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RESULTS UNDER $\log A \geqslant \log B$ CAN BE DERIVED FROM ONES UNDER $A \geqslant B \geqslant 0$ BY UCHIYAMA'S METHOD – ASSOCIATED WITH FURUTA AND KANTOROVICH TYPE OPERATOR INEQUALITIES

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Dedicated to
Professor Masanori Fukamiya
with respect and affection
on his 88th birthday

(*communicated by J. Pečarić*)

Abstract. In what follows, a capital letter means a bounded linear operator on a Hilbert space H. We discuss some parallelism between $A \geqslant B \geqslant 0$ and $\log A \geqslant \log B$ on generalized Furuta inequality and Kantorovich type inequalities. Precisely speaking, several results under the chaotic order $\log A \geqslant \log B$ on Furuta type inequalities and Kantorovich type inequalities can be both derived from ones under the usual order $A \geqslant B \geqslant 0$ by using Uchiyama's method [23].

1. Introduction

We start this section by introducing the following order preserving operator inequalities.

THEOREM F. (Furuta inequality)

If $A \geqslant B \geqslant 0$, *then for each* $r \geqslant 0$,

(*i*) $$(B^{\frac{r}{2}}A^pB^{\frac{r}{2}})^{\frac{1}{q}} \geqslant (B^{\frac{r}{2}}B^pB^{\frac{r}{2}})^{\frac{1}{q}}$$

and

(*ii*) $$(A^{\frac{r}{2}}A^pA^{\frac{r}{2}})^{\frac{1}{q}} \geqslant (A^{\frac{r}{2}}B^pA^{\frac{r}{2}})^{\frac{1}{q}}$$

hold for $p \geqslant 0$ *and* $q \geqslant 1$ *with* $(1+r)q \geqslant p+r$.

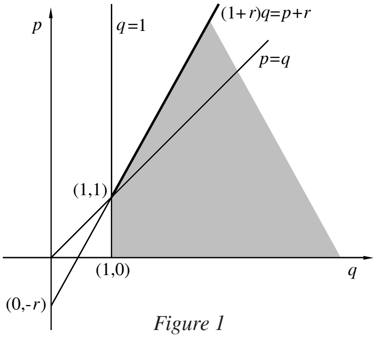

Figure 1

The original proof of Theorem F is in [9], mean theoretic proofs in [4][17] and one page proof in [10]. The domain drawn for p, q and r in the Figure 1 is the best possible one for (i) and (ii) of Theorem F in [21].

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On the other hand, the celebrated Kantorovich inequality asserts that if A is positive operator on H such that $M \geqslant A \geqslant m > 0$, then $(A^{-1}x, x)(Ax, x) \leqslant \dfrac{(m+M)^2}{4mM}$ holds for every unit vector x in H. Many authors have been investigating several types extensions of Kantorovich inequality and there is a series of papers on Kantorovich inequality by Mond-Pecaric and two of them are [18] and [19].

We can summarize the parallelism associated with several results between $A \geqslant B$ and $\log A \geqslant \log B$ in this paper. It is interesting to point out that results on chaotic order can be derived from ones on usual order by using Uchiyama's method [23] in Theorem 1, Theorem 2, Theorem 3 and Theorem 4 underbelow.

2. A characterization of chaotic order can be derived from Furuta type inequality

We introduce the following parallelism between $A \geqslant B \geqslant 0$ and $\log A \geqslant \log B$ associated with Furuta inequality and Furuta type inequality;

THEOREM A. (Parallelism on Furuta type inequalities) *Let A and B be invertible positive operators. Then the following* (F1), (F2) *and* (F3) *hold.*

(F1) *$A \geqslant B$ holds if and only if $A^{1+r} \geqslant (A^{\frac{r}{2}} B^p A^{\frac{r}{2}})^{\frac{1+r}{p+r}}$ holds for any $p \geqslant 1$ and $r \geqslant 0$.*

(F2) *$\log A \geqslant \log B$ holds if and only if $A^r \geqslant (A^{\frac{r}{2}} B^p A^{\frac{r}{2}})^{\frac{r}{p+r}}$ holds for any $p \geqslant 0$ and $r \geqslant 0$.*

(F3) *$\log A \geqslant \log B$ holds if and only if $A^r \geqslant (A^{\frac{r}{2}} B^p A^{\frac{r}{2}})^{\frac{r}{p+r}}$ holds for any $p \geqslant 1$ and $r \geqslant 0$.*

(F1) is Furuta inequality [9][4][17] and one page proof in [10], and (F2) is well known in ([11][5] and etc.) (F2) in case $p = r$ in [1] and (F3) in ([11] and etc.).

On the other hand, the following Theorem B interpolates Furuta inequality itself and a useful inequality equivalent to the main theorem of log majorization by Ando-Hiai [2].

THEOREM B. (Generalized Furuta inequality) *If $A \geqslant B > 0$, then for each $t \in [0,1]$ and $p \geqslant 1$,*

$$G_{p,t}(A,B,r,s) = A^{\frac{-r}{2}}\{A^{\frac{r}{2}}(A^{\frac{-t}{2}}B^pA^{\frac{-t}{2}})^sA^{\frac{-r}{2}}\}^{\frac{1-t+r}{(p-t)s+r}}A^{\frac{-r}{2}}$$

is decreasing for r and s such that $r \geqslant t$ and $s \geqslant 1$. Moreover the following (GF1) *holds.*

(GF1) *$A \geqslant B$ holds if and only if $A^{1-t+r} \geqslant \{A^{\frac{r}{2}}(A^{\frac{-t}{2}}B^pA^{\frac{-t}{2}})^sA^{\frac{r}{2}}\}^{\frac{1-t+r}{(p-t)s+r}}$ holds for any $p \geqslant 1$, $1 \geqslant t \geqslant 0$, $r \geqslant t$ and $s \geqslant 1$.*

The original proof of Theorem B is in [12], the mean theoretic one in [6] and one page proof of (GF1) in [13]. An excellent and tough proof of the best possibility of $\dfrac{1-t+r}{(p-t)s+r}$ is in [22], and simple proofs are in [24] and [8].

Uchiyama [23] gives a simplified proof of the following Theorem C equivalent to (F2) by only using Furuta inequality with his skillful technique.

THEOREM C. *Let A and B be self adjoint operators. Then $A \geqslant B$ ensures*

$$e^{rA} \geqslant (e^{\frac{rA}{2}} e^{pB} e^{\frac{rA}{2}})^{\frac{r}{p+r}}. \tag{2.0}$$

for any $p \geqslant 0$ and $r \geqslant 0$.

THEOREM 1. (Furuta type inequalities) *Let A and B be invertible positive operators. Then the following parallelism holds and* (F2) *can be derived from* (GF1) *directly.*

(GF1) *$A \geqslant B$ implies $A^{1-t+r} \geqslant \{A^{\frac{r}{2}}(A^{\frac{-t}{2}} B^p A^{\frac{-t}{2}})^s A^{\frac{r}{2}}\}^{\frac{1-t+r}{(p-t)s+r}}$ holds for any $p \geqslant 1 \geqslant t \geqslant 0$, $r \geqslant t$ and $s \geqslant 1$.*

(F2) *$\log A \geqslant \log B$ implies $A^r \geqslant (A^{\frac{r}{2}} B^p A^{\frac{r}{2}})^{\frac{r}{p+r}}$ holds for any $p \geqslant 0$ and $r \geqslant 0$.*

Proof of Theorem 1. We have only to show a proof of (F2) via (GF1) since (GF1) is shown in Theorem B. Let us try its calculation along Uchiyama's idea used in Theorem C. The hypothesis $\log A \geqslant \log B$ ensures $A_1 = 1 + \dfrac{\log A}{n} \geqslant 1 + \dfrac{\log B}{n} = B_1 > 0$ for sufficiently large natural number n. By substituting np, nr and nt to p, r and t in (GF1), we have

$$A_1^{1-nt+nr} \geqslant \{A_1^{\frac{nr}{2}}(A_1^{\frac{-nt}{2}} B_1^{np} A_1^{\frac{-nt}{2}})^s A_1^{\frac{nr}{2}}\}^{\frac{1-nt+nr}{(np-nt)s+nr}} \tag{2.1}$$

holds for any $np \geqslant 1 \geqslant nt \geqslant 0$, $nr \geqslant nt$ and $s \geqslant 1$.

We recall the following obvious and crucial formula

$$\lim_{n\to\infty} (I + \frac{1}{n}\log X)^n = X \text{ for any } X > 0. \tag{$\clubsuit$}$$

When $n \to \infty$, then $t \to 0$ since $1 \geqslant nt \geqslant 0$, so that

$$A_1^{\frac{-nt}{2}} = \left(1 + \frac{1}{n}\log A\right)^{\frac{-nt}{2}} \longrightarrow I \text{ as } n \to \infty \text{ since } 1 \geqslant nt \geqslant 0$$

$$A_1^{1-nt+nr} = \left(1 + \frac{1}{n}\log A\right)^{n(\frac{1}{n}-t+r)} \longrightarrow A^r \text{ by } (\clubsuit) \text{ as } n \to \infty$$

$$B_1^{-np} = \left(1 + \frac{1}{n}\log B\right)^{np} \longrightarrow B^p \text{ by } (\clubsuit) \text{ as } n \to \infty$$

and

$$\frac{1-nt+nr}{(np-nt)s+nr} = \frac{\frac{1}{n}-t+r}{(p-t)s+r} \longrightarrow \frac{r}{ps+r} \text{ as } n \to \infty,$$

therefore (2.1) ensures the following (2.2) for $p \geqslant 0$ and $s \geqslant 1$

$$A^r \geqslant (A^{\frac{r}{2}} B^{ps} A^{\frac{r}{2}})^{\frac{r}{ps+r}} \text{ holds for any } ps \geqslant 0 \text{ and } r \geqslant 0, \tag{2.2}$$

so the proof of (F2) via (GF1) is complete.

REMARK 1. We might expect to get more precise result than Theorem C by using (GF1) which is an extension of Furuta inequality. However the operator inequality (2.2) is nothing but (2.0) of Theorem C and this result is caused by the following fact;

$$A_1^{\frac{-nt}{2}} = (1+\frac{1}{n}\log A)^{\frac{-nt}{2}} \to I \text{ as } n \to \infty \text{ since } 1 \geqslant nt \geqslant 0.$$

3. A counterexample to a question associated with generalized Furuta inequality

Motivated by (GF1) of Theorem B and the parallelism among (F1), (F2) and (F3) in Theorem A, we might apt to conjecture the parallelism in the following Question 1.

QUESTION 1. *Let A and B be invertible positive operators. Then*

(Q1) $\log A \geqslant \log B$ *holds if and only if* $A^r \geqslant \{A^{\frac{r}{2}}(A^{\frac{-t}{2}}B^pA^{\frac{-t}{2}})^sA^{\frac{r}{2}}\}^{\frac{r}{(p-t)s+r}}$ *holds for any* $p \geqslant t$, $1 \geqslant t \geqslant 0$, $r \geqslant t$ *and* $s \geqslant 1$?

(Q2) $\log A \geqslant \log B$ *holds if and only if* $A^{r-t} \geqslant \{A^{\frac{r}{2}}(A^{\frac{-t}{2}}B^pA^{\frac{-t}{2}})^sA^{\frac{r}{2}}\}^{\frac{r-t}{(p-t)s+r}}$ *holds for any* $p \geqslant 1$, $1 \geqslant t \geqslant 0$, $r \geqslant t$ *and* $s \geqslant 1$?

BACKGROUND OF QUESTION 1. The exponent $\frac{r}{(p-t)s+r}$ of the right hand side of (Q1) can be expressed as $\frac{r}{(p-t)s+r} = \frac{r}{p'+r}$, where $p' = (p-t)s \geqslant 0$ and $r \geqslant 0$ since $p \geqslant t$, $1 \geqslant t \geqslant 0$, $r \geqslant t$ and $s \geqslant 1$, so that we might be apt to guess that (Q1) just might correspond to an extension form of (F2). In fact, put $t = 0$ in (Q1), then

$\log A \geqslant \log B$ *holds if and only if* $A^r \geqslant (A^{\frac{r}{2}}B^{ps}A^{\frac{r}{2}})^{\frac{r}{ps+r}}$ *holds for any* $ps \geqslant 0$ *and* $r \geqslant 0$.

This is just equivalent to (F2).

The exponent $\frac{r-t}{(p-t)s+r}$ of the right hand side of (Q2) can be expressed as $\frac{r-t}{(p-t)s+r} = \frac{r-t}{(p-t)s+t+(r-t)} = \frac{r'}{p'+r'}$, where $p' = (p-t)s+t \geqslant (p-t)+t = p \geqslant 1$ and $r' = r-t \geqslant 0$ since $p \geqslant 1$, $1 \geqslant t \geqslant 0$, $r \geqslant t$ and $s \geqslant 1$, so that we might be apt to guess that (Q2) just might correspond to an extension form of (F3). In fact, put $t = 0$ in (Q2), then

$\log A \geqslant \log B$ *holds if and only if* $A^r \geqslant (A^{\frac{r}{2}}B^{ps}A^{\frac{r}{2}})^{\frac{r}{ps+r}}$ *holds for any* $ps \geqslant 1$ *and* $r \geqslant 0$.

This is just equivalent to (F3).

We pose Question 1 by considering *Background of Question* 1. Although ($\Longleftarrow$) in (Q1) and (Q2) are both correct, but we cite a counterexample to ($\Longrightarrow$) in (Q1) and (Q2) as follows. ($\Longleftarrow$) in (Q1) is valid. In fact, we put $t = 0$ at the right hand side of (Q1). Then $A^r \geqslant (A^{\frac{r}{2}}B^{ps}A^{\frac{r}{2}})^{\frac{r}{ps+r}}$ holds for any $r \geqslant 0$ and $ps \geqslant 0$. Taking logarithm of both sides,

$$(ps+r)\log A \geqslant \log(A^{\frac{r}{2}}B^{ps}A^{\frac{r}{2}}),$$

tending $r \to 0$, so we have $\log A \geqslant \log B$. ($\Longleftarrow$) in (Q2) is also valid by the same way as ($\Longleftarrow$) in (Q1).

COUNTEREXAMPLE TO ($\Longrightarrow$) IN (Q2). *There exists positive invertible operators A and B such that $\log A \geqslant \log B$ and*

$$A^{r-t} \ngeqslant \{A^{\frac{r}{2}}(A^{\frac{-t}{2}}B^pA^{\frac{-t}{2}})^sA^{\frac{r}{2}}\}^{\frac{r-t}{(p-t)s+r}}$$

for $r=2$, $t=1$, $s=2$ and $p=2$.

In fact, take A and B as:

$$\log A = \begin{pmatrix} 2 & 2 \\ 2 & -1 \end{pmatrix} \qquad \text{and} \qquad \log B = \begin{pmatrix} 1 & 3 \\ 3 & -2 \end{pmatrix}.$$

Then

$$\log A - \log B = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \geqslant 0,$$

so that $\log A \geqslant \log B$. Next $\log A$ is diagonalized as follows;

$$U(\log A)U = \begin{pmatrix} -2 & 0 \\ 0 & 3 \end{pmatrix}, \quad \text{where} \quad U = \begin{pmatrix} \frac{-1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{pmatrix}.$$

Also $\log B$ is diagonalized as follows;

$$V(\log B)V = \begin{pmatrix} \frac{-1-3\sqrt{5}}{2} & 0 \\ 0 & \frac{-1+3\sqrt{5}}{2} \end{pmatrix}, \text{where } V = \begin{pmatrix} \frac{1-\sqrt{5}}{\sqrt{2}\sqrt{5-\sqrt{5}}} & \frac{\sqrt{2}}{\sqrt{5-\sqrt{5}}} \\ \frac{\sqrt{2}}{\sqrt{5-\sqrt{5}}} & \frac{\sqrt{2}}{\sqrt{5+\sqrt{5}}} \end{pmatrix}.$$

Therefore we obtain the following A and B by somewhat tough calculation

$$A = \begin{pmatrix} \frac{1}{5e^2} + \frac{4e^3}{5} & \frac{-2}{5e^2} + \frac{2e^3}{5} \\ \frac{-2}{5e^2} + \frac{2e^3}{5} & \frac{4}{5e^2} + \frac{e^3}{5} \end{pmatrix}$$

and

$$B = \begin{pmatrix} \frac{(1-\sqrt{5})^2 e^{\frac{-1-3\sqrt{5}}{2}}}{2(5-\sqrt{5})} + \frac{(1+\sqrt{5})^2 e^{\frac{-1+3\sqrt{5}}{2}}}{2(5+\sqrt{5})} & \frac{(1-\sqrt{5})e^{\frac{-1-3\sqrt{5}}{2}}}{5-\sqrt{5}} + \frac{(1+\sqrt{5})e^{\frac{-1+3\sqrt{5}}{2}}}{5+\sqrt{5}} \\ \frac{(1-\sqrt{5})e^{\frac{-1-3\sqrt{5}}{2}}}{5-\sqrt{5}} + \frac{(1+\sqrt{5})e^{\frac{-1+3\sqrt{5}}{2}}}{5+\sqrt{5}} & \frac{2e^{\frac{-1-3\sqrt{5}}{2}}}{5-\sqrt{5}} + \frac{2e^{\frac{-1+3\sqrt{5}}{2}}}{5+\sqrt{5}} \end{pmatrix}.$$

But the computer shows that

$$X = A^{r-t} - \{A^{\frac{r}{2}}(A^{\frac{-t}{2}}B^pA^{\frac{-t}{2}})^sA^{\frac{r}{2}}\}^{\frac{r-t}{(p-t)s+r}} = \begin{pmatrix} -0.64014... & -0.53427... \\ -0.53427... & -0.22833... \end{pmatrix}$$

for $r=2$, $t=1$, $s=2$, $p=2$, and the eigenvalues of X are $-1.0068...$ and $0.1383...$. Hence we obtain

$$A^{r-t} \ngeqslant \{A^{\frac{r}{2}}(A^{\frac{-t}{2}}B^pA^{\frac{-t}{2}})^sA^{\frac{r}{2}}\}^{\frac{r-t}{(p-t)s+r}}$$

for $r=2$, $t=1$, $s=2$ and $p=2$ under the hypothesis $\log A \geqslant \log B$ and this is a counterexample to ($\Longrightarrow$) in (Q2).

($\Longrightarrow$) *in* (Q1) *is also invalid.* Assume that ($\Longrightarrow$) in (Q1) is valid, that is,

$$\log A \geqslant \log B \text{ implies } A^r \geqslant \{A^{\frac{r}{2}}(A^{\frac{-t}{2}}B^pA^{\frac{-t}{2}})^sA^{\frac{r}{2}}\}^{\frac{r}{(p-t)s+r}}$$

holds for any $p \geqslant t$, $1 \geqslant t \geqslant 0$, $r \geqslant t$ and $s \geqslant 1$. Then we have the following result by Löwner-Heinz theorem since $\frac{r-t}{r} \in [0,1]$ holds.

$$\log A \geqslant \log B \textit{ ensures } A^{r-t} \geqslant \{A^{\frac{r}{2}}(A^{\frac{-t}{2}}B^pA^{\frac{-t}{2}})^sA^{\frac{r}{2}}\}^{\frac{r-t}{(p-t)s+r}}$$

holds for any $p \geqslant 1$, $1 \geqslant t \geqslant 0$, $r \geqslant t$ *and* $s \geqslant 1$, that is, ($\Longrightarrow$) in (Q2) holds. But ($\Longrightarrow$) in (Q2) is invalid as stated before, so that ($\Longrightarrow$) in (Q1) is also invalid.

(F2) *of Theorem* A *can be regarded as limit form of* (GF1$'$) *which is a variation of* (GF1)

THEOREM B$'$. (A variation of generalized Furuta inequality) *Let* A *and* B *be invertible positive operators. Then the following* (GF1$'$) *holds.*

(GF1$'$) *For a fixed* $\alpha > 0$,

$$A^\alpha \geqslant B^\alpha \quad \textit{holds if and only if} \quad A^{\alpha-t+r} \geqslant \{A^{\frac{r}{2}}(A^{\frac{-t}{2}}B^pA^{\frac{-t}{2}})^sA^{\frac{r}{2}}\}^{\frac{\alpha-t+r}{(p-t)s+r}}$$

holds for any $p \geqslant \alpha \geqslant t \geqslant 0$, $r \geqslant t$ *and* $s \geqslant 1$.

Proof of (GF1$'$). Put $A_1 = A^\alpha$ and $B_1 = B^\alpha$. Then $A_1 \geqslant B_1$. Put $r' = \dfrac{r}{\alpha}$, $t' = \dfrac{t}{\alpha}$ and $p' = \dfrac{p}{\alpha}$. Then $p' \geqslant 1 \geqslant t' \geqslant 0$ and $r' \geqslant t'$ hold. Applying (GF1) of Theorem B to A_1 and B_1,

$$\begin{aligned} A^{\alpha-t+r} &= A_1^{1-t'+r'} \\ &\geqslant \{A_1^{\frac{r'}{2}}(A_1^{\frac{-t'}{2}}B_1^{p'}A_1^{\frac{-t'}{2}})^sA_1^{\frac{r'}{2}}\}^{\frac{1-t'+r'}{(p'-t')s+r'}} \\ &= \{A^{\frac{r}{2}}(A^{\frac{-t}{2}}B^pA^{\frac{-t}{2}})^sA^{\frac{r}{2}}\}^{\frac{\alpha-t+r}{(p-t)s+r}}. \end{aligned}$$

We show the following natural understanding that (F2) *can be regarded as limit form of* (GF1$'$). We recall the following obvious fact (3.1)

$$\log T = \lim_{\alpha \to 0} \frac{T^\alpha - I}{\alpha} \text{ for an operator } T > 0. \tag{3.1}$$

The left hand hypothesis $A^\alpha \geqslant B^\alpha$ in (GF1$'$) implies $\frac{A^\alpha - I}{\alpha} \geqslant \frac{B^\alpha - I}{\alpha}$ and when $\alpha \to 0$, by (3.1) we have $\log A \geqslant \log B$ which is the left hand in (F2).

On the other hand, the right hand side of (F2) can be obtained by the following process via the right hand side of (GF1$'$); in fact $\alpha \to 0$ at the following right hand side of (GF1$'$)

$$A^{\alpha-t+r} \geqslant \{A^{\frac{r}{2}}(A^{\frac{-t}{2}}B^pA^{\frac{-t}{2}})^sA^{\frac{r}{2}}\}^{\frac{\alpha-t+r}{(p-t)s+r}}$$

holds for any $p \geqslant \alpha \geqslant t \geqslant 0$, $r \geqslant t$ and $s \geqslant 1$. Then $t \to 0$, $p \geqslant 0$ and $r \geqslant 0$, so that we have

$$A^r \geqslant (A^{\frac{r}{2}}B^{ps}A^{\frac{r}{2}})^{\frac{r}{ps+r}}$$

holds for $r \geqslant 0$, $p \geqslant 0$ and $s \geqslant 1$, therefore replacing ps by $p \geqslant 0$, we obtain the right hand side of (F2). Consequently (F2) *can be regarded as limit form of* (GF1$'$).

PROPOSITION 1. (Parallelism on generalized Furuta inequality) *Let A and B be invertible positive operators. Then the following* (GF1), (GF1′) *and* (F2) *hold.*

(GF1) $A \geqslant B$ *holds if and only if*

$$A^{1-t+r} \geqslant \{A^{\frac{r}{2}}(A^{\frac{-t}{2}}B^pA^{\frac{-t}{2}})^sA^{\frac{r}{2}}\}^{\frac{1-t+r}{(p-t)s+r}}.$$

holds for any $p \geqslant 1 \geqslant t \geqslant 0$, $r \geqslant t$ and $s \geqslant 1$.

(GF1′) *For a fixed $\alpha > 0$, $A^\alpha \geqslant B^\alpha$ holds if and only if*

$$A^{\alpha-t+r} \geqslant \{A^{\frac{r}{2}}(A^{\frac{-t}{2}}B^pA^{\frac{-t}{2}})^sA^{\frac{r}{2}}\}^{\frac{\alpha-t+r}{(p-t)s+r}}$$

holds for any $p \geqslant \alpha \geqslant t \geqslant 0$, $r \geqslant t$ and $s \geqslant 1$.

(F2) $\log A \geqslant \log B$ *holds if and only if $A^r \geqslant (A^{\frac{r}{2}}B^pA^{\frac{r}{2}})^{\frac{r}{p+r}}$ holds for any $p \geqslant 0$ and $r \geqslant 0$.*

When $t = 0$, Proposition 1 becomes the following Corollary 2.

COROLLARY 2. (Parallelism on Furuta type inequalities) *Let A and B be invertible positive operators. Then the following* (F1), (F1′) *and* (F2) *hold.*

(F1) $A \geqslant B$ *holds if and only if $A^{1+r} \geqslant (A^{\frac{r}{2}}B^pA^{\frac{r}{2}})^{\frac{1+r}{p+r}}$ holds for any $p \geqslant 1$ and $r \geqslant 0$.*

(F1′) *For a fixed $\alpha > 0$, $A^\alpha \geqslant B^\alpha$ holds if and only if $A^{\alpha+r} \geqslant (A^{\frac{r}{2}}B^pA^{\frac{r}{2}})^{\frac{\alpha+r}{p+r}}$ holds for any $p \geqslant \alpha$ and $r \geqslant 0$.*

(F2) $\log A \geqslant \log B$ *holds if and only if $A^r \geqslant (A^{\frac{r}{2}}B^pA^{\frac{r}{2}})^{\frac{r}{p+r}}$ holds for any $p \geqslant 0$ and $r \geqslant 0$.*

Corollary 2 can be expressed in the following form with explanation of graphic meaning.

COROLLARY 2′ [7].
Let A and B be positive invertible operators. Then the following (F1-g), (F1′-g) *and* (F2-g) *hold.*

(F1-g) $A \geqslant B$ *holds if and only if*

$$A^{\frac{p+r}{q}} \geqslant (A^{\frac{r}{2}}B^pA^{\frac{r}{2}})^{\frac{1}{q}}$$

for $p \geqslant 0$, $r \geqslant 0$ and $q \geqslant 1$ with $(1+r)q \geqslant p+r$.

(F1′-g) *For a fixed $\alpha \geqslant 0$, $A^\alpha \geqslant B^\alpha$ holds if and only if*

$$A^{\frac{p+r}{q}} \geqslant (A^{\frac{r}{2}}B^pA^{\frac{r}{2}})^{\frac{1}{q}}$$

for $p \geqslant 0$, $r \geqslant 0$ and $q \geqslant 1$ with $(\alpha+r)q \geqslant p+r$.

(F2-g) $\log A \geqslant \log B$ *holds if and only if*

$$A^{\frac{p+r}{q}} \geqslant (A^{\frac{r}{2}}B^pA^{\frac{r}{2}})^{\frac{1}{q}}$$

for $p \geqslant 0$, $r \geqslant 0$ and $q \geqslant 1$ with $rq \geqslant p+r$.

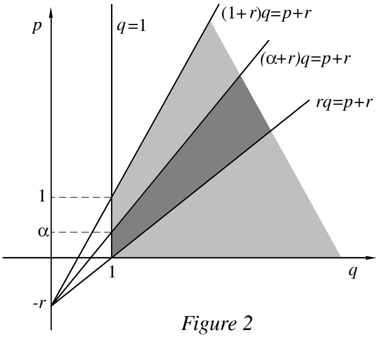

Figure 2

4. Kantorovich type inequalities under $\log A \geqslant \log B$ can be derived from ones under $A \geqslant B \geqslant 0$, I

THEOREM 2. (Kantorovich type inequalities, I) *Let* $A > 0$ *and* $M \geqslant B \geqslant m > 0$. *Then the following parallelism holds and* (4.2) *can be derived from* (4.1) *directly.*

(4.1) $A \geqslant B$ *implies* $K(m,M,p)A^p \geqslant B^p$ *for any* $p \geqslant 1$,

(4.2) $\log A \geqslant \log B$ *implies* $M_h(p)A^p \geqslant B^p$ *for any* $p > 0$, *where* $K(m,M,p)$ *and* $M_h(p)$ *are defined as follows:*

$$K(m,M,p) = \frac{(p-1)^{p-1}}{p^p}\frac{(M^p-m^p)^p}{(M-m)(mM^p-Mm^p)^{p-1}}$$

and

$$M_h(p) = \frac{h^{\frac{p}{h^p-1}}}{e\log h^{\frac{p}{h^p-1}}} \quad \textit{for} \quad h = \frac{M}{m} > 1.$$

REMARK 2. $K(m,M,p)$ of Theorem 2 just coincides with the following $K_1(h,p)$;

$$K_1(h,p) = \frac{(p-1)^{p-1}}{p^p}\frac{(h^p-1)^p}{(h-1)(h^p-h)^{p-1}} \text{ for } h = \tfrac{M}{m} > 1 \ . \tag{4.3}$$

In fact, (4.1) of Theorem 2 is shown in [15] and (4.2) in [27] and [3]. But a long and tough proof of (4.2) is given in [27] by using (4.1) and also by applying (F2) of Theorem A which is a characterization of chaotic order. Here we show a direct proof of (4.2) as just only an application of (4.1), that is, applying Uchiyama's method to (4.1) of Theorem 2, we shall show a direct proof of (4.1) $\Longrightarrow$ (4.2) in Theorem 2. We prepare the following Proposition 2 to prove Theorem 2.

PROPOSITION 2. *Let* $K(m,M,p)$ *be the same as in Theorem 2 and* $K_1(h,p)$ *be the same as in* (4.3), $h = \frac{M}{m} > 1$ *and* $h_n = \dfrac{1+\frac{1}{n}\log M}{1+\frac{1}{n}\log m}$ *for natural number n. Then the following* (4.4) *and* (4.5) *hold*

$$\lim_{n\to\infty}\left(\frac{h_n^{np}-1}{h_n^{np}-h_n}\right)^n = h^{\frac{1}{h^p-1}} \tag{4.4}$$

$$\lim_{n\to\infty} K(1+\frac{1}{n}\log m, 1+\frac{1}{n}\log M, np) = \lim_{n\to\infty} K_1(h_n, np) = M_h(p). \tag{4.5}$$

Again we recall the obvious and crucial formula (♣) cited in the proof of Theorem 1

$$\lim_{n\to\infty}(I+\frac{1}{n}\log X)^n = X \text{ for any } X > 0. \tag{♣}$$

Proof of Proposition 2.

Proof of (4.4). Let $f(n) = \left(\frac{h_n^{np}-1}{h_n^{np}-h_n}\right)^n$. Then tending $n \longrightarrow \infty$, then $h_n \longrightarrow 1$ and $h_n^n \longrightarrow h$ since $h_n^n = \frac{(1+\frac{1}{n}\log M)^n}{(1+\frac{1}{n}\log m)^n} \longrightarrow \frac{M}{m} = h$ by (♣), so that $\log f(n) = n\log\left(\frac{h_n^{np}-1}{h_n^{np}-h_n}\right)$ is $\frac{0}{0}$ form as $n \to \infty$, so applying L'Hospital theorem,

$$
\begin{aligned}
\lim_{n\to\infty}\frac{\frac{d}{dn}\left(\log\frac{h_n^{np}-1}{h_n^{np}-h_n}\right)}{\frac{d}{dn}\left(\frac{1}{n}\right)} &= \lim_{n\to\infty}\frac{n^2[(h_n^{np})'(1-h_n)+(h_n)'(h_n^{np}-1)]}{-(h_n^{np}-1)(h_n^{np}-h_n)} \\
&= \frac{-1}{(h^p-1)^2}\lim_{n\to\infty}h_n^{np}\left[p\log h_n^n+\frac{n^2p\log h^{-1}}{(n+\log M)(n+\log m)}\right]\frac{n\log h^{-1}}{(n+\log m)} \\
&\quad+\frac{-1}{(h^p-1)^2}\lim_{n\to\infty}\frac{n^2\log h^{-1}(h_n^{np}-1)}{(n+\log m)^2} \\
&= \frac{-1}{(h^p-1)^2}h^p(p\log h-p\log h)\log h^{-1}+\frac{-1}{(h^p-1)^2}(h^p-1)\log h^{-1} \\
&= \log h^{\frac{1}{(h^p-1)}},
\end{aligned}
$$

so the proof of (4.4) is complete.

Proof of (4.5).

$$
\begin{aligned}
\lim_{n\to\infty}& K\left(1+\frac{1}{n}\log m, 1+\frac{1}{n}\log M, np\right) \\
&= \lim_{n\to\infty}K_1(h_n, np) \qquad \text{by Remark 2} \\
&= \lim_{n\to\infty}\frac{(np-1)^{np-1}}{(np)^{np}(h_n-1)}\frac{(h_n^{np}-1)^{np}}{(h_n^{np}-h_n)^{np-1}} \\
&= \lim_{n\to\infty}\frac{(1+\frac{1}{n}(\frac{-1}{p}))^{np}}{(np-1)(h_n-1)}\left[(h_n^{np}-h_n)\left(\frac{h_n^{np}-1}{h_n^{np}-h_n}\right)^{np}\right] \\
&= \lim_{n\to\infty}\frac{(1+\frac{1}{n}(\frac{-1}{p}))^{np}(1+\frac{1}{n}\log m)}{(np-1)\frac{1}{n}\log\frac{M}{m}}\left[(h_n^{np}-h_n)\left(\frac{h_n^{np}-1}{h_n^{np}-h_n}\right)^{np}\right] \\
&= \frac{1}{pe\log h}(h^p-1)h^{\frac{p}{h^p-1}} \qquad \text{by (♣) and (4.4)} \\
&= \frac{h^{\frac{p}{h^p-1}}}{e\log h^{\frac{p}{h^p-1}}} \\
&= M_h(p),
\end{aligned}
$$

so the proof of (4.5) in Proposition 2 is complete.

Proof of Theorem 2. (4.1) is shown in [15], so we have only to show a direct proof of (4.1) $\Longrightarrow$ (4.2).

Proof of (4.1) $\Longrightarrow$ (4.2) *in Theorem* 2. As $I+\frac{1}{n}\log A \geqslant I+\frac{1}{n}\log B > 0$ for sufficiently large natural number n and

$$I+\frac{1}{n}\log M \geqslant I+\frac{1}{n}\log B \geqslant I+\frac{1}{n}\log m$$

for any natural number n.

Substituting $1+\frac{1}{n}\log M$, $1+\frac{1}{n}\log m$ and np for M, m and p in (4.1), we have

$$K\left(1+\frac{1}{n}\log m, 1+\frac{1}{n}\log M, np\right)\left(I+\frac{1}{n}\log A\right)^{np} \geqslant \left(I+\frac{1}{n}\log B\right)^{np}$$

for $np \geqslant 1$. Tending $n \longrightarrow \infty$, we obtain the following desired result

$$M_h(p)A^p \geqslant B^p$$

by (4.5) of Proposition 2 and (♣), so the proof of (4.1) $\Longrightarrow$ (4.2) is complete.

REMARK 3. We remark that Izumino and Nakamoto [16] obtain an excellent operator inequality on a functional order interpolating (4.1) and (4.2) by applying a convex inequality due to Mond-Pecaric [18].

5. Kantorovich type inequalities under $\log A \geqslant \log B$ can be derived from ones under $A \geqslant B \geqslant 0$, II

By the same way as in §4, we shall show the parallelism between usual order and chaotic order. Seo [20] shows the following nice result as an extension of [14] by using (GF1).

THEOREM 3. (Kantorovich type inequalities, II) *Let* $A > 0$ *and* $M \geqslant B \geqslant m > 0$. *Then the following parallelism holds and* (ii) *can be derived from* (i) *directly.*

(i) $A \geqslant B$ *implies* $\frac{(M^{p-1}+m^{p-1})^2}{4m^{p-1}M^{p-1}}A^p \geqslant B^p$ *for all* $p \geqslant 2$.

(ii) $\log A \geqslant \log B$ *implies* $\frac{(M^p+m^p)^2}{4m^pM^p}A^p \geqslant B^p$ *for all* $p \geqslant 0$.

We remark that more general result than (i) of Theorem 3 is given in [20] and (ii) is shown in [27].

Simple proof of (i) *in Theorem 3.* By Furuta inequality, $A \geqslant B \geqslant 0$ ensures

$$A_1 = (B^{\frac{p-2}{2}}A^pB^{\frac{p-2}{2}})^{\frac{1}{2}} \geqslant B^{p-1} = B_1 \qquad \text{for all } p \geqslant 2.$$

By (4.1) of Theorem 2, we have $K(m^{p-1}, M^{p-1}, 2)A_1^2 \geqslant B_1^2$ since $A_1 \geqslant B_1$ and $M^{p-1} \geqslant B_1 \geqslant m^{p-1} > 0$, that is,

$$K(m^{p-1}, M^{p-1}, 2)B^{\frac{p-2}{2}}A^pB^{\frac{p-2}{2}} \geqslant B^{2p-2},$$

and

$$\frac{(M^{p-1}+m^{p-1})^2}{4m^{p-1}M^{p-1}}A^p \geqslant B^p \text{ for all } p \geqslant 2.$$

Proof of $(i) \Longrightarrow (ii)$ *in Theorem* 3. We may assume that $p > 0$ in (ii) of Theorem 3. As $I + \frac{1}{n}\log A \geqslant I + \frac{1}{n}\log B > 0$ for sufficiently large natural number n and

$$I + \frac{1}{n}\log M \geqslant I + \frac{1}{n}\log B \geqslant I + \frac{1}{n}\log m$$

for any natural number n.

Applying (i) of Theorem 3, we have

$$\frac{((1+\frac{1}{n}\log M)^{np-1} + (1+\frac{1}{n}\log m)^{np-1})^2}{4(1+\frac{1}{n}\log m)^{np-1}(1+\frac{1}{n}\log M)^{np-1}}\left(I+\tfrac{1}{n}\log A\right)^{np}$$
$$\geqslant \left(I+\tfrac{1}{n}\log B\right)^{np} \text{ for } np \geqslant 2.$$

Tending $n \longrightarrow \infty$, we obtain by (♣)

$$\frac{(M^p+m^p)^2}{4m^pM^p}A^p \geqslant B^p \qquad \text{for all } p > 0$$

so the proof of $(i) \Longrightarrow (ii)$ is complete.

6. Kantorovich type inequalities under $\log A \geqslant \log B$ can be derived from ones under $A \geqslant B \geqslant 0$, III

By the same way as in §4 and §5, we shall show the parallelism between usual order and chaotic order. Yamazaki shows the following nice result.

THEOREM 4. (Kantorovich type inequalities, III) *Let $A > 0$ and $M \geqslant B \geqslant m > 0$. Then the following parallelism holds and (ii) can be derived from (i) directly.*

(i) *$A \geqslant B$ implies* $A^p + \dfrac{mM^p - Mm^p}{M-m}\{K(m,M,p)^{\frac{1}{p-1}} - 1\}I \geqslant B^p$ *for all $p \geqslant 1$.*

(ii) *$\log A \geqslant \log B$ implies* $A^p + \dfrac{M^p - m^p}{\log M^p - \log m^p}\log M_h(p) I \geqslant B^p$ *for all $p \geqslant 0$.*

Proof of Theorem 4. (i) is shown in Yamazaki [25] and (ii) in [26], so we have only to a direct proof of (ii) via (i).

Proof of $(i) \Longrightarrow (ii)$ *in Theorem* 4. We may assume that $p > 0$ in (ii) of Theorem 4. As $A_1 = I + \frac{1}{n}\log A \geqslant I + \frac{1}{n}\log B = B_1 > 0$ for sufficiently large natural number n and

$$I + \frac{1}{n}\log M \geqslant B_1 \geqslant I + \frac{1}{n}\log m$$

for any natural number n. Applying (i) of Theorem 4, we have

$$A_1^{np} + \frac{m_1 M_1^{np} - M_1 m_1^{np}}{M_1 - m_1}\{K(m_1, M_1, np)^{\frac{1}{np-1}} - 1\}I \geqslant B_1^{np} \tag{6.1}$$

for all $np > 1$, where $M_1 = 1 + \dfrac{1}{n}\log M$ and $m_1 = 1 + \dfrac{1}{n}\log m$. We have only to prove the following (6.2) since $A_1^{np} = \left(I + \frac{1}{n}\log A\right)^{np} \longrightarrow A^p$ and $B_1^{np} = \left(I + \frac{1}{n}\log B\right)^{np} \longrightarrow B^p$ as $n \longrightarrow \infty$ in (6.1) by (♣).

$$\begin{aligned} &\lim_{n\to\infty} \frac{m_1 M_1^{np} - M_1 m_1^{np}}{M_1 - m_1}\{K(m_1, M_1, np)^{\frac{1}{np-1}} - 1\} \\ &= \frac{M^p - m^p}{\log M - \log m}\lim_{n\to\infty} n(K(m_1, M_1, np)^{\frac{1}{np-1}} - 1) \text{ by } (♣) \\ &= \frac{M^p - m^p}{\log M - \log m}\lim_{n\to\infty} \frac{1}{(p - \frac{1}{n})}(K(m_1, M_1, np)^{\frac{1}{np-1}} - 1)(np - 1) \\ &= \frac{M^p - m^p}{\log M^p - \log m^p}\log M_h(p) \text{ for all } p > 0 \text{ by } (4.5) \end{aligned} \tag{6.2}$$

since $\lim_{n\to\infty}(a_n^{\frac{1}{n}} - 1)n = \log a$ for any $a > 0$ when positive sequence $a_n \to a$. Whence the proof of $(i) \Longrightarrow (ii)$ is complete.

7. Concluding Remark

We can summarize the parallelism between $A \geqslant B$ and $\log A \geqslant \log B$ associated with Furuta type inequalities and Kantorovich type inequalities.

It is interesting to point out that results on chaotic order can be derived from ones on usual order by using Uchiyama's method [23] as follows.

(**F₁**) *Furuta type inequalities under* $\log A \geqslant \log B$ *can be derived from ones under* $A \geqslant B \geqslant 0$.

Let A and B be invertible positive operators. Then the following parallelism holds and (F2″) *can be derived from* (GF1″) *directly.*

(GF1″) $A \geqslant B$ *implies* $A^{1-t+r} \geqslant \{A^{\frac{r}{2}}(A^{\frac{-t}{2}}B^pA^{\frac{-t}{2}})^sA^{\frac{r}{2}}\}^{\frac{1-t+r}{(p-t)s+r}}$ *holds for any* $p \geqslant 1 \geqslant t \geqslant 0$, $r \geqslant t$ *and* $s \geqslant 1$.

(F2″) $\log A \geqslant \log B$ *implies* $A^r \geqslant (A^{\frac{r}{2}}B^pA^{\frac{r}{2}})^{\frac{r}{p+r}}$ *holds for any* $p \geqslant 0$ *and* $r \geqslant 0$.

(**K₁**) *Kantorovich type inequalities under* $\log A \geqslant \log B$ *can be derived from ones under* $A \geqslant B \geqslant 0$, I.

Let $A > 0$ *and* $M \geqslant B \geqslant m > 0$. *Then the following parallelism holds and* (CK1) *can be derived from* (K1) *directly.*

(K1) $A \geqslant B$ *implies* $K(m, M, p)A^p \geqslant B^p$ *for any* $p \geqslant 1$, (CK1) $\log A \geqslant \log B$ *implies* $M_h(p)A^p \geqslant B^p$ *for any* $p > 0$*, where* $K(m,M,p)$ *and* $M_h(p)$ *are defined as follows:*

$$K(m,M,p) = \frac{(p-1)^{p-1}}{p^p} \frac{(M^p - m^p)^p}{(M-m)(mM^p - Mm^p)^{p-1}}$$

and

$$M_h(p) = \frac{h^{\frac{p}{h^p-1}}}{e \log h^{\frac{p}{h^p-1}}} \text{ for } h = \frac{M}{m} > 1.$$

($\mathbf{K_2}$) *Kantorovich type inequalities under* $\log A \geqslant \log B$ *can be derived from ones under* $A \geqslant B \geqslant 0$, II.

Let $A > 0$ *and* $M \geqslant B \geqslant m > 0$. *Then the following parallelism holds and* (CK2) *can be derived from* (K2) *directly.*

(K2) $A \geqslant B$ *implies* $\dfrac{(M^{p-1} + m^{p-1})^2}{4m^{p-1}M^{p-1}} A^p \geqslant B^p$ *for all* $p \geqslant 2$.

(CK2) $\log A \geqslant \log B$ *implies* $\dfrac{(M^p + m^p)^2}{4m^p M^p} A^p \geqslant B^p$ *for all* $p \geqslant 0$.

($\mathbf{K_3}$) *Kantorovich type inequalities under* $\log A \geqslant \log B$ *can be derived from ones under* $A \geqslant B \geqslant 0$, III.

Let $A > 0$ *and* $M \geqslant B \geqslant m > 0$. *Then the following parallelism holds and* (CK3) *can be derived from* (K3) *directly.*

(K3) $A \geqslant B$ *implies* $A^p + \dfrac{mM^p - Mm^p}{M-m}\{K(m,M,p)^{\frac{1}{p-1}} - 1\}I \geqslant B^p$ *for all* $p \geqslant 1$.

(CK3) $\log A \geqslant \log B$ *implies* $A^p + \dfrac{M^p - m^p}{\log M^p - \log m^p} \log M_h(p) I \geqslant B^p$ *for all* $p \geqslant 0$.

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