Mathematical
Inequalities
& Applications
Volume 3, Number 4 (2000), 459–472

ANALYTIC ISOPERIMETRIC INEQUALITIES

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(*communicated by V. Volenec*)

Abstract. In this paper we introduce the concept of area functions for a given function. We then apply these area functions to establish new analytic and geometric isoperimetric inequalities.

1. Introduction

The study of analytic and geometric inequalities is important in both analysis and geometry. Moreover these inequalities are closely related to each other. For instance, by using Fourier series and Wirtinger inequality, Hurwitz (cf. [3, pp. 97–98] [5,15]) showed that the classical geometric isoperimetric inequality for piecewise smooth closed curves in the plane is equivalent to an analytic isoperimetric inequality, that is, Wirtinger inequality. More generally, for a Riemannian manifold, Federer and Fleming verified that the classical isoperimetric inequality is equivalent to one type of Sobolev inequality (cf. [2, p. 97] [14, p. 98]). It is well-known that these inequalities are also useful in mathematical physics [11], statistics and other applied sciences.

For cyclic polygons in the plane, Tang [16] proved that analytic and geometric inequalities are also equivalent. Let us discuss these inequalities below. The classical isoperimetric inequality for polygons states that

$$L^2(P_n) \geqslant 4d_nA(P_n), \quad d_n = n\tan(\pi/n) \tag{1}$$

with equality if and only if P_n is regular [4,6,10], where P_n is an n-sided polygon in the plane with length $L(P_n)$ which enclosed a domain of area $A(P_n)$. It is well-known that if we want to establish any isoperimetric inequality for polygons in the plane such as inequality (1), it suffices to consider P_n cyclic [6,9], that is, it is inscribed in a circle. For simplicity, we shall assume that the circle is of radius 1. For cyclic polygons the inequalty (1) is equivalent to the following analytic isoperimetric inequality [16]

$$\left(\sum_{i=1}^{n} \sin\theta_i\right)^2 \geqslant d_n(\sigma)\sum_{i=1}^{n} \sin\theta_i\cos\theta_i, \quad d_n(\sigma) = n\tan\sigma, \tag{2}$$

Mathematics subject classification (1991): 26B25, 26D20, 26D05, 51M16, 51M25.

Key words and phrases: cyclic polygon, analytic and geometric isoperimetric inequalities, Jensen's inequality, quadratic Jensen's inequality.

Paper MIA-03-45

where $0 < \theta_i < \pi/2, 1 \leqslant i \leqslant n$, and $\sigma = \sum_{i=1}^{n} \theta_i/n$ is a constant. Equality holds if and only if $\theta_1 = \theta_2 = \cdots = \theta_n = \sigma$. Tang [16] also showed that

$$\left(\sum_{i=1}^{n} \cos\theta_i\right)^2 \geqslant \delta_n(\sigma) \sum_{i=1}^{n} \sin\theta_i \cos\theta_i, \quad \delta_n(\sigma) = n\cot\sigma. \tag{3}$$

Equality occurs in (3) if and only if $\theta_1 = \theta_2 = \cdots = \theta_n = \sigma$. There is a geometric isoperimetric inequality for polygons naturally associated with the inequality (3) (See[16]).

The inequalities (2) and (3) are generalized further in [8,17,18]. For instances, Zhang [17,18] proved that

$$\left(\sum_{i=1}^{n} \sin\theta_i\right)^2 \geqslant d_n(\sigma) \sum_{i=1}^{n} \sin\theta_i \cos\theta_i + (n\sin\sigma - \sum_{i=1}^{n} \sin\theta_i)^2. \tag{4}$$

$$\left(\sum_{i=1}^{n} \cos\theta_i\right)^2 \geqslant \delta_n(\sigma) \sum_{i=1}^{n} \sin\theta_i \cos\theta_i + (n\cos\sigma - \sum_{i=1}^{n} \cos\theta_i)^2. \tag{5}$$

Equality is valid in (4) (resp.(5)) if and only if $\theta_1 = \theta_2 = \cdots = \theta_n = \sigma$.
The geometric counterparts for plane polygons corresponding to (4) and (5) can be easily formulated. For example, inequality (4) is equivalent to

$$L^2(P_n) \geqslant 4d_nA(P_n) + \left\{n\sin\frac{\pi}{n} - L(P_n)\right\}^2. \tag{6}$$

with equality holding if and only if P_n is regular. Notice that the term $n\sin\frac{\pi}{n}$ is equal to the length of regular n- gon inscribed in the same unit circle as P_n.

Throughout this paper, let $S : I \longrightarrow R^+ = \{x \in R : x > 0\}$ be a C^2-differentiable function with derivative $S' \neq 0$, where I is an open interval of the real line R. Assume, as we may, that $I \subset R^+$. Analytic isoperimetric inequalities should exist for many other functions. In order to have an analytic isoperimetric inequality for a function S, we must have a suitable area function defined by S. Thus, we defined an area function $A_n(\theta)$ in [7] by

$$A_n(\theta) = \sum_{i=1}^{n} S(\theta_i)S'(\theta_i) \tag{7}$$

where $\theta = (\theta_1, \theta_2, \cdots, \theta_n), \theta_i \in I$. Since the function $A_n(\theta)$ involves the derivative of S, we proved some anlytic isoperimetric inequalities in [7,18] for very special types of functions which are solutions of particular second order ordinary differential equations. In this paper, we shall avoid using the function S' in the definition of area function. Instead, we prove new area formulas for general polygons which generalize Macnab formula (see section 2). These area formulas inspired us to define new area functions $A_S^r(\alpha,\beta;c), A_r^S(\alpha,\beta;c), A_S^{k,m}(\alpha,\beta;c)$ and $A_{k,m}^S(\alpha,\beta;c)$. In doing so, we are able to establish new analytic isoperimetric inequalities involving these area functions. One of our basic results is the following. For notations, see sections 3 and 4.

THEOREM 1.1. *For any* $c, 1 > c \geqslant 0$, *we have*

(i) $$\left\{\sum_{i=1}^{r}[S(\alpha_i)+S(\beta_i)]\right\}^2 \geqslant A_S^r(\alpha,\beta;c), \quad \alpha_i,\beta_i \in I.$$

(ii) *Assume that* k *and* m *are positive integers,* $n=k+m$, $\alpha \geqslant \beta$ $(\alpha,\beta \in I)$, $a=S(\alpha)$ *and* $b=S(\beta)$. *If* $S'>0$ *and* $\frac{na}{2m} \geqslant S(\sigma) \geqslant \frac{nb}{2k}$, *or* $S'<0$ *and* $\frac{na}{2m} \leqslant S(\sigma) \leqslant \frac{nb}{2k}$, *where* $\sigma=(k\alpha+m\beta)/n$, *then*

$$\{kS(\alpha)+mS(\beta)\}^2 \geqslant A_S^{k,m}(\alpha,\beta;c)+\{nS(\sigma)-[kS(\alpha)+mS(\beta)]\}^2. \tag{8}$$

The sign of equality holds in (i) (resp.(8)) if and only if $\alpha=\beta$.

Theorem 1.1 is very general. It holds for most functions S with non-zero derivative S' on an open interval I. When $m=k$, the hypotheses in (ii) simply states that $a \geqslant S(\sigma) \geqslant b$, if $S'>0$, and $a \leqslant S(\sigma) \leqslant b$ if $S'<0$. But these inequalities always hold. Therefore, the inequality (8) is a best possible result. We like to point out that our new analytic isoperimetric inequalities can be used to obtain new geometric inequalities, especially for geodesic cyclic polygons in the spaces of constant curvatures by applying the techniques used in [8].

2. Area formulas for polygons

Since ancient Greek, mathematicians and scientists are interested in finding reasonable formulas for the area $A(P_n)$ of the polygons P_n. For triangles and cyclic quadrilaterals, there are famous Heron's formula and Brahmagupta formula [1,6]. It is natural to expect that similar formulas for $A(P_n)$ might exists for cyclic polygons $P_n, n \geqslant 5$. This is false. Robbin [12] has proved that similar formulas for $A(P_n), n \geqslant 7$, do not exist. Yet, we are still able to find some simple elegant formulas for $A(P_n)$. We shall discuss such formulas in this section.

First, let us introduce some notations which we will use throughout this paper. Let P_n be a n-sided convex cyclic polygon so that the origin O is inside the closure of the domain bounded by the polygon. Let $2a_i$ be the length of the i-th side of P_n and θ_i be the half of the center angle subtended by the i-th side. Hence $\sum_{i=1}^{n}\theta_i=\pi$, and $a_i=\sin\theta_i, 0<\theta_i<\pi/2, 1 \leqslant i \leqslant n$. Set $u_i=\cos\theta_i, 1 \leqslant i \leqslant n$. There are two special types of polygons:

(i) $P_n(\alpha,\beta), n=2rm$: There are m-sides of lengths $2a_i$ and $2b_i$, where $a_i = \sin\alpha_i, b_i=\sin\beta_i, (\alpha_i,\beta_i$ are equal to some θ_j's above), $1 \leqslant i \leqslant r, \alpha = (\alpha_i,\cdots,\alpha_r)$ and $\beta=(\beta_1,\cdots,\beta_r)$. We write $P_n(\alpha,\beta)$ simply as $P_{m,m}$ if $r=1$.

(ii) $P_{k,m}(\alpha,\beta), n=k+m$: There are k-sides of length 2a, m-sides of length $2b, a = \sin\alpha$, and $b=\sin\beta$. It becomes $P_{m,m}$ if $k=m$.

In [9], Macnab proved the following beautiful formula.

THEOREM 2.1.

$$A(P_{m,m}) = \frac{m}{\sin(\pi/m)}\{2ab + (a^2+b^2)\cos(\pi/m)\}.$$

In order to extend this formula to the general polygons, we need a lemma.
For $0 < \alpha, \beta, \alpha+\beta < \pi/2$, define

$$A(\alpha,\beta) = \sin\alpha\cos\alpha + sin\beta\cos\beta.$$

For simplicity of statements, let us set $\alpha+\beta = 2\eta$, $a = \sin\alpha$, $b = \sin\beta$, $c = \cos 2\eta$, $u = \cos\alpha$ and $v = \cos\beta$.

LEMMA 2.2.
(i) $A(\alpha,\beta) = \{2ab + (a^2+b^2)c\}/(sin2\eta)$.
(ii) $A(\alpha,\beta) = \{2uv - (u^2+v^2)c\}/(\sin 2\eta)$.

Proof. (i) We claim the followings:

$$\cot\alpha = \left(\frac{b}{a} + c\right)/(\sin 2\eta). \tag{9}$$

$$\cot\beta = \left(\frac{a}{b} + c\right)/(\sin 2\eta). \tag{10}$$

Notice that

$$\frac{b}{a}\sin\alpha = \sin\beta = \sin(2\eta-\alpha) = \sin 2\eta\cos\alpha - cos2\eta\sin\alpha,$$

and so,

$$\left(\frac{b}{a} + c\right)\sin\alpha = \cos\alpha\sin 2\eta.$$

This proves (9). The proof of (10) is identical. It follows from (9) and (10) that

$$A(\alpha,\beta) = a^2\cot\alpha + b^2\cot\beta = \{2ab + (a^2+b^2)c\}/(\sin 2\eta).$$

(ii) The proof is similar. Instead of (9) and (10), we use

$$\tan\alpha = \left(\frac{v}{u} - c\right)/(\sin 2\eta). \tag{11}$$

$$\tan\beta = \left(\frac{u}{v} - c\right)/(\sin 2\eta). \tag{12}$$

REMARK. From (ii) we have the following inequality:

$$2\cos\alpha\cos\beta > (\cos^2\alpha + \cos^2\beta)\cos(\alpha+\beta). \tag{13}$$

THEOREM 2.3.

(i)
$$A(P_n(\alpha,\beta)) = m\sum_{i=1}^{r}\{2a_ib_i + (a_i^2+b_i^2)\cos 2\sigma\}/\sin 2\sigma$$
$$= m\sum_{i=1}^{r}\{2u_iv_i - (u_i^2+v_i^2\cos 2\sigma\}/\sin 2\sigma.$$

where $2\sigma = \alpha_i + \beta_i = \pi/(mr)$, $u_i = \cos\alpha_i$ *and* $v_i = \cos\beta_i$, $1 \leqslant i \leqslant r$.

(ii)
$$A(P_{k,m}(\alpha,\beta)) = \{nab + (ka^2+mb^2)\cos 2\eta\}/sin2\eta$$
$$= \{nuv - (ku^2+mv^2)\cos 2\eta\}/\sin 2\eta,$$

where $2\eta = \alpha+\beta$, $u = \cos\alpha$ *and* $v = \cos\beta$.

(iii)
$$A(P_n) = \frac{1}{n-1}\sum_{i<j}\{2a_ia_j + (a_i^2+a_j^2)\cos 2\eta_{ij}\}/\sin 2\eta_{ij}$$
$$= \frac{1}{n-1}\sum_{i<j}\{2u_iu_j - (u_i^2+u_j^2)\cos 2\eta_{ij}\}/\sin 2\eta_{ij},$$

where $2\eta_{ij} = \theta_i + \theta_j$, $i<j$, $1 \leqslant i,\ j \leqslant n$.

Proof. Observe that we have

$$A(P_n(\alpha,\beta)) = m\sum_{i=1}^{r}A(\alpha_i,\beta_i), \tag{14}$$

$$\begin{aligned} A(P_{k,m}(\alpha,\beta)) &= ka^2\cot\alpha + mb^2\cot\beta \\ &= ku^2\tan\alpha + mv^2\tan\beta, \end{aligned} \tag{15}$$

and

$$A(P_n) = \frac{1}{n-1}\sum_{i<j}A(\theta_i,\theta_j). \tag{16}$$

Hence the theorem follows immediately from Lemma 2.2, (9), (10), (11) and (12).

3. Isoperimetric Inequalities

In this section we shall use the area formula of the polygon $P_n(\alpha,\beta)$ obtained in section 2 as a model to introduce new area functions $A_S^r(\alpha,\beta;c)$ and $A_r^S(\alpha,\beta;c)$, and to prove new analytic isoperimetric inequalities.

A direct calculation using Theorem 2.3(i), we get

$$d_nA(P_n(\alpha,\beta)) = m^2\left\{\frac{2r}{1+c}\sum_{i=1}^{r}[2a_ib_i + (a_i^2+b_i^2)c]\right\},$$

and

$$\delta_n A(P_n(\alpha,\beta)) = m^2\left\{\frac{2r}{1-c}\sum_{i=1}^{r}[2u_iv_i-(u_i^2+v_i^2)c]\right\},$$

where $0<c=\cos 2\sigma<1$.

DEFINITION 3.1.. For $\alpha=(\alpha_1,\cdots,\alpha_r), \beta=(\beta_1,\cdots,\beta_r)\in I^r$ satisfying $\alpha_i+\beta_i=2\sigma, 1\leqslant i\leqslant r, \sigma$ a constant, define area functions $A_S^r(\alpha,\beta;c)$ and $A_r^S(\alpha,\beta;c), c\geqslant 0,$ as follows:

$$A_S^r(\alpha,\beta;c)=\frac{2r}{1-c}\sum_{i=1}^{r}\{2a_ib_i-(a_i^2+b_i^2)c\},$$

if $2a_ib_i>(a_i^2+b_i^2)c, 1\leqslant i\leqslant r, 0\leqslant c<1,$ and

$$A_r^S(\alpha,\beta;c)=\frac{2r}{1+c}\sum_{i=1}^{r}\{2a_ib_i+(a_i^2+b_i^2)c\},$$

where $a_i=S(\alpha_i)$ and $b_i=S(\beta_i), 1\leqslant i\leqslant r.$

In order to simplify the statements, we introduce the functions

$$B_S(\alpha,\beta)=2r\sum_{i=1}^{r}(a_i^2+b_i^2)-\left\{\sum_{i=1}^{r}(a_i+b_i)\right\}^2,$$

and

$$D_S(\alpha,\beta)=\left\{\sum_{i=1}^{r}(a_i+b_i)\right\}^2-4r\sum_{i=1}^{r}a_ib_i.$$

A simple calculation shows that

$$B_S(\alpha,\beta)=\sum_{i=1}^{r}(a_i-b_i)^2+\sum_{\substack{i,j=1\\ i<j}}^{r}\{(a_i-a_j)^2+(a_j-b_i)^2+(a_i-b_j)^2+(b_i-b_j)^2\}. \tag{17}$$

Hence

$$B_S(\alpha,\beta)\geqslant 0, \tag{18}$$

with equality if and only if $a_1=a_2=\cdots=a_r=b_1=\cdots=b_r$.

THEOREM 3.2.

(i) For any c, $1>c\geqslant 0$,

$$\left\{\sum_{i=1}^{r}(a_i+b_i)\right\}^2\geqslant A_S^r(\alpha,\beta;c). \tag{19}$$

(ii) Let $c\geqslant 0$ and $2r\sum_{i=1}^{r}(a_i-b_i)^2\geqslant(1+c)B_S(\alpha,\beta)$. Then

$$\left\{\sum_{i=1}^{r}(a_i+b_i)\right\}^2\geqslant A_r^S(\alpha,\beta;c), \tag{20}$$

and if $\alpha_i \neq \beta_i$ *for some* $i, c < 1$.

Equality holds in (19) *(resp.*(20) *) if and only if* $a_1 = \cdots = a_r = b_1 = \cdots = b_r$.

Proof. First, we prove

$$D_S(\alpha,\beta) = \sum_{i=1}^{r}(a_i - b_i)^2 + 2\sum_{i<j}\{(a_i - b_j)(a_j - b_i) + (a_i - a_j)(b_j - b_i)\}. \tag{21}$$

Since

$$4(r-1)\sum_{i=1}^{r} a_i b_i = 2\sum_{i<j}(2a_i b_i + 2a_j b_j),$$

$$\begin{aligned} D_S(\alpha,\beta) &= \sum_{i=1}^{r}(a_i - b_i)^2 - 4(r-1)\sum_{i=1}^{r} a_i b_i + 2\sum_{i<j}(a_i a_j + a_i b_j + a_j b_i + b_i b_j) \\ &= \sum_{i=1}^{r}(a_i - b_i)^2 + 2\sum_{i<j}(a_i a_j + a_i b_j + a_j b_i + b_i b_j - 2a_i b_i - 2a_j b_j)) \\ &= \sum_{i=1}^{r}(a_i - b_i)^2 + 2\sum_{i<j}\{(a_i - b_j)(a_j - b_i) + (a_i - a_j)(b_j - b_i)\} \end{aligned}$$

as desired. Now we proceed to show that

$$D_S(\alpha,\beta) \geqslant 0, \tag{22}$$

with equality holding if and only if $a_1 = \cdots = a_r = b_1 = \cdots = b_r$. Without loss of generality, we may assume that $\alpha_i \geqslant \beta_i$ for $1 \leqslant i \leqslant r$, and by rearranging the subscripts if necessary, assuming $\alpha_1 \geqslant \alpha_2 \geqslant \cdots \geqslant \alpha_r$. Since $\alpha_i + \beta_i = 2\sigma$ for $1 \leqslant i \leqslant r$,

$$\alpha_1 \geqslant \alpha_2 \geqslant \cdots \geqslant \alpha_r \geqslant \beta_r \geqslant \beta_{r-1} \geqslant \cdots \geqslant \beta_1.$$

This will imply that

$$a_1 \geqslant a_2 \geqslant \cdots \geqslant \alpha_r \geqslant b_r \geqslant b_{r-1} \geqslant \cdots \geqslant b_1.$$

if $S' > 0$, and if $S' < 0$,

$$a_1 \leqslant a_2 \leqslant \cdots \leqslant a_r \leqslant b_r \leqslant b_{r-1} \leqslant \cdots \leqslant b_1,$$

Therefore, if $S' \neq 0$, and $i < j$, we have

$$(a_i - b_j)(a_j - b_i) + (a_i - a_j)(b_j - b_i) \geqslant 0,$$

which proves (22). From (18) and (22)

$$\left\{\sum_{i=1}^{r}(a_i+b_i)\right\}^2 - A_S^r(\alpha,\beta;c)$$
$$= \frac{1}{1-c}\left\{\left[\sum_{i=1}^{r}(a_i+b_i)\right]^2 - 4r\sum_{i=1}^{r}a_ib_i + \left(2r\sum_{i=1}^{r}(a_i^2+b_i^2) - \left[\sum_{i=1}^{r}(a_i+b_i)\right]^2\right)c\right\}$$
$$= \frac{1}{1-c}\{D_S(\alpha,\beta)+B_S(\alpha,\beta)c\} \geqslant 0.$$

This completes the proof of (19).

To prove the inequality (20), let us observe that the hypothesis

$$2r\sum_{i=1}^{r}(a_i-b_i)^2 \geqslant (1+c)B_S(\alpha,\beta)$$

is equivalent to the following inequality

$$D_S(\alpha,\beta) - B_S(\alpha,\beta)c \geqslant 0, \tag{23}$$

because $D_S(\alpha,\beta)+B_S(\alpha,\beta) = 2r\sum_{i=1}^{r}(a_i-b_i)^2$. Therefore,

$$\left\{\sum_{i=1}^{r}(a_i+b_i)\right\}^2 - A_r^S(\alpha,\beta;c) = \frac{1}{1+c}\{D_S(\alpha,\beta)-B_S(\alpha,\beta)c\} \geqslant 0,$$

which concludes the proof of (20). From (17) and (21)

$$B_S(\alpha,\beta) \geqslant D_S(\alpha,\beta). \tag{24}$$

If $\alpha_i \neq \beta_i$ for some i, $B_S(\alpha,\beta) > 0$ by (18) which leads to $c < 1$ by (23) and (24), thereby proving the theorem.

In (19), if S is the cosine function, $I = (0,\pi/2)$ and $c = \cos 2\sigma$, then $A_S^r(\alpha,\beta;c)$ is well-defined by (13), and the inequality (19) is exactly the inequality of (3) with $(\theta_1,\theta_2,\cdots,\theta_n) = (\alpha_1,\beta_1,\cdots,\alpha_r,\beta_r), n = 2r$ and $\alpha_i+\beta_i = 2\sigma, 1 \leqslant i \leqslant k$.

4. Quadratic Jensen's inequalities

If the function S is convex (resp. concave), Jensen's inequality states that

$$2rS(\sigma) - \sum_{i=1}^{r}(a_i+b_i) \leqslant 0, (resp.\ 2rS(\sigma) - \sum_{i=1}^{r}(a_i+b_i) \geqslant 0) \tag{25}$$

with equality holding if and only if $a_1 = \cdots = a_r = b_1 = \cdots = b_r$.

Thus, the analytic isoperimetric inequalities proved in this section may be regarded as *quadratic Jensen's inequalities* because they contain the terms defined by Jensen's inequality such as $\left\{2rS(\sigma) - \sum_{i=1}^{r}(a_i + b_i)\right\}^2$ in (26) and (27).

Since $\alpha_i + \beta_i = 2\sigma$, we may assume that $\alpha_i \geqslant \beta_i$, $1 \leqslant i \leqslant r$.

THEOREM 4.1. *Let* $0 \leqslant c < 1$.

(i) *Let* $1 \leqslant i \leqslant r$. *Suppose* $\alpha_i \geqslant \beta_i$ *and the function* S *satisfying one of the following 4 conditions.*

(a) $S'' < 0$, $S' > 0$ *and* $S'(a_i) \geqslant cS'(\beta_i)$*;*
(b) $S'' < 0$, $S' < 0$ *and* $S'(\beta_i) \leqslant cS'(\alpha_i)$*;*
(c) $S'' > 0$, $S' > 0$ *and* $S'(\beta_i) \geqslant cS'(\alpha_i)$*;*
(d) $S'' > 0$, $S' < 0$ *and* $S'(\alpha_i) \leqslant cS'(\beta_i)$.

Then we have quadratic Jensen's inequality

$$\left\{\sum_{i=1}^{r}(a_i + b_i)\right\}^2 \geqslant A_r^S(\alpha, \beta; c) + \left\{2rS(\sigma) - \sum_{i=1}^{r}(a_i + b_i)\right\}^2. \tag{26}$$

We have analytic isoperimetric inequality

$$\left\{\sum_{i=1}^{r}(a_i + b_i)\right\}^2 \geqslant A_S^r(\alpha, \beta; c) + \left\{2rS(\sigma) - \sum_{i=1}^{r}(a_i + b_i)\right\}^2. \tag{27}$$

Equality is valid in (26) *(resp.* (27) *) if and only if* $a_1 = \cdots = a_r = b_i = \cdots = b_r$.

Proof. (i) For $1 \leqslant i \leqslant r$, we define the function

$$\Phi^S(\alpha_i, \beta_i; c) = 2\{a_i - S(\sigma)\}\{S(\sigma) - b_i\} - \{[a_i - S(\sigma)]^2 + [S(\sigma) - b_i]^2\}c.$$

We claim that

$$\Phi^S(\alpha_i, \beta_i; c) \geqslant 0,\ 1 \leqslant i \leqslant r \tag{28}$$

with equality if and only if $\alpha_i = \beta_i = \sigma$. It is clear that equality is true if $\alpha_i = \beta_i$. It remains to verify that $\Phi^S(\alpha_i, \beta_i; c) > 0$ if $\alpha_i > \beta_i$. We shall show that the function $\Phi^S(\alpha_i, \beta_i; c)$ is strictly Schur convex for each $i = 1, 2, \cdots, r$. This means that (cf. [13, p. 259], [18, p. 462], [19])

$$(\alpha_i - \beta_i)\left(\frac{\partial}{\partial \alpha_i}\Phi^S - \frac{\partial}{\partial \beta_i}\Phi^S\right) > 0,\ for\ \alpha_i > \beta_i. \tag{29}$$

It is well-known [13] (see also [18, p.426]) that if $\Phi(\alpha_i, \beta_i; c)$ is strictly Schur convex,

$$\Phi^S(\alpha_i, \beta_i; c) > \Phi^S((\alpha_i, \beta_i)\begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix}; c) \tag{30}$$

But

$$\Phi^S((\alpha_i, \beta_i)\begin{pmatrix} 1/2 & 1/2 \\ 1/2 & 1/2 \end{pmatrix}; c) = \Phi^S(\sigma, \sigma; c) = 0.$$

This will imply that $\Phi^S(\alpha_i,\beta_i;c) > 0$ by (30). Thus, to complete the proof of (30), we need to verify (29). However

$$\frac{\partial}{\partial\alpha_i}\Phi^S(\alpha_i,\beta_i;c) = 2S'(\alpha_i)\{(1+c)S(\sigma) - a_i c - b_i\}.$$

$$\frac{\partial}{\partial\beta_i}\Phi^S(\alpha_i,\beta_i;c) = 2S'(\beta_i)\{(1+c)S(\sigma) - b_i c - a_i\},$$

hence

$$\begin{aligned}(\alpha_i-\beta_i)&\left(\frac{\partial}{\partial\alpha_i}\Phi^S - \frac{\partial}{\partial\beta_i}\Phi^S\right)\\ &= 2(\alpha_i-\beta_i)\{S'(\alpha_i)[S(\sigma)-b_i] + S'(\beta_i)[a_i - S(\sigma)]\\ &\qquad -c[S'(\alpha_i)(a_i - S(\sigma)) + S'(\beta_i)(S(\sigma)-b_i)]\}.\end{aligned}$$

Since the proof of (29) for four cases are similar, we shall only prove the case (a). From $S'' < 0$ and $\alpha_i > \beta_i$, we get $S'(\alpha_i) < S'(\beta_i)$. As $S' > 0$ and $\alpha_i > \sigma > \beta_i$, we have $a_i - S(\sigma) > 0$ and $S(\sigma) - b_i > 0$. Therefore

$$\begin{aligned}(\alpha_i-\beta_i)&\left(\frac{\partial}{\partial\alpha_i}\Phi^S - \frac{\partial}{\partial\beta_i}\Phi^S\right)\\ &> 2(\alpha_i-\beta_i)\{S'(\alpha_i)([S(\sigma)-b_i] + [a_i - S(\sigma)])\\ &\qquad - cS'(\beta_i)([a_i - S(\sigma)] + [S(\sigma)-b_i])\}\\ &= 2(\alpha_i-\beta_i)(a_i-b_i)\{S'(\alpha_i) - cS'(\beta_i)\} \geqslant 0\end{aligned}$$

by hypotheses, which proves (29), whence, (28). Now we have

$$\begin{aligned}\left\{\sum_{i=1}^r (a_i+b_i)\right\}^2 &- A_r^S(\alpha,\beta;c) - \left\{2rS(\sigma) - \sum_{i=1}^r (a_i+b_i)\right\}^2\\ &= \frac{2r}{1+c}\sum_{i=1}^r \Phi^S(\alpha_i,\beta_i;c).\end{aligned}\tag{31}$$

This proves (26) by using (28).

(ii) For $1 \leqslant i \leqslant r$, we define another function

$$\Phi_S(\alpha_i,\beta_i;c) = 2\{a_i - S(\sigma)\}\{S(\sigma)-b_i\} + \{[a_i - S(\sigma)]^2 + [S(\sigma)-b]^2\}c.$$

Since $S' \neq 0, \{a_i - S(\sigma)\}\{S(\sigma)-b_i\} \geqslant 0, 1 \leqslant i \leqslant r$. Hence $\Phi_S(\alpha_i,\beta_i;c) \geqslant 0$ with equality holding if and only if $\alpha_i = \beta_i$. Thus,

$$\begin{aligned}\left\{\sum_{i=1}^r (a_i+b_i)\right\}^2 &- A_S^r(\alpha,\beta;c) - \left\{2rS(\sigma) - \sum_{i=1}^r (a_i+b_i)\right\}^2\\ &= \frac{2r}{1-c}\sum_{i=1}^r \Phi_S(\alpha_i,\beta_i;c) \geqslant 0.\end{aligned}\tag{32}$$

This completes the proof of (27).

COROLLARY 4.2.

$$L^2(P_n(\alpha,\beta)) - 4d_nA(P_n(\alpha,\beta)) - \left\{2n\sin\frac{\pi}{n} - L(P_n(\alpha,\beta)\right\}^2,$$
$$-\frac{4mn}{1+\cos\frac{2\pi}{n}}\sum_{i=1}^{r}\Phi^S\left(\alpha_i,\beta_i;\cos\frac{2\pi}{n}\right) \geqslant 0, \tag{33}$$

where $\sum_{i=1}^{r}\Phi^S\left(\alpha_i,\beta_i;\cos\frac{2\pi}{n}\right) \geqslant 0$. *With equality holds in* (33) *if and only if* $P_n(\alpha,\beta)$ *is regular.*

Proof. We can easily check that

$$\sin\alpha_i \geqslant c\sin\beta_i, \text{ for } \alpha_i \geqslant \beta_i,\ c=\cos 2\sigma \text{ and } \sigma=\pi/n.$$

Applying (31) to sine function and $I=(0,\pi/2)$ we get

$$\begin{aligned}\left\{\sum_{i=1}^{r}(\sin\alpha_i+\sin\beta_i)\right\}^2 &= \frac{2r}{1+c}\sum_{i=1}^{r}A(\alpha_i,\beta_i)\\ &+\left\{2r\sin\sigma-\sum_{i=1}^{r}(\sin\alpha_i+\sin\beta_i)\right\}^2\\ &+\frac{2r}{1+\cos 2\sigma}\sum_{i=1}^{r}\Phi^S(\alpha_i,\beta_i;\cos 2\sigma),\end{aligned} \tag{34}$$

which is equivalent to (33).

The geometric isoperimetric inequality (33) is much shaper than (6) for polygon $P_n(\alpha,\beta)$. If we apply (32) to cosine function we obtain inequality.

$$\begin{aligned}\left\{\sum_{i=1}^{r}(\cos\alpha_i+\cos\beta_i)\right\}^2 &- \frac{2r}{1-c}\sum_{i=1}^{r}A(\alpha_i,\beta_i)\\ &-\left\{2r\cos\sigma-\sum_{i=1}^{r}(\cos\alpha_i+\cos\beta_i)\right\}^2\\ &-\frac{2r}{1-\cos 2\sigma}\sum_{i=1}^{r}\Phi_S(\alpha_i,\beta_i;\cos 2\sigma)\geqslant 0,\end{aligned} \tag{35}$$

where $\sum_{i=1}^{r}\Phi_S(\alpha_i,\beta_i;\cos 2\sigma)\geqslant 0$, with equality holding in (35) if and only if $\alpha_i = \beta_i = \sigma$, $1\leqslant i\leqslant r$.

Next, we like to introduce area functions which model after the area of $P_{k,m}(\alpha,\beta)$. Simple calculations, using Theorem 2.3 (ii), lead to

$$\delta_nA(P_{k,m}(\alpha,\beta)) = (\cot\sigma\tan\eta)\left\{\frac{n}{1-c}[nuv-(ku^2+mv^2)c]\right\}, \tag{36}$$

$$d_nA(P_{k,m}(\alpha,\beta)) = (\tan\sigma\cot\eta)\left\{\frac{n}{1+c}[nab+(ka^2+mb^2)c]\right\}, \tag{37}$$

where $c=\cos 2\eta$, and $\sigma=(k\alpha+m\beta)/n=\pi/n$. Hence we have

$$nuv > (ku^2+mv^2)c, \tag{38}$$

$$\delta_nA(P_{k,m}(\alpha,\beta)) \leqslant \frac{n}{1-c}\left\{nuv-(ku^2+mv^2)c\right\},\ \textit{if}\ \ \sigma\geqslant\eta; \tag{39}$$

$$d_nA(P_{k,m}(\alpha,\beta)) \leqslant \frac{n}{1+c}\left\{nab+(ka^2+mb^2)c\right\},\ \textit{if}\ \ \sigma\leqslant\eta. \tag{40}$$

The equalities (36), (37) and inequalities (39) and (40) motivate us to define

DEFINITION 4.3. Let $k,m>0$ be integers, $n=k+m$, and $c>0$. For $\alpha,\beta\in I$, define *area functions* $A_S^{k,m}(\alpha,\beta;c)$ and $A_{k,m}^S(\alpha,\beta;c)$ by

$$A_S^{k,m}(\alpha,\beta,c) = \frac{n}{1-c}\left\{nab-(ka^2+mb^2)c\right\},$$

if $0\leqslant c<1$ and $nab>(ka^2+mb^2)c$, and

$$A_{k,m}^S(\alpha,\beta;c) = \frac{n}{1+c}\left\{nab+(ka^2+mb^2)c\right\}.$$

We shall assume that $\sigma=(k\alpha+m\beta)/n$ is a constant, and $\alpha\geqslant\beta$. Let us recall that if S is concave (resp. convex), Jensen's inequality state that

$$nS(\sigma)\geqslant ka+mb \ \ (\text{resp. } \ nS(\sigma)\leqslant ka+mb),$$

with equality if and and only if $a=\beta=\sigma$.

THEOREM 4.4. *Let* $\alpha\geqslant\beta$. *Suppose that we have* $S'>0$ *and* $\frac{na}{2m}\geqslant S(\sigma)\geqslant\frac{nb}{2k}$, *or* $S'<0$ *and* $\frac{na}{2m}\leqslant S(\sigma)\leqslant\frac{nb}{2k}$. *Then*

$$(ka+mb)^2 \geqslant A_S^{k,m}(\alpha,\beta;c)+\{nS(\sigma)-(ka+mb)\}^2. \tag{41}$$

Equality occurs if and only if $a=b=S(\sigma)$.

Proof. Here we consider only the case of $S'>0$. The case $S'<0$ can be easily modified. Notice that

$$\begin{aligned}&(ka+mb)^2-A_S^{k,m}(\alpha,\beta;c)-\{nS(\sigma)-(ka+mb)\}^2\\ &=\frac{n}{1-c}\left\{2(ka+mb)S(\sigma)-nab-nS^2(\sigma)+c[k(a-S(\sigma))^2+m(S(\sigma)-b)^2]\right\}.\end{aligned}$$

Since $\beta=(n\sigma-k\alpha)/m$, define the function $F(\alpha)$ by

$$F(\alpha)=2(ka+mb)S(\sigma)-nab-nS^2(\sigma).$$

Then $F(\sigma)=0$. As $S'>0$, and $\frac{na}{2m}\geqslant S(\sigma)\geqslant\frac{nb}{2k}$, we have

$$F'(\alpha)=S'(\alpha)\{2kS(\sigma)-nb\}+kS'(\beta)\left\{\frac{na}{m}-2S(\sigma)\right\}\geqslant 0,$$

and so, $F(\alpha)$ is an increasing function of α. Therefore $F(\alpha)\geqslant F(\sigma)=0$, thereby proving (41).

REMARK. The hypothesis of the theorem is true if $(k-m)(a-b)\geqslant 0$.

THEOREM 4.5. *Suppose that* $\alpha \geqslant \beta$, *and* S *satisfies one of the following 4 conditions:*

(*i*) $S' > 0$ *and* $\frac{na}{2m} \geqslant S(\sigma) \geqslant \frac{nb}{2k}$; *and if* $S'' < 0$,

$$S'(\alpha)(ka - mb) \geqslant \frac{2km}{n}(a-b)\left\{cS'(\beta)\right\},$$

and if $S'' > 0$,

$$S'(\beta)(ka - mb) \geqslant \frac{2km}{n}(a-b)\left\{cS'(\alpha)\right\}.$$

(*ii*) $S' < 0$ *and* $\frac{na}{2m} \leqslant S(\sigma) \leqslant \frac{nb}{2k}$; *and if* $S'' < 0$,

$$S'(\beta)(mb - ka) \leqslant \frac{2km}{n}(b-a)\left\{cS'(\alpha)\right\},$$

and if $S'' > 0$

$$S'(\alpha)(mb - ka) \leqslant \frac{2km}{n}(b-a)\{cS'(\beta)\}.$$

Then we have quadratic Jensen's inequality.

$$(ka+mb)^2 \geqslant A^S_{k,m}(\alpha,\beta;c) + \{nS(\sigma) - (ka+mb)\}^2. \tag{42}$$

The sign of equality holds if and only if $a = b = S(\sigma)$.

Proof. Again, we shall only give the proof under the hypotheses $S'' < 0$ and $S' > 0$. Define the function

$$G(\alpha) = (ka+mb)^2 - A^S_{k,m}(\alpha,\beta;c) - \{nS(\sigma) - (ka+mb)\}^2.$$

A direct calculations give $G(\sigma) = 0$ and

$$G(\alpha) = 2n(ka+mb)S(\sigma) - n^2S^2(\sigma) - \frac{n}{1+c}\left\{nab + (ka^2+mb^2)c\right\}.$$

So we may assume $\alpha > \beta$, hence $a > S(\sigma) > b$, and $0 < S'(\alpha) < S'(\beta)$. Thus $0 > -S'(\alpha) > -S'(\beta)$. Differentiating,

$$\begin{aligned}
G'(\alpha) = \frac{n}{1+c}&\left\{S'(\alpha)[2kS(\sigma) - nb] + S'(\beta)\left[\frac{kna}{m} - 2kS(\sigma)\right]\right. \\
&\left.+c[-S'(\alpha)(2ka - 2kS(\sigma)) - S'(\beta)(2kS(\sigma) - 2kb)]\right\} \\
&> \frac{n}{1+c}\left\{S'(\alpha)[2kS(\sigma) - nb] + S'(\alpha)\left[\frac{kna}{m} - 2kS(\sigma)\right]\right. \\
&\left.+c[-S'(\beta)(2k)(a - S(\sigma)) - S'(\beta)(2k)(S(\sigma) - b)]\right\} \\
&= \frac{n}{1+c}\left\{S'(\alpha)\frac{n}{m}(ka - mb) - 2kS'(\beta)(a-b)c\right\} \\
&\geqslant 0
\end{aligned}$$

by hypotheses. Thus, $G(\alpha) > G(\sigma) = 0$, and concludes the proof of (42).

It is clear that we can apply Theorems 4.4 and 4.5 to some polygons of type $P_{k,m}(\alpha,\beta)$ to obtain improved geometric isoperimetric inequalities. In the case of $k=m$, this gives another proof of Theorem 4.1 for $r=1$. We also can obtain reverse inequality. For instance, by a modification of the proof of (42) we can show that

$$(ka+mb)^2 \leqslant A^S_{k,m}(\alpha,\beta;c) + \{nS(\sigma)-(ka+mb)\}^2 \tag{43}$$

with equality holding if and only if $\alpha=\beta=\sigma$ under the hypotheses: $S''<0$, $S'>0$, $\frac{na}{2m} \geqslant S(\sigma) \geqslant \frac{nb}{2k}$ and

$$S'(\beta)(ka-mb) \leqslant \frac{2km}{n}(a-b)\{cS'(\alpha)\}.$$

To conclude this paper, let us remark that we can use other types of polygons, or even arbitrary polygon P_n as models to obtain various forms of area functions, and to establish some other types of analytic and geometric isoperimetric inequalities.

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(Received February 23, 1999)

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