\rangle|\}.
\end{aligned}$$

Proof. This is an immediate result of Proposition 3 and the proof is similar to that of Theorem 2.

5. Applications and remarks

(I) Suppose $(\mathscr{B},\mathscr{A},P)$ has a lifting property. For A in $\mathscr{B}$, the Toeplitz operator T_A is defined by $T_Af = P(Af)$ for f in H. Then we can give an operator version of Theorem 2 in [6]. That is, if $\inf\{\|Af\| \; ; \; f \in H, \; \|f\| = 1\} = \inf\{\|AF\| \; ; \; F \in K, \; \|K\| = 1\}$, then

$$\inf_{f\in H \ \|f\|=1} \|T_Af\|^2 = \sup_{D\in\mathscr{A}} \{\inf_{F\in K \ \|F\|=1} (\|AF\|^2 - \|(A+D)F\|^2)\}.$$

This has an application. That is, T_A is left invertible if and only if there exist D in $\mathscr{A}$ and $\varepsilon > 0$ such that

$$\|AF\|^2 \geqslant \varepsilon + \|(A+D)F\|^2$$

for any F in K with $\|F\| = 1$. This is equivalent to

$$A^*A \geqslant \varepsilon + (A+D)^*(A+D).$$

It may be interesting that we did not use the factorization theorem in the proof of the result above. When A is a unitary operator, T_A is left invertible if and only if there exists a D in $\mathscr{A}$ such that $\|A+D\| < 1$. This implies Theorem 2 in [7] where $\mathscr{B} = \mathscr{L}(K)$, that is, Example (1) in Section 2, and Theorem 7.30 in [3] where $\mathscr{B} = L^\infty(T)$, that is , the classical case.

(II) Suppose $(\mathscr{B},\mathscr{A},P)$ has a lifting property, We give an application of Corollary 5. Let A be a nonzero operator in $\mathscr{B}$ with $\inf\{\|AF\| \; ; \; F \in K, \; \|F\| = 1\} = \inf\{\|Af\| \; ; \; f \in H, \; \|f\| = 1\}$. $S_{A,I}$ is left invertible if and only if there exist D in $\mathscr{A}$ and $\varepsilon > 0$ such that

$$\|AF\|^2 \geqslant \varepsilon + \|(A+D)F\|^2$$

for any F in K with $\|F\| = 1$.

We give a proof. If $S_{A,I}$ is left invertible, by Corollary 5 there exist $D \in \mathscr{A}$ and $1 > \delta > 0$ such that

$$\frac{\|AF\|^2+1}{2} - \sqrt{\|(A+D)F\|^2 + \left(\frac{\|AF\|^2-1}{2}\right)^2} \geqslant \delta$$

for any $F \in K$ with $\|F\| = 1$. This implies that

$$(1-\delta)\|AF\|^2 \geqslant (1-\delta)\delta + \|(A+D)F\|^2$$

for any $F \in K$ with $\|F\| = 1$. Setting $\varepsilon = (1-\delta)\delta$ the necessity follows. Conversely if there exist D in $\mathscr{A}$ and $\varepsilon > 0$ such that

$$\|AF\|^2 \geqslant \varepsilon + \|(A+D)F\|^2$$

for any F in K with $\|F\| = 1$,

$$\begin{aligned}
&\frac{\|AF\|^2+1}{2} - \sqrt{\|(A+D)F\|^2 + \left(\frac{\|AF\|^2-1}{2}\right)^2} \\
&\qquad \geqslant \frac{\|AF\|^2+1}{2} - \sqrt{\|AF\|^2 - \varepsilon + \left(\frac{\|AF\|^2-1}{2}\right)^2} \\
&\qquad = \frac{\|AF\|^2+1}{2} - \sqrt{\left(\frac{\|AF\|^2+1}{2}\right)^2 - \varepsilon^2} \\
&\qquad = \varepsilon^2 \Big/ \left\{ \frac{\|AF\|^2+1}{2} + \sqrt{\left(\frac{\|AF\|^2+1}{2}\right)^2 - \varepsilon^2} \right\}
\end{aligned}$$

for any $F \in K$ with $\|F\| = 1$. Now Corollary 5 implies that $\inf_{F\in K\ \|F\|=1} \|S_{A,I}F\| > 0$.

In an abstract situation (see Example (1) in Section 2), Shinbrot [7] studied singular integral operators.

(III) We don't assume that $(\mathscr{B}, \mathscr{A}, P)$ has a lifting property. We can prove the following :

(1) Let A and B be in $\mathscr{B}$. Then

$$\|S_{A,B}\|^2 = \sup_{\substack{f\in H,\ g\in H^\perp \\ \|f\|=\|g\|=1}} \left\{ \frac{\|Af\|^2+\|Bg\|^2}{2} + \sqrt{|\langle B^*Af, g\rangle|^2 + \left(\frac{\|Af\|^2-\|Bg\|^2}{2}\right)^2} \right\}.$$

Hence if $A^*A = |a|^2 I$ and $B^*B = |b|^2 I$ then

$$\|S_{A,B}\|^2 = \frac{|a|^2+|b|^2}{2} + \sqrt{\|QB^*AP\|^2 + \left(\frac{|a|^2-|b|^2}{2}\right)^2}.$$

(2) Let A and B be in $\mathscr{B}$. Then

$$\begin{aligned}
&\inf_{F\in K\ \|F\|=1} \|S_{A,B}F\|^2 \\
&\qquad = \inf_{\substack{f\in H\ g\in H^\perp \\ \|f\|=\|g\|=1}} \left\{ \frac{\|Af\|^2+\|Bg\|^2}{2} - \sqrt{|\langle B^*Af, g\rangle|^2 + \left(\frac{\|Af\|^2-\|Bg\|^2}{2}\right)^2} \right\}.
\end{aligned}$$

Hence if $A^*A = |a|^2 I$ and $B^*B = |b|^2 I$ then

$$\inf_{F\in K\ \|F\|=1} \|S_{A,B}F\|^2 = \frac{|a|^2+|b|^2}{2} - \sqrt{\|QB^*A\|^2 + \left(\frac{|a|^2-|b|^2}{2}\right)^2}$$

(IV) Let $\mathscr{S}$ be a set of unitary operators on K and suppose the set $\{Vf\ ;\ V \in \mathscr{S}, f \in H\}$ is dense in K. If A is in the commutant of $\mathscr{S}$, then

$$\|A\| = \|AP\|$$

and

$$\inf_{F\in K\ \|F\|=1} \|AF\| = \inf_{F\in H\ \|f\|=1} \|Af\|.$$

This can be proved as in the proof of $(viii)$ of Theorem 4 in [2].

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