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# ESTIMATING RANDOM POLYNOMIALS BY MEANS OF METRIC ENTROPY METHODS

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(*communicated by Goran Peskir*)

*Abstract.* The purpose of this paper is to indicate how easy, classical metric entropy methods arising from the theory of stochastic processes, apply to get uniform estimates for random polynomials like the well-known Salem-Zygmund's bound [7]. As an application, we give a criterion for uniform convergence of some random Fourier series.

## 1. Introduction

Let $(p_k)_{k\geqslant 1}$ be a strictly increasing sequence of positive integers greater than $1$; let also $(\theta_k)_{k\geqslant 1}$ be a sequence of reals and consider two sequences of real random variables $\mathscr{X} = \{X_1, X_2, \ldots\}$ and $\mathscr{Y} = \{Y_1, Y_2, \ldots\}$ defined on a common probability space $(\Omega, \mathscr{A}, \mathbf{P})$. Associate to these datas the following sequence of random polynomials

$$(1) \qquad \forall N \geqslant 1, \qquad Z_N(\omega, t) = \sum_{k=1}^{N} \theta_k \left\{X_k(\omega)\cos 2\pi p_k t + Y_k(\omega)\sin 2\pi p_k t\right\}$$

In this paper, we show that the metric entropy methods arising from theory of stochastic processes, are an efficient tool for estimating the total extremums

$$(2) \qquad \forall N \geqslant 1, \qquad Q_N := \sup_{0\leqslant t\leqslant 1} |Z_N(t)|\,.$$

These estimates are known to be very efficient when studying randomly perturbed ergodic sums, we may refer for instance to [PSW], [SW]. We will see that this reduces to apply the metric entropy method in the simplest possible case: the real line provided with the usual distance. And this is also the reason for which we believe that it is likely the most elementary possible approach. We recover as a particular case of a more general estimate, the well-known estimate from Salem-Zygmund [SZ] (Theorem 7). We could condense the arguments on one page; we have chosen on the contrary, to display them completely, in order to make the method more accessible.

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Let us first observe in the particular case when $\mathscr{X}$ and $\mathscr{Y}$ are independent, identically distributed random variables with $\mathbf{E}X_1 = \mathbf{E}Y_1 = 0$ and $\mathbf{E}X_1^2 = \mathbf{E}Y_1^2 = 1$, that

$$\begin{aligned}\mathbf{E}\left(Z_N(s) - Z_N(t)\right)^2 &= \mathbf{E}\Bigg(\sum_{k=1}^{N} \theta_k \Big\{X_k \left[\cos 2\pi p_k t - \cos 2\pi p_k s\right] \\ &\qquad + Y_k \left[\sin 2\pi p_k t - \sin 2\pi p_k s\right]\Big\}\Bigg)^2 \\ &= \sum_{k=1}^{N} \theta_k^2 \left(\left[\cos 2\pi p_k t - \cos 2\pi p_k s\right]^2 + \left[\sin 2\pi p_k t - \sin 2\pi p_k s\right]\right)^2 \\ &= 2\sum_{k=1}^{N} \theta_k^2 \left[1 - \cos 2\pi p_k (t-s)\right] = 4\sum_{k=1}^{N} \theta_k^2 \sin^2 \pi p_k (t-s)\end{aligned}$$

Therefore, if we put

$$\forall s,t \in [0,1], \qquad d_N(s,t) = 2\left(\sum_{k=1}^{N} \theta_k^2 \sin^2 \pi p_k (s-t)\right)^{\frac{1}{2}}, \tag{3}$$

we define like this a pseudo-metric on $[0,1]$, since here $d_N(s,t) = \|Z_N(s) - Z_N(t)\|_2$. This pseudo-metric will play a central role in what follows.

We introduce now an assumption concerning the increments of the process $Z_N(\cdot)$. Consider the Young function $G(t) = \exp(t^2) - 1$, $t$ real, together with the associated Orlicz's space $L^G(\mathbf{P})$, that is, the set of $\mathscr{A}$-measurable functions $f : \Omega \to \mathbf{R}$, such that $\mathbf{E}G(af) < \infty$ for some real $0 < a < \infty$. We recall that $L^G(\mathbf{P})$ is provided with the following norm

$$\forall f \in L^G(\mathbf{P}), \qquad \|f\|_G = \inf\{c > 0 \ : \ \mathbf{E}G(\frac{f}{c}) \leqslant 1\}$$

and that $(L^G(\mathbf{P}), \|\cdot\|_G)$ is a Banach space. The space $L^G(\mathbf{P})$ is often called the gauge space. We will assume

$$\forall 0 \leqslant s,t \leqslant 1 \qquad \begin{cases} \|Z_N(s) - Z_N(t)\|_G \leqslant C d_N(s,t) \\ \\ \|Z_N(s)\|_G \leqslant C\left(\sum_{k=1}^{N} \theta_k^2\right)^{\frac{1}{2}}, \end{cases} \tag{4}$$

where by $C$ we denote here and in what follows a universal constant, which may change its value at each occurence. These assumptions are satisfied in the usual settings: $\mathscr{X}$ and $\mathscr{Y}$ are independent, identically distributed, Rademacher or Gaussian random variables; but also in other settings (see Examples 1-3). We will prove the following result

THEOREM 1. *Under assumption* (4)*, there exists a universal constant $C$ (which is a function of the constant $C$ from* (4)*) such that*

$$\forall N \geqslant 1, \qquad \|Q_N\|_G \leqslant C\left(\log p_N\right)^{\frac{1}{2}}\left(\sum_{k=1}^{N} \theta_k^2\right)^{1/2}. \tag{5}$$

This estimate is optimal. Indeed, assume that $X_n = \xi_{2n}$, $Y_n = \xi_{2n+1}$ where $(\xi_n)_{n\geqslant 0}$ is a sequence of independent Rademacher random variables. Assume also that $\theta_k = 1$, $p_k = k$ $(k \geqslant 1)$. Then, refering for instance to [KS] Proposition 2, p.129, we have

$$\forall N \geqslant 1, \qquad \mathbf{E}\, Q_N \geqslant C\,(N\log N)^{\frac{1}{2}},$$

where $C$ is a universal constant.

In order to motivate the reader, we shall immediately give three classes of examples.

EXAMPLE 1. Assume that $\mathscr{X}$ and $\mathscr{Y}$ are two stationary centered gaussian sequences, with finite decoupling coefficient, that is:

$$p(\mathscr{X}) = \sum_{k=1}^{\infty}\left|\frac{\mathbf{E}X_1X_k}{\mathbf{E}(X_1)^2}\right| < \infty, \qquad p(\mathscr{Y}) = \sum_{k=1}^{\infty}\left|\frac{\mathbf{E}Y_1Y_k}{\mathbf{E}(Y_1)^2}\right| < \infty. \tag{6}$$

Then, assumption (4) is satisfied. More precisely, for any $0 \leqslant s,t \leqslant 1$

$$\begin{cases} \|Z_N(s) - Z_N(t)\|_G \leqslant 18\sqrt{2}\max\left(p(\mathscr{X}),p(\mathscr{Y})\right)^{\frac{1}{2}}\left(\sum_{k=1}^N \theta_k^2 \sin^2 \pi p_k(s-t)\right)^{\frac{1}{2}} \\ \|Z_N(s)\|_G \leqslant 9\sqrt{2}\max\left(p(\mathscr{X}),p(\mathscr{Y})\right)^{\frac{1}{2}}\left(\sum_{k=1}^N \theta_k^2\right)^{\frac{1}{2}} \end{cases} \tag{7}$$

So, Theorem 1 does apply in that case. Note that the decoupling assumption is trivially satisfied when both $\mathscr{X}$ and $\mathscr{Y}$ consist of independent $N(0,1)$ distributed random variables. Observe also that no assumption on the correlation between $\mathscr{X}$ and $\mathscr{Y}$ is required, and consequently the elaborated Gaussian spectral approach proposed in [F] is unsufficient here, since $Z_N$ is not necessarily Gaussian. Finally, recall that the Ornstein-Uhlenbeck process $U_k = W(e^k)e^{-k/2}$ $k = 1,2,\ldots$ is the typical example of stationary Gaussian sequence with finite decoupling coefficient. We show (7) and will invoke the following lemma due to Klein-Landau-Shucker.

LEMMA 2. ([KLS], Theorem 1) *Let $T = (T_1,T_2,\ldots)$ be a stationary, centered Gaussian sequence with finite decoupling coefficient $p(T)$. Let $(f_k, k=1,2,\ldots)$ be a sequence of complex-valued Borel-measurable functions. Then, for each finite subset $J$ of* $\mathbf{N}$,

$$\left|\mathbf{E}\prod_{j\in J} f_j(T_j)\right| \leqslant \prod_{j\in J}\|f_j(T_1)\|_{p(T)}. \tag{8}$$

Let $\lambda$ be some fixed real. By means of Cauchy-Schwarz's inequality

$$\begin{aligned}\mathbf{E}e^{\lambda(Z_n(s)-Z_N(t))} &= \mathbf{E}e^{\lambda\sum_{k=1}^N \theta_k\{X_k(\cos 2\pi p_k s - \cos 2\pi p_k t) + Y_k(\sin 2\pi p_k s - \sin 2\pi p_k t)\}} \\ &\leqslant \left(\mathbf{E}e^{2\lambda\sum_{k=1}^N \theta_k X_k(\cos 2\pi p_k s - \cos 2\pi p_k t)}\mathbf{E}e^{2\lambda\sum_{k=1}^N \theta_k Y_k(\sin 2\pi p_k s - \sin 2\pi p_k t)}\right)^{\frac{1}{2}}\end{aligned}$$

Put $f_k^{\mathscr{X}}(x) = e^{2\lambda\theta_k x(\cos 2\pi p_k s - \cos 2\pi p_k t)}$, $f_k^{\mathscr{Y}}(x) = e^{2\lambda\theta_k x(\sin 2\pi p_k s - \sin 2\pi p_k t)}$, $k = 1,\ldots,N$ and apply lemma 2. We obtain, since $\mathbf{E}e^{\lambda N(0,1)} = e^{\lambda^2/2}$,

$$\mathbf{E}e^{2\lambda\sum_{k=1}^N \theta_k X_k(\cos 2\pi p_k s - \cos 2\pi p_k t)} \leqslant e^{2\lambda^2 p(\mathscr{X})\sum_{k=1}^N \theta_k^2(\cos 2\pi p_k s - \cos 2\pi p_k t)^2}$$
$$\mathbf{E}e^{2\lambda\sum_{k=1}^N \theta_k Y_k(\sin 2\pi p_k s - \sin 2\pi p_k t)} \leqslant e^{2\lambda^2 p(\mathscr{Y})\sum_{k=1}^N \theta_k^2(\sin 2\pi p_k s - \sin 2\pi p_k t)^2}$$

Hence

$$\begin{aligned}\mathbf{E}e^{\lambda(Z_n(s)-Z_N(t))} &\leqslant e^{2\lambda^2\max(p(\mathscr{X}),p(\mathscr{Y}))\sum_{k=1}^N \theta_k^2\{(\cos 2\pi p_k s-\cos 2\pi p_k t)^2+(\sin 2\pi p_k s - \sin 2\pi p_k t)^2\}}\\ &\leqslant e^{8\lambda^2\max(p(\mathscr{X}),p(\mathscr{Y}))\sum_{k=1}^N \theta_k^2\sin^2\pi p_k(s-t)}\end{aligned}$$

Now, we shall use the fact that if $U$ is a real random variable such that: $\mathbf{E}e^{\lambda U} \leqslant e^{\lambda^2 C^2}$ ($\forall\lambda\in\mathbf{R}$), then $||U||_G \leqslant 9C$. Thus, it follows from the previous estimates that

$$\|Z_n(s) - Z_N(t)\|_G \leqslant 18\sqrt{2}\max\left(p(\mathscr{X}),p(\mathscr{Y})\right)^{\frac12}\left(\sum_{k=1}^N \theta_k^2\sin^2\pi p_k(s-t)\right)^{\frac12}$$

Hence the first inequality in (7). The second one is deduced by a similar reasoning.

EXAMPLE 2. Assume that both $\mathscr{X}$ and $\mathscr{Y}$ are sequences of independent, centered real random variables, and that there exists a real constant $M$ such that

$$\forall k \geqslant 1, \qquad |X_k| \leqslant M, \qquad |Y_k| \leqslant M.$$

Then, assumption (4) is satisfied. More precisely, for any $0 \leqslant s,t \leqslant 1$

$$(9)\quad \begin{cases} \|Z_N(s) - Z_N(t)\|_G \leqslant 4M\left(\sum_{k=1}^N \theta_k^2\sin^2\pi p_k(s-t)\right)^{\frac12} \\ \|Z_N(s)\|_G \leqslant 4M\left(\sum_{k=1}^N \theta_k^2\right)^{\frac12}\end{cases}$$

We don't prove that result, which is rather a direct consequence of Theorem 3.5.1 p.77 of [G]. Our Theorem 1 thus applies in that case as well.

EXAMPLE 3. Let $\mathscr{A}_0 \subset \mathscr{A}_1 \subset \ldots \mathscr{A}$ be an increasing filtration of $\mathscr{A}$ ($\mathscr{A} = \bigvee_{i=0}^\infty \mathscr{A}_i$), and assume that $\mathscr{X}$ is a sequence of martingale differences adapted to that filtration, with

$$\forall k \geqslant 1, \qquad ||X_k||_\infty \leqslant 1.$$

Assume that $\mathscr{Y} \equiv 0$ (for instance!). Then assumption (4) is satisfied. Indeed, $Z_n(t) = \sum_{k=1}^N d_k^{(t)}$ where $d_k^{(t)} = \theta_k X_k \cos 2\pi p_k t$. Thus, $Z_n(t)$ is a sum of martingales differences satisfying: $||d_k^{(t)}||_\infty \leqslant \theta_k$. But, we know from [LT], Lemma 1.5 p. 31 for instance, that

$$(10)\qquad \forall v \geqslant 0, \qquad \mathbf{P}\left\{\ \left|\sum_{k=1}^N d_k^{(t)}\right| > v\ \right\} \leqslant 2\exp\left(-\frac{v^2}{2\sum_{k=1}^N ||d_k^{(t)}||_\infty^2}\right).$$

This, of course implies that $\|Z_N(s)\|_G \leqslant C\left(\sum_{k=1}^N \theta_k^2\right)^{\frac{1}{2}}$ for some universal constant $C$. Similarly, we have

$$\|Z_N(s) - Z_N(t)\|_G \leqslant C\left(\sum_{k=1}^N \theta_k^2 (\cos 2\pi p_k s - \cos 2\pi p_k t)^2\right)^{\frac{1}{2}} \leqslant C d_N(s,t). \tag{11}$$

Consequently, Theorem 1 applies in that case as well.

## 2. Proof of Theorem 1

The key point of the proof is contained in the following elementary observation: the pseudo-metric $d_N(.,.)$ is locally comparable to the usual distance. Indeed, since $|\sin x| \leqslant (|x| \wedge 1)$, we thus have

$$d_N^2(s,t) \leqslant 4\sum_{k=1}^N \theta_k^2 \left((\pi p_k|s-t|)^2 \wedge 1\right) \leqslant 4\pi^2|s-t|^2 \sum_{k=1}^N \theta_k^2 \left(p_k^2 \wedge \frac{1}{\pi^2|s-t|^2}\right); \tag{12}$$

We thus deduce that if $\pi|s-t| \leqslant 1/p_N$, then $\left(p_k^2 \wedge \frac{1}{\pi^2|s-t|^2}\right) = p_k^2$, $k = 1,\dots,N$. And consequently $d_N(s,t) \leqslant 2\pi|s-t|\left(\sum_{k=1}^N \theta_k^2 p_k^2\right)^{1/2}$.

We divide the interval $[0,1[$ in sub-intervals:

$$I_{N,j} = \left[\frac{j-1}{4p_N}, \frac{j}{4p_N}\right[, \qquad j = 1,2,\dots,4p_N \tag{13}$$

Since $s,t \in I_{N,j} \quad \Rightarrow \quad |s-t| \leqslant \frac{1}{4p_N} \leqslant \frac{1}{\pi p_N}$, it follows from the previous estimate

$$\forall j = 1,2,\dots,4p_N, \quad \forall s,t \in I_{N,j}, \qquad d_N(s,t) \leqslant 2\pi|s-t|\left(\sum_{k=1}^N \theta_k^2 p_k^2\right)^{1/2}. \tag{14}$$

Introduce now the auxiliary process

$$\forall j = 1,2,\dots,4p_N,\ \forall t \in I_{N,j}, \qquad \mathscr{Y}_N(t) = \frac{\left[Z_N(t) - Z_N(\frac{j-1}{4p_N})\right]}{2\pi\left(\sum_{k=1}^N \theta_k^2 p_k^2\right)^{1/2}}. \tag{15}$$

Then, we bound $Q_N$ relatively to the partition of $[0,1[$ as follows:

$$Q_N \leqslant \sup_{j=1,2,\dots,4p_N}\left|Z_N(\frac{j-1}{4p_N})\right| + 2\pi\left(\sum_{k=1}^N \theta_k^2 p_k^2\right)^{1/2} \sup_{j=1,2,\dots,4p_N}\sup_{t\in I_{N,j}} |\mathscr{Y}_N(t)|. \tag{16}$$

We are now in an easy setting, since we have to estimate the local extremums $\sup_{t\in I_{N,j}} |\mathscr{Y}_N(t)|$ of a stochastic process of which the increments are locally bounded by the usual distance. Indeed, from (14): for any $s,t \in I_{N,j}$, $||\mathscr{Y}_N(s) - \mathscr{Y}_N(t)||_G \leqslant C|s-t|$, $j = 1,2,\dots,4p_N$.

In order to estimate $Q_N$, we will need two simple tools: the first one is a classical inequality (see for instance [GPW], inequality (3.5) p.62):

$$(17)\qquad \forall n \geqslant 2,\ \forall f_1,\dots,f_n \qquad \|\sup_{1\leqslant j\leqslant n} |f_j|\,\|_G \leqslant ([2/\log 2]\log n)^{\frac{1}{2}} \sup_{1\leqslant j\leqslant n} \|\,|f_j|\,\|_G.$$

Observe at once from (4) and (17) that

$$(18)\qquad \begin{aligned} \|Q_N\|_G &\leqslant ([2/\log 2]\log 4p_N)^{\frac{1}{2}} \Bigg\{ \sup_{j=1,\dots,4p_N} \left\| |Z_N(\frac{j-1}{4p_N})| \right\|_G \\ &\quad + 2\pi \left( \sum_{k=1}^{N} \theta_k^2 p_k^2 \right)^{1/2} . \sup_{j=1,\dots,4p_N} \left\| \sup_{t\in I_{N,j}} |\mathscr{Y}_N(t)| \right\|_G \Bigg\} \\ &\leqslant ([2/\log 2]\log 4p_N)^{\frac{1}{2}} \Bigg\{ C \left( \sum_{k=1}^{N} \theta_k^2 \right)^{1/2} \\ &\quad + 2\pi \left( \sum_{k=1}^{N} \theta_k^2 p_k^2 \right)^{1/2} . \sup_{j=1,\dots,4p_N} \left\| \sup_{t\in I_{N,j}} |\mathscr{Y}_N(t)| \right\|_G \Bigg\} \end{aligned}$$

The second tool is nothing else but a reformulation of the well-known integral criterion of Dudley in theory of processes, that we slightly adapt to our purpose, in order to estimate $\left\|\sup_{t\in I_{N,j}} |\mathscr{Y}_N(t)|\right\|_G$. We give an elementary proof of it in Appendix, for the sake of completeness. Introduce a notation: let $(E,d)$ be a set provided with a pseudo-metric and a positive real $u$; we note by $N(E,d,u)$ the smallest covering number (possibly infinite) of $E$ by open $d$-balls of radius $u$.

THEOREM 4. ([D], Theorem 2.1) *Let $E$ be a countable set, provided with a pseudo-metric $d$ and let $X = \{X(\omega,t), \omega\in\Omega,\ t\in E\}$ be a stochastic process indexed on $E$, with basic probability space $(\Omega,\mathscr{A},\mathbf{P})$, and satisfying the following increment's condition*

$$(19)\qquad \forall s,t\in E, \qquad \|X_s - X_t\|_G \leqslant d(s,t).$$

*Assume that the following integral*

$$(20)\qquad \mathbf{I}(E,d) = \int_0^{diam(E,d)} (\log N(E,d,u))^{\frac{1}{2}}\, du$$

*is convergent. Then, there exists a universal constant $C$ such that*

$$(21)\qquad \left\| \sup_{s,t\in E} (X_s - X_t) \right\|_G \leqslant C\,\mathbf{I}(E,d).$$

Now, estimate $\left\|\sup_{t\in I_{N,j}} |\mathscr{Y}_N(t)|\right\|_G$. By taking account of (14) and the previous theorem, and since $diam(I_{N,j},|.|) = 1/4p_N$, we must first estimate $N(I_{N,j},|.|,u)$ for $0 < u \leqslant 1/4p_N$, which is obvious:

$$N(I_{N,j},|.|,u) \leqslant 1 + \left[\frac{1/4p_N}{2u}\right] \leqslant 1 + \frac{1/4p_N}{2u} \leqslant \frac{1}{2up_N}.$$

Thus

$$(22) \qquad \mathbf{I}(I_{N,j},|.|) \leqslant \int_0^{\frac{1}{4p_N}} \left(\log \frac{2}{4up_N}\right)^{\frac{1}{2}} du \overset{\left(u=\frac{v}{4p_N}\right)}{=} \frac{1}{4p_N}\int_0^1 \left(\log\frac{2}{v}\right)^{\frac{1}{2}} dv \leqslant \frac{C}{p_N}.$$

It follows from (4) and Theorem 4, and from the fact that $\mathscr{Y}(\frac{j-1}{4p_N}) = 0$, that for any countable subset $E$ of $I_{N,j}$

$$(23) \qquad \left\|\sup_{t\in E} |\mathscr{Y}_N(t)|\right\|_G \leqslant \left\|\sup_{s,t\in E} |\mathscr{Y}_N(s)-\mathscr{Y}_N(t)|\right\|_G \leqslant \frac{C}{p_N}.$$

We shall now make use of the following useful observation: the $\omega$-trajectories $t \longrightarrow Z_N(t,\omega)$ are continuous for each $\omega \in \Omega$, and consequently, those of the auxiliary process $\mathscr{Y}_N$ are continuous too. By specifying estimate (23) for a countable dense subset of $I_{N,j}$, we have in fact shown

$$\left\|\sup_{t\in I_{N,j}} |\mathscr{Y}_N(t)|\right\|_G \leqslant \frac{C}{p_N}.$$

By putting this estimate in (18), we thus obtain

$$\begin{aligned} \|Q_N\|_G &\leqslant C\,(\log 4p_N)^{\frac{1}{2}} \left\{ \left(\sum_{k=1}^N \theta_k^2\right)^{1/2} + \frac{1}{p_N}\left(\sum_{k=1}^N \theta_k^2 p_k^2\right)^{1/2} \right\} \\ &\leqslant C\,(\log p_N)^{\frac{1}{2}} \left(\sum_{k=1}^N \theta_k^2\right)^{1/2} \end{aligned}$$

We have therefore proved Theorem 1. □

## 3. Applications

In this section, we give four applications of Theorem 1, the first one establishes a precise uniform estimate of complex random polynomials of the form

$$\sum_{k=1}^N U_k\theta_k \exp 2i\pi p_k t \qquad\qquad N = 1,2,\ldots$$

where $\mathscr{U} = (U_k)_{k=1}^\infty$ is a sequence of weakly dependent random variables; the second one provides a global uniform estimate of the sequence formed by the differences of these polynomials. In that case, we will assume that the sequence $U$ is Gaussian. The third application provides a similar global uniform estimate for sequences of independent symmetric random variables. A fourth application to a variant of the initial problem is given in Theorem 9. We first establish:

COROLLARY 5.

*(a) Let $\mathscr{U} = (U_k)_{k=1}^{\infty}$ be a sequence of independent, centered real random variables. We assume that there exists a real $M < \infty$ such that: $|U_k| \leqslant M$ a.s. for any $k \geqslant 1$. Then*

$$\Big\| \sup_{0\leqslant t\leqslant 1}\Big|\sum_{k=1}^{N} U_k\theta_k e^{2i\pi p_k t}\Big| \Big\|_G \leqslant CM\Big(\log p_N \sum_{k=1}^{N}\theta_k^2\Big)^{\frac12} \tag{24a}$$

*where $C$ is a universal constant.*

*(b) Let $\mathscr{V} = (V_k)_{k=1}^{\infty}$ be a centered, stationary Gaussian sequence with finite decoupling coefficient $p(\mathscr{V})$ (see Example 2). Then*

$$\Big\| \sup_{0\leqslant t\leqslant 1}\Big|\sum_{k=1}^{N} V_k\theta_k e^{2i\pi p_k t}\Big| \Big\|_G \leqslant C\sqrt{p(\mathscr{V})}\Big(\log p_N \sum_{k=1}^{N}\theta_k^2\Big)^{\frac12} \tag{24b}$$

*where $C$ is a universal constant.*

*Proof.* For establishing (24a), we apply Theorem 1 to $\mathscr{X} = \mathscr{U}$, $\mathscr{Y} = 0$, next to $\mathscr{X} = 0$, $\mathscr{Y} = \mathscr{U}$. This provides the desired estimate for both the imaginary and real part. Hence, the result by putting together these estimates. We operate similarly for establishing (24b), by applying Theorem 1 to $\mathscr{X} = \mathscr{V}$, $\mathscr{Y} = 0$, next to $\mathscr{X} = 0$, $\mathscr{Y} = \mathscr{V}$. □

Estimate (24b) can be considerably strenghtened. This is the object of the next Corollary.

COROLLARY 6. *Let $\mathscr{V} = (V_k)_{k=1}^{\infty}$ be a centered, stationary Gaussian sequence with finite decoupling coefficient $p(\mathscr{V})$. Then,*

$$\Bigg\| \sup_{N<M}\sup_{0\leqslant t\leqslant 1}\frac{\big|\sum_{k=N+1}^{M} V_k\theta_k e^{2i\pi p_k t}\big|}{\big(\log p_M \sum_{k=N+1}^{M}\theta_k^2\big)^{\frac12}} \Bigg\|_G \leqslant C\sqrt{p(\mathscr{V})}, \tag{25}$$

*where $C$ is a universal constant.*

*Proof.* It is enough to establish a similar estimate for each of the imaginary and real parts. Put, to that effect

$$\begin{cases} L_{N,M}^{(\cos)} = \sup_{0\leqslant t\leqslant 1}\dfrac{\big|\sum_{k=N+1}^{M} V_k\theta_k\cos 2\pi p_k t\big|}{\big(\log p_M \sum_{k=N+1}^{M}\theta_k^2\big)^{\frac12}} & (N<M),\\[2ex] L_{N,M}^{(\sin)} = \sup_{0\leqslant t\leqslant 1}\dfrac{\big|\sum_{k=N+1}^{M} V_k\theta_k\sin 2\pi p_k t\big|}{\big(\log p_M \sum_{k=N+1}^{M}\theta_k^2\big)^{\frac12}} & (N<M),\\[2ex] L^{(\cos)} = \sup_{N<M} L_{N,M}^{(\cos)} \qquad L^{(\sin)} = \sup_{N<M} L_{N,M}^{(\sin)} \end{cases} \tag{26}$$

It suffices to show that

$$\mathbf{E}L^{(\cos)} \leqslant C\sqrt{p(\mathscr{V})} \qquad \mathbf{E}L^{(\sin)} \leqslant C\sqrt{p(\mathscr{V})}. \tag{27}$$

It will follow from that, by taking into account the strong integrability properties of Gaussian semi-norms (see for instance [LT], inequality 3.5 p.59),

$$\left\|L^{(\cos)}\right\|_G \leqslant C\sqrt{p(\mathscr{V})} \qquad \left\|L^{(\sin)}\right\|_G \leqslant C\sqrt{p(\mathscr{V})}.$$

Hence, the wished result follows by combining together these estimates. We prove now (27). By means of the following inequality, which is easily derived from a simple form of Borell-Sudakov-Tsirelson inequality (see for instance [LT] Lemma 3.1, p.57):

*if $G_1,\dots,G_N$ are Gaussian random vectors with values in a separable Banach space $(B,||\cdot||)$, then*

$$\mathbf{E}\sup_{1\leqslant k\leqslant N}||G_k|| \leqslant C\left\{\sup_{1\leqslant k\leqslant N}\mathbf{E}||G_k|| + \mathbf{E}\sup_{1\leqslant k\leqslant N}\sigma_k|g_k|\right\}$$

*where $\sigma_k = \sup_{f\in B^*,\ ||f||\leqslant 1}\left(\mathbf{E}<f,G_k>^2\right)^{\frac12}$, $k=1,\dots,N$, $(g_k)_{k=1}^N$ is a sequence of independent $N(0,1)$ distributed random variables, and $C$ is a universal constant,*

$$\mathbf{E}L^{(\cos)} \leqslant C\left\{\sup_{N<M}\mathbf{E}L^{(\cos)}_{N,M} + \mathbf{E}\sup_{N<M}|\lambda_{N,M}|\sigma_{N,M}\right\}$$

where

$$\sigma_{N,M} = \sup_{0\leqslant t\leqslant 1}\left\|\frac{\left|\sum_{k=N+1}^M V_k\theta_k\cos 2\pi p_k t\right|}{\left(\log p_M\sum_{k=N+1}^M\theta_k^2\right)^{\frac12}}\right\|_2,$$

and $(\lambda_{N,M})_{N<M}$ is a sequence of independent $N(0,1)$ distributed random variables. By a computation similar to the one made in example 1, we also obtain

$$\left\|\sum_{k=N+1}^M V_k\theta_k\cos 2\pi p_k t\right\|_G \leqslant C\sqrt{p(\mathscr{V})}\Big(\sum_{k=N+1}^M\theta_k^2\cos^2(2\pi p_k t)\Big)^{\frac12} \leqslant C\sqrt{p(\mathscr{V})}\Big(\sum_{k=N+1}^M\theta_k^2\Big)^{\frac12}.$$

Hence, $\left\|\sum_{k=N+1}^M V_k\theta_k\cos 2\pi p_k t\right\|_2 \leqslant C\sqrt{p(\mathscr{V})}\left(\sum_{k=N+1}^M\theta_k^2\right)^{\frac12}$, and therefore

$$\sigma_{N,M} \leqslant C\sqrt{p(\mathscr{V})}\left(\log p_M\right)^{-\frac12}$$

By Theorem 1, we already know that $\sup_{N<M}\mathbf{E}L^{(\cos)}_{N,M} \leqslant C\sqrt{p(\mathscr{V})}$. Consider now the other part. For that, we re-index the sequence as follows: put $m_1 = 1$, $m_k = 1+\sum_{j=2}^k(j-1)$ $(k\geqslant 2)$. Next, put for any $M\geqslant 1$ and any $l\in[m_M,m_{M+1}[$, $g_l := \lambda_{l-m_M,M}$, $s_l := (\log p_M)^{\frac12}$. Observe that $s_l \geqslant (\log M)^{\frac12} \geqslant C(\log l)^{\frac12}$. Thus

$$\begin{aligned}\mathbf{E}\sup_{N<M}|\lambda_{N,M}|\sigma_{N,M} &\leqslant C\mathbf{E}\sup_{l\geqslant 1}\frac{|g_l|}{s_l}\\ &\leqslant C\sup_{l\geqslant 1}\sqrt{\frac{\log l}{s_l}}\mathbf{E}\sup_{l\geqslant 1}\frac{|g_l|}{\sqrt{\log l}} \leqslant C<\infty.\end{aligned}$$

Hence $\mathbf{E}L^{(\cos)} \leqslant C\sqrt{p(\mathscr{V})}$. By arguing identically, we establish an estimate of the same order for $\mathbf{E}L^{(\sin)}$. Hence (27). The Corollary is thus proved. □

We will now prove the following result

THEOREM 7. *Let $\mathscr{W} = (W_k)_{k=1}^{\infty}$ be a sequence of independent, symmetric real random variables. Then,*

$$\left\| \sup_{N<M} \sup_{0\leqslant t\leqslant 1} \frac{\left|\sum_{k=N+1}^{M} W_k e^{2i\pi p_k t}\right|}{\left(\log p_M \sum_{k=N+1}^{M} W_k^2\right)^{\frac{1}{2}}} \right\|_G \leqslant C, \tag{28}$$

*where $C$ is a universal constant.*

Observe that, by means of Cauchy-Schwarz's inequality

$$\frac{\left|\sum_{k=N+1}^{M} W_k e^{2i\pi p_k t}\right|}{\left(\log p_M \sum_{k=N+1}^{M} W_k^2\right)^{\frac{1}{2}}} \leqslant \frac{\sum_{k=N+1}^{M} |W_k|}{\left(\log p_M \sum_{k=N+1}^{M} W_k^2\right)^{\frac{1}{2}}} \leqslant \frac{(M-N)^{\frac{1}{2}}}{(\log p_M)^{\frac{1}{2}}}.$$

In particular, if $(p_m)_{m\geqslant 1}$ is $\lambda$-lacunary ($\lambda > 1$), that is $p_{m+1} \geqslant \lambda p_m$ for all $m \geqslant 1$, then

$$\sup_{N<M} \sup_{0\leqslant t\leqslant 1} \frac{\left|\sum_{k=N+1}^{M} W_k e^{2i\pi p_k t}\right|}{\left(\log p_M \sum_{k=N+1}^{M} W_k^2\right)^{\frac{1}{2}}} \leqslant C,$$

where $C$ is a constant depending on $\lambda$ only. So that, Theorem 7 is only interesting when $(p_m)_{m\geqslant 1}$ grows at most geometrically.

*Proof.* Since the sequence $\mathscr{W}$ is symmetric, it has the same distribution as the sequence $\mathscr{W}' = (\varepsilon_k W_k)_{k=1}^{\infty}$, where $\varepsilon = (\varepsilon_k)_{k=1}^{\infty}$ is a sequence of independent Rademacher random variables, which is also independent from the sequence $\mathscr{W}$. Let $P$ be some fixed nonnegative integer. Let also $g = (g_k)_{k=1}^{\infty}$ be a sequence of independent $N(0,1)$ distributed random variables, also independent from the sequence $\mathscr{W}$. Since $|g| = (|g_k|)_{k=1}^{\infty}$ and $sign(g) = (sign(g_k))_{k=1}^{\infty}$ are independent sequences (or more briefly, by means of the contraction principle)

$$\left\| \sup_{N<M\leqslant P} \sup_{0\leqslant t\leqslant 1} \frac{\left|\sum_{k=N+1}^{M} \varepsilon_k W_k e^{2i\pi p_k t}\right|}{\left(\log p_M \sum_{k=N+1}^{M} W_k^2\right)^{\frac{1}{2}}} \right\|_G$$

$$\leqslant \sqrt{\frac{\pi}{2}} \left\| \sup_{N<M\leqslant P} \sup_{0\leqslant t\leqslant 1} \frac{\left|\sum_{k=N+1}^{M} g_k W_k e^{2i\pi p_k t}\right|}{\left(\log p_M \sum_{k=N+1}^{M} W_k^2\right)^{\frac{1}{2}}} \right\|_G$$

By applying now Corollary 6 to the sequence $g$ conditionnally to $\mathscr{W}$, next integrating with respect to the law of $\mathscr{W}$ and finally letting $P$ tend to infinity, we obtain the announced result. □

It is now easy to deduce from Theorem 7 (except for the constant 2 in (29)), the well-known estimate of Salem-Zygmund (see [K] or [SZ]) that we recall now for the convenience of the reader

THEOREM 8. (Salem-Zygmund's estimate) *Let $(n_k)_{k\in\mathbf{N}}$, $(p_k)_{k\in\mathbf{N}}$ be two increasing sequences of integers and a sequence $(a_n)_{n\in\mathbf{N}}$ of reals. Let also $\epsilon=(\epsilon_k)_{k\in\mathbf{N}}$ be a sequence of independent Rademacher random variables defined on a probability space that we note $(\Omega,\mathscr{B},\mathbf{P})$. Then,*

$$\mathbf{P}\left\{\limsup_{k\to+\infty}\sup_{0\leqslant t\leqslant 1}\frac{\max_{n_k<n\leqslant n_{k+1}}|\sum_{j=n_k+1}^{n}a_j\epsilon_j e^{2i\pi p_j t}|}{\left(\log p_{n_{k+1}}\sum_{j=n_k+1}^{n_{k+1}}a_j^2\right)^{\frac{1}{2}}}\leqslant 2\right\}=1. \tag{29}$$

This is lemma 4.4.1 obtained from theorem 4.3.1 in [SZ]. It is worth observing here that the lines of proof of Theorem 4.3.1 in [SZ] already contain the following useful estimate valid for all positive integers $N$,

$$\left\|\sup_x\left|\sum_{n=1}^{N}a_n\epsilon_n\exp(2i\pi nx)\right|\right\|_1\leqslant C\sqrt{\left(\sum_{n=1}^{N}a_n^2\right)\log N},$$

where the universal constant $C$ can be estimated. Their proof is based on Bernstein's inequality for polynomials and exponential integrability properties of Rademacher sums.

The method we have used, as well as Theorem 7, also allows to study the following variant of the initial problem. Let us consider a sequence $\mathscr{P}$:

$$P_1,P_2,\dots$$

of $\mathbf{Z}$-valued, independent random variables defined on a probability space $(\Omega,\mathscr{B},\mathbf{P})$, and satisfying

$$\mathbf{P}\{p_i+P_i\geqslant 0\}=1\qquad i=1,2,\dots \tag{30}$$

Introduce the following sequence of random polynomials

$$U_N(t)=\sum_{k=1}^{N}e^{2i\pi t(p_k+P_k)}-\mathbf{E}e^{2i\pi t(p_k+P_k)},\qquad N=1,2,\dots \tag{31}$$

By means of a classical symmetrization argument, the study of the extremums of these polynomials may be reduced to the study of the following "symmetrized" sequence

$$V_N(t)=\sum_{k=1}^{N}\varepsilon_k e^{2i\pi t(p_k+P_k)},\qquad N=1,2,\dots,$$

where $\varepsilon_1,\varepsilon_2,\dots$ is a Rademacher sequence defined on another probability space $(\Omega_\varepsilon,\mathscr{B}_\varepsilon,\mathbf{P}_\varepsilon)$. We denote by $\mathbf{E}_\varepsilon$ the corresponding symbol of integration. But, *conditionally* to the sequence $(P_k)_k$ these polynomials are exactly of the same type as those examined in the previous Sections. And so, our method may be applied to the study of their extremal properties. For this, we will assume that the following condition in which $\Phi:\mathbf{N}\to\mathbf{N}$ is some increasing map, is satisfied

$$C(\mathscr{P},\Phi)=\mathbf{E}\sup_{M\geqslant 1}\frac{\left[\log_+(p_M+P_M)\right]^{\frac{1}{2}}}{\Phi(M)}<\infty. \tag{32}$$

Then, we have the following result

THEOREM 9. *There exists a universal constant $C$ such that*

$$\mathbf{E}\sup_{N<M}\sup_{0\leqslant t\leqslant 1}\frac{|U_M(t)-U_N(t)|}{(M-N)^{\frac{1}{2}}\Phi(M)}\leqslant C.C(\mathscr{P},\Phi). \tag{33}$$

*Proof.* Consider the "symmetrized" sequence $(V_N)_{N\geqslant 1}$. By virtue of Theorem 7, one has

$$\left\|\sup_{N<M}\sup_{0\leqslant t\leqslant 1}\frac{|V_M(t)-V_N(t)|}{(M-N)^{\frac{1}{2}}\log_+(p_M+P_M)}\right\|_{G,\mathbf{P}_\varepsilon}\leqslant C,$$

where $C$ is a universal constant. Hence $\mathbf{E}_\varepsilon\sup_{N<M}\sup_{0\leqslant t\leqslant 1}\frac{|V_M(t)-V_N(t)|}{(M-N)^{\frac{1}{2}}\log_+(p_M+P_M)}\leqslant C$. And thus,

$$\begin{aligned}
&\mathbf{E}\,\mathbf{E}_\varepsilon\sup_{N<M}\sup_{0\leqslant t\leqslant 1}\frac{|V_M(t)-V_N(t)|}{(M-N)^{\frac{1}{2}}\Phi(M)}\\
&\quad\leqslant \mathbf{E}\,\mathbf{E}_\varepsilon\sup_{N<M}\sup_{0\leqslant t\leqslant 1}\frac{|V_M(t)-V_N(t)|}{(M-N)^{\frac{1}{2}}\log_+(p_M+P_M)}.\sup_M\frac{\left[\log_+(p_M+P_M)\right]^{\frac{1}{2}}}{\Phi(M)}\\
&\quad\leqslant C\mathbf{E}\sup_M\frac{\left[\log_+(p_M+P_M)\right]^{\frac{1}{2}}}{\Phi(M)}\leqslant C\,C(\mathscr{P},\Phi).
\end{aligned}$$

It remains to observe in order to conclude, that by means of usual symmetrization procedure

$$\begin{aligned}
&\mathbf{E}\,\mathbf{E}_\varepsilon\sup_{N<M}\sup_{0\leqslant t\leqslant 1}\frac{|U_M(t)-U_N(t)|}{(M-N)^{\frac{1}{2}}\Phi(M)}\\
&\qquad=\mathbf{E}\sup_{N<M}\sup_{0\leqslant t\leqslant 1}\frac{\left|\sum_{k=N+1}^M e^{2i\pi t(p_k+P_k)}-\mathbf{E}'e^{2i\pi t(p_k+P_k')}\right|}{(M-N)^{\frac{1}{2}}\Phi(M)}\\
&\qquad\leqslant\mathbf{E}\mathbf{E}'\sup_{N<M}\sup_{0\leqslant t\leqslant 1}\frac{\left|\sum_{k=N+1}^M e^{2i\pi t(p_k+P_k)}-e^{2i\pi t(p_k+P_k')}\right|}{(M-N)^{\frac{1}{2}}\Phi(M)}\\
&\qquad\leqslant 2\mathbf{E}\sup_{N<M}\sup_{0\leqslant t\leqslant 1}\frac{\left|\sum_{k=N+1}^M\varepsilon_k e^{2i\pi t(p_k+P_k)}\right|}{(M-N)^{\frac{1}{2}}\Phi(M)}\\
&\qquad=\mathbf{E}\,\mathbf{E}_\varepsilon\sup_{N<M}\sup_{0\leqslant t\leqslant 1}\frac{|V_M(t)-V_N(t)|}{(M-N)^{\frac{1}{2}}\Phi(M)}\leqslant C.C(\mathscr{P},\Phi),
\end{aligned}$$

where $P_1',P_2',\dots$ is an independent copy of the sequence $P_1,P_2,\dots$ defined on another probability space $(\Omega',\mathscr{B}',\mathbf{P}')$, with $\mathbf{E}'$ as corresponding symbol of integration. □

## 4. Uniform convergence of random Fourier series

Let $\mathscr{C}$ be the space of $\mathbf{C}$-valued continuous functions on $[0,1]$ provided with the supremum-norm $||f|| = \sup_{0\leqslant t\leqslant 1} |f(t)|$, $f \in \mathscr{C}$. We investigate in this section the uniform convergence of random Fourier series of type

$$\sum_{k=1}^{n} W_k(\omega)e^{2i\pi p_k t} \qquad n = 1, 2, \ldots,$$

where $\mathscr{W} = (W_k)_{k=1}^{\infty}$ is a sequence of independent, symmetric real random variables, and $\{p_k\}_{k\geqslant 1}$ is a non-decreasing sequence of non-negative integers (with $p_1 > 1$). Our aim is to indicate how Theorem 7 can be used to get a simple sufficient condition for uniform convergence of random Fourier series. This condition is expressed by means of the convergence of a series whose terms are depending on the sequence $(p_k)$. When the order of the size of this sequence is known, this condition can be easier to check than the remarkable characterization of that property by Marcus and Pisier, in terms of the so-called Dudley's entropy integral. It is why it seemed us interesting to indicate it here. We refer to [LT] Chapter 13 Theorem 13.6 and Corollary 13.9 concerning Marcus-Pisier's Theorem. We will prove the following result

THEOREM 10. *Let $\mathscr{W} = (W_k)_{k=1}^{\infty}$ be a sequence of independent, symmetric real random variables, defined on the probability space $(\Omega, \mathscr{A}, \mathbf{P})$, and let $\{p_k\}_{k\geqslant 1}$ be a non-decreasing sequence of non-negative integers (with $p_1 > 1$). Suppose that there exist integers $0 := n_0 < n_1 < n_2 < \ldots$ such that the following condition is satisfied:*

$$(34) \qquad \sum_{i=0}^{\infty} \sqrt{\log(p_{n_{i+1}})}\mathbf{E}\left[\sum_{k=n_i+1}^{n_{i+1}} |W_k|^2\right]^{\frac{1}{2}} \quad \textit{converges.}$$

*Then, the sequence of partial sums $\sum_{k=1}^{n} W_k(\omega)e^{2i\pi p_k t}$ $n = 1, 2, \ldots$ converges in $\mathscr{C}$, for $\mathbf{P}$-almost all $\omega$.*

*Proof.* Put $S_n(\omega, t) := \sum_{k=1}^{n} W_k(\omega)e^{2i\pi p_k t}$ and

$$R = \sup_{N<M} \sup_{0\leqslant t\leqslant 1} \frac{\left|\sum_{k=N+1}^{M} W_k e^{2i\pi p_k t}\right|}{\left(\log p_M \sum_{k=N+1}^{M} W_k^2\right)^{\frac{1}{2}}}.$$

By Theorem 7, $\mathbf{E}R < \infty$, so that

$$\forall i \geqslant 1, \qquad \|S_{n_{i+1}} - S_{n_i}\|_{\mathscr{C}} \leqslant R\sqrt{\log(p_{n_{i+1}})}\left[\sum_{k=n_i+1}^{n_{i+1}} |W_k|^2\right]^{\frac{1}{2}}.$$

Moreover,

$$\forall i \geqslant 1, \qquad \sup_{n_i\leqslant n\leqslant n_{i+1}} \|S_n - S_{n_i}\|_{\mathscr{C}} \leqslant R \sup_{n_i\leqslant n\leqslant n_{i+1}} \sqrt{\log(p_n)}\left[\sum_{k=n_i+1}^{n} |W_k|^2\right]^{\frac{1}{2}}$$

$$= R\sqrt{\log(p_{n_{i+1}})}\left[\sum_{k=n_i+1}^{n_{i+1}} |W_k|^2\right]^{\frac{1}{2}}.$$

Thus, by means of the triangle inequality

$$\forall r \geqslant 1, \qquad \sup_{u,v\geqslant r} \|S_u - S_v\|_{\mathscr{C}} \leqslant R \sum_{i\geqslant r} \sqrt{\log(p_{n_{i+1}})} \left[\sum_{k=n_i+1}^{n_{i+1}} |W_k|^2\right]^{\frac{1}{2}}.$$

This last inequality shows, when combined with the assumption made in (34) and Fatou's lemma, that

$$\sup_{u,v\geqslant r} \|S_u - S_v\|_{\mathscr{C}} \qquad \longrightarrow \qquad 0$$

when $r$ tends to infinity, almost surely. The result easily follows. □

## 5. Appendix

We give here as proposed in paragraph 2, an elementary proof of Theorem 4. We identify in what follows $E$ with $\mathbf{N}$. We note by $D$ the diameter of $(E,d)$. We can assume $D > 0$ otherwise the result is obvious. Let for any integer $n = 0,1,2,\dots$, $S_n \subset E$ be a sequence of centers of balls corresponding to a minimal covering of $E$ of size $2^{-n}D$, ($S_0 = i_0$). We note $S = \cup_{n=0}^{\infty} S_n$; then $S$ is a $d$-dense subset of $E$. Note also formally by $i \mapsto \tilde{\imath}$, the map which sends $i \in S_n$ to the smallest integer $\tilde{\imath} \in S_{n-1}$ such that $||X_i - X_{\tilde{\imath}}|| < 2^{-n+1}D$. Finally, put

$$\forall n \geqslant 0, \qquad M_n = \sup_{i\in S_n} |X_i - X_{i_0}|, \qquad \mathscr{M}_n = \sup_{0\leqslant j\leqslant n} M_j.$$

We note that $\mathscr{M}_0 = M_0 = 0$. Then,

$$0 \leqslant \mathscr{M}_n - \mathscr{M}_{n-1} \leqslant \sup_{i\in S_n} |X_i - X_{\tilde{\imath}}|.$$

Indeed, either $\mathscr{M}_n = \mathscr{M}_{n-1}$, in which case there is nothing to prove; or $\mathscr{M}_n > \mathscr{M}_{n-1}$, and thus $\mathscr{M}_n = M_n > \mathscr{M}_{n-1}$. Let then $i_s \in S_n$ be an indice such that $M_n = |X_{i_s} - X_{i_0}|$. Then,

$$\mathscr{M}_n - \mathscr{M}_{n-1} = |X_{i_s} - X_{i_0}| - \mathscr{M}_{n-1} \leqslant |X_{i_s} - X_{i_0}| - |X_{\tilde{\imath}_s} - X_{i_0}| \leqslant |X_{i_s} - X_{\tilde{\imath}_s}|.$$

Thus, by using (17), and observing that $N(E,d,u) \geqslant 2$ if $0 \leqslant u < D$,

$$\begin{aligned} \|\mathscr{M}_n - \mathscr{M}_{n-1}\|_G &\leqslant C\left(\log N(E,d,2^{-n}D)\right)^{\frac{1}{2}} \sup_{i\in S_n} \|X_i - X_{\tilde{\imath}}\|_G \\ (n \geqslant 1) \qquad &\leqslant C2^{-(n-1)}D\left(\log N(E,d,2^{-n}D)\right)^{\frac{1}{2}}. \end{aligned}$$

As $\mathscr{M}_n = \mathscr{M}_n - \mathscr{M}_0 = \sum_{k=1}^n \mathscr{M}_k - \mathscr{M}_{k-1}$, it follows that

$$\|\mathscr{M}_n\|_G \leqslant \sum_{k=1}^n \|\mathscr{M}_k - \mathscr{M}_{k-1}\|_G \leqslant C\sum_{k=1}^n 2^{-k+1}D\left(\log N(E,d,2^{-k}D)\right)^{\frac{1}{2}}$$

$$\leqslant C\sum_{k=1}^{\infty} 2^{-k+1}D\left(\log N(E,d,2^{-k}D)\right)^{\frac{1}{2}} \leqslant C\int_0^D (\log N(E,d,u))^{\frac{1}{2}}\,du,$$

where $C$ is a universal constant. When $n$ tends to infinity $\mathscr{M}_n$ tends to $\sup_{i\in S}|X_i - X_{i_0}|$. By using the triangle inequality, we have thus shown

$$\left\|\sup_{i,j\in S}|X_i - X_j|\right\|_G \leqslant C\int_0^D (\log N(E,d,u))^{\frac{1}{2}}\,du.$$

But assumption (19) shows $X$ is $d$-continuous in probability. Since $S$ is $d$-dense in $E$, for each $t\in E$, we can exhibit a sequence $(t_n)$ of $S$ such that: $\lim_{n\to\infty} d(t_n,t) = 0$ and $\mathbf{P}\{\lim_{n\to\infty} X_{t_n} = X_t\} = 1$.

It follows that $\mathbf{P}\{\sup_{i,j\in S}|X_i - X_j| = \sup_{i,j\in E}|X_i - X_j|\} = 1$. Hence the theorem. □

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