Mathematical
Inequalities
& Applications
Volume 3, Number 4 (2000), 497–510

HILBERT INTEGRAL OPERATOR INEQUALITIES

KUANG JICHANG AND THEMISTOCLES M. RASSIAS

(*communicated by J. Pečarić*)

Abstract. In this paper, we establish new Hilbert integral operator inequalities with the general kernel. They are significant extensions and improvements of some known results.

1. Introduction

The integral version of the remarkable Hilbert's inequality is the following: If $f, g \in L^2[0,\infty)$, then

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x+y}dxdy \leqslant \pi\left(\int_0^\infty f^2(x)dx \int_0^\infty g^2(x)dx\right)^{\frac{1}{2}} \tag{1.1}$$

and

$$\int_0^\infty \int_0^\infty \frac{f(x)f(y)}{x+y}dxdy \leqslant \pi \int_0^\infty f^2(x)dx, \tag{1.2}$$

where π is the best value (cf. [1, Chap. 9]).

In 1936, A. E. Ingham [2] proved

THEOREM A. *If* $a_n > 0$, $n = 0,1,\ldots$, $0 < \sum_{n=0}^{\infty} a_n^2 < \infty$ *and* $\lambda > 0$, *then*

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \frac{a_m a_n}{m+n+\lambda} \leqslant M(\lambda) \sum_{n=0}^{\infty} a_n^2 \tag{1.3}$$

where

$$M(\lambda) = \begin{cases} \dfrac{\pi}{\sin \lambda\pi}, & 0 < \lambda \leqslant \dfrac{1}{2} \\ M(\frac{1}{2}) = \pi, & \lambda > \dfrac{1}{2}. \end{cases} \tag{1.4}$$

In recent years, various improvements and extensions of the inequality (1.1) have been considered (see e.g. [3–6] and the references cited therein). But the integral

Mathematics subject classification (1991): 26D10.
Key words and phrases: Hilbert inequality, integral operator, kernel.

Paper MIA-03-48

analogues of (1.3) do not appear to have been investigated for general values of λ. The aim of this paper is to establish some new inequalities related to the Hilbert integral operator

$$T_\lambda(f,g) = \int_a^b \int_a^b K(x+\lambda, y+\lambda) f(x) g(y) dxdy$$

with the general kernel $K(x,y)$. Our main results can be stated as follows.

THEOREM 1. *Let* $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $0 < a < b$, $\lambda \geqslant 0$, $0 < 1 - 2\frac{\lambda}{r} < t$, *(if* $t \leqslant 1$, *then take* $\lambda > 0$*),* $K(x,y)$ *be nonnegative, symmetrical and homogeneous of degree* $-t$ *and let* $K(1,y)$ *be a strictly decreasing function of* y, *and*

$$I(r,\lambda) = \int_0^\infty K(1,y) y^{-2\frac{\lambda}{r}} dy < \infty, \quad r = p, q. \tag{1.6}$$

Let f *and* g *be nonnegative measurable functions defined on* $[0,\infty)$. *We then have*

(1) *If* $0 \leqslant \lambda \leqslant \frac{1}{2}$, *then*

$$T_\lambda(f,g) \leqslant \left\{ \int_a^b [I(q,\lambda) - \varphi_1(q,x,t,\lambda)](x+\lambda)^{1-t} f^p(x) dx \right\}^{\frac{1}{p}} \times$$
$$\times \left\{ \int_a^b [I(p,\lambda) - \varphi_1(p,x,t,\lambda)](x+\lambda)^{1-t} g^q(x) dx \right\}^{\frac{1}{q}}, \tag{1.7}$$

where

$$\varphi_1(r,x,t,\lambda) = \left(\frac{a+\lambda}{x+\lambda}\right)^{1-2\frac{\lambda}{r}} \int_0^1 K(1,u) u^{-2\frac{\lambda}{r}} du$$
$$+ \left(\frac{x+\lambda}{b+\lambda}\right)^{t+2\frac{\lambda}{r}-1} \int_0^1 K(1,u) u^{t+2\frac{\lambda}{r}-2} du; \tag{1.8}$$

$$\int_a^\infty \int_a^\infty K(x+\lambda, y+\lambda) f(x) g(y) dxdy$$
$$\leqslant \left\{ \int_a^\infty \left[I(q,\lambda) - \left(\frac{a+\lambda}{x+\lambda}\right)^{1-2\frac{\lambda}{q}} \int_0^1 K(1,u) du \right] (x+\lambda)^{1-t} f^p(x) dx \right\}^{\frac{1}{p}} \times$$
$$\times \left\{ \int_a^\infty \left[I(p,\lambda) - \left(\frac{a+\lambda}{x+\lambda}\right)^{1-2\frac{\lambda}{p}} \int_0^1 K(1,u) du \right] (x+\lambda)^{1-t} g^q(x) dx \right\}^{\frac{1}{q}}; \tag{1.9}$$

$$\int_0^\infty\int_0^\infty K(x+\lambda,y+\lambda)f(x)g(y)dxdy$$

$$\leqslant I(q,\lambda)^{\frac{1}{p}}I(p,\lambda)^{\frac{1}{q}}\left\{\int_0^\infty (x+\lambda)^{1-t}f^p(x)dx\right\}^{\frac{1}{p}}\times\left\{\int_0^\infty (x+\lambda)^{1-t}g^q(x)dx\right\}^{\frac{1}{q}}. \tag{1.10}$$

(2) *If* $\lambda > \frac{1}{2}$, *then*

$$T_\lambda(f,g)\leqslant T_{\frac{1}{2}}(f,g)\leqslant\left\{\int_a^b\Big[I\big(q,\frac{1}{2}\big)-\varphi_1\big(q,x,t,\frac{1}{2}\big)\Big]\big(x+\frac{1}{2}\big)^{1-t}f^p(x)dx\right\}^{\frac{1}{p}}\times$$

$$\times\left\{\int_a^b\big[I\big(p,\frac{1}{2}\big)-\varphi_1\big(p,x,t,\frac{1}{2}\big)\big(x+\frac{1}{2}\big)^{1-t}g^q(x)dx\right\}^{\frac{1}{q}}; \tag{1.11}$$

$$\int_a^\infty\int_a^\infty K(x+\lambda,y+\lambda)f(x)g(y)dxdy$$

$$\leqslant\left\{\int_a^\infty\left[I\big(q,\frac{1}{2}\big)-\left(\frac{2a+1}{2x+1}\right)^{\frac{1}{p}}\int_0^1 K(1,u)du\right]\big(x+\frac{1}{2}\big)^{1-t}f^p(x)dx\right\}^{\frac{1}{p}}\times$$

$$\times\left\{\int_a^\infty\left[I\big(p,\frac{1}{2}\big)-\left(\frac{2a+1}{2x+1}\right)^{\frac{1}{q}}\int_0^1 K(1,u)du\right]\big(x+\frac{1}{2}\big)^{1-t}g^q(x)dx\right\}^{\frac{1}{q}}; \tag{1.12}$$

$$\int_0^\infty\int_0^\infty K(x+\lambda,y+\lambda)f(x)g(y)dxdy$$

$$\leqslant\left\{\int_0^\infty\left[I\big(q,\frac{1}{2}\big)-(2x+1)^{-\frac{1}{p}}\int_0^1 K(1,u)du\right]\big(x+\frac{1}{2}\big)^{1-t}f^p(x)dx\right\}^{\frac{1}{p}}\times$$

$$\times\left\{\int_0^\infty\left[I\big(p,\frac{1}{2}\big)-(2x+1)^{-\frac{1}{q}}\int_0^1 K(1,u)du\right]\big(x+\frac{1}{2}\big)^{1-t}g^q(x)dx\right\}^{\frac{1}{q}}. \tag{1.13}$$

REMARK 1. In Theorem 1, we have assumed $0 < 1-2\frac{\lambda}{r} < t$. Hence, $\lambda = 0$ implies $t > 1$. For $t \leqslant 1$, we may assume that $\max\left\{\frac{1}{p},\frac{1}{q}\right\} < t$. Thus, we get the following result:

THEOREM 2. *Let* $p > 1$, $\frac{1}{p}+\frac{1}{q} = 1$, $0 < a < b$, $K(x,y)$ *be nonnegative, symmetrical and homogeneous of degree* $-t$, $\max\left\{\frac{1}{p},\frac{1}{q}\right\} < t$, $K(1,y)$ *be a strictly* *decreasing function of* y*, and*

$$I(r) = \int_0^\infty K(1,y)y^{-\frac{1}{r}}dy < \infty, \quad r = p, q. \tag{1.14}$$

If f *and* g *are nonnegative measurable functions defined on* $[a,b]$*, then*

$$T_0(f,g) \leqslant \left\{\int_a^b [I(q) - \varphi_2(q,x)]x^{1-t}f^p(x)dx\right\}^{\frac{1}{p}} \times \\ \times \left\{\int_a^b [I(p) - \varphi_2(p,x)]x^{1-t}g^q(x)dx\right\}^{\frac{1}{q}}; \tag{1.15}$$

$$\int_a^\infty\int_a^\infty K(x,y)f(x)g(y)dxdy \leqslant \left\{\int_a^\infty \left[I(q) - \left(\frac{a}{x}\right)^{\frac{1}{p}} \int_0^1 K(1,u)u^{-\frac{1}{q}}du\right]x^{1-t}f^p(x)dx\right\}^{\frac{1}{p}} \times \\ \times \left\{\int_a^\infty \left[I(p) - \left(\frac{a}{x}\right)^{\frac{1}{q}} \int_0^1 K(1,u)u^{-\frac{1}{p}}du\right]x^{1-t}g^q(x)dx\right\}^{\frac{1}{q}}; \tag{1.16}$$

$$\int_0^\infty\int_0^\infty K(x,y)f(x)g(y)dxdy \leqslant I(q)^{\frac{1}{p}}I(p)^{\frac{1}{q}}\left\{\int_0^\infty x^{1-t}f^p(x)dx\right\}^{\frac{1}{p}}\left\{\int_0^\infty x^{1-t}g^q(x)dx\right\}^{\frac{1}{q}}, \tag{1.17}$$

$$\varphi_2(r,x) = \left(\frac{a}{x}\right)^{1-\frac{1}{r}} \int_0^1 K(1,u)u^{-\frac{1}{r}}du + \left(\frac{x}{b}\right)^{t+\frac{1}{r}-2} \int_0^1 K(1,u)u^{t+\frac{1}{r}-2}du. \tag{1.18}$$

If $K(x,y)$ is homogeneous of degree -1 and $I(q) = I(p)$, then by (1.17), we get

$$\int_0^\infty\int_0^\infty K(x,y)f(x)g(y)dxdy \leqslant I(p)\left(\int_0^\infty f^p(x)dx\right)^{\frac{1}{p}}\left(\int_0^\infty g^q(x)dx\right)^{\frac{1}{q}}, \tag{1.19}$$

which is Theorem 319 in [1]. In section 2 we prove two lemmas. In section 3, we prove Theorem 1–2. Finally, in section 4 we provide some applications.

2. Some lemmas

For the proof of the above theorems we need two lemmas.

LEMMA 1. *Under the same conditions as those of Theorem 1, define the weight function* ω *as*

$$\omega(r,x,\lambda) = (x+\lambda)^{2\frac{\lambda}{r}} \int_a^b K(x+\lambda, y+\lambda)(y+\lambda)^{-2\frac{\lambda}{r}}dy, \quad r > 1,\ a \leqslant x \leqslant b. \tag{2.1}$$

Then

$$\omega(r,x,\lambda) \leqslant (x+\lambda)^{1-t}[I(r,\lambda)-\varphi_1(r,x,t,\lambda)], \tag{2.2}$$

where $I(r,\lambda), \varphi_1(r,x,t,\lambda)$ *are indicated as* (1.6), (1.8)*, respectively.*

Proof. Let $u=\frac{y+\lambda}{x+\lambda}$, then we have

$$\begin{aligned}\omega(r,x,\lambda) &= (x+\lambda)^{1-t}\int_{\frac{a+\lambda}{x+\lambda}}^{\frac{b+\lambda}{x+\lambda}} K(1,u)u^{-2\frac{\lambda}{r}}\,du \\ &= (x+\lambda)^{1-t}\Bigg[\int_0^{\infty} - \int_0^{\frac{a+\lambda}{x+\lambda}} - \int_{\frac{b+\lambda}{x+\lambda}}^{\infty}\Bigg]K(1,u)u^{-2\frac{\lambda}{r}}\,du \\ &= (x+\lambda)^{1-t}[I(r,\lambda)-I_2-I_3]. \end{aligned}\tag{2.3}$$

Let $v=\frac{1}{u}$, then

$$I_3 = \int_0^{\frac{x+\lambda}{b+\lambda}} K(1,v)v^{t+2\frac{\lambda}{r}-2}dv. \tag{2.4}$$

Define h_1, h_2 by

$$h_1(y) = y^{2\frac{\lambda}{r}-1}\int_0^y K(1,u)u^{-2\frac{\lambda}{r}}\,du, \tag{2.5}$$

$$h_2(y) = y^{1-t-2\frac{\lambda}{r}}\int_0^y K(1,u)u^{t+2\frac{\lambda}{r}-2}du. \tag{2.6}$$

Taking derivation of h_1 and h_2 and using integration by parts, we obtain

$$\begin{aligned} h_1'(y) &= \Big(\frac{2\lambda}{r}-1\Big)y^{2\frac{\lambda}{r}-2}\int_0^y K(1,u)u^{-2\frac{\lambda}{r}}\,du + \frac{K(1,y)}{y} \\ &= ty^{2\frac{\lambda}{r}-2}\int_0^y u^{1-2\frac{\lambda}{r}}K_u'(1,u)du < 0, \end{aligned}$$

where $K(1,u)$ is a strictly decreasing function of u. It follows that $K_u'(1,u)<0$.

Similarly we get

$$h_2'(y) = ty^{-t-2\frac{\lambda}{r}}\int_0^y u^{t+2\frac{\lambda}{r}-1}K_u'(1,u)du < 0.$$

Hence, h_1 and h_2 are strictly monotonically decreasing function for $0 \leqslant y \leqslant 1$, that is

$$h_1(y) \geqslant h_1(1) = \int_0^1 K(1,u)u^{-2\frac{\lambda}{r}}du, \tag{2.7}$$

$$h_2(y) \geqslant h_2(1) = \int_0^1 K(1,u)u^{t+2\frac{\lambda}{r}-2}du. \tag{2.8}$$

Let $y = \frac{a+\lambda}{x+\lambda}$ in (2.7) for $a \leqslant x \leqslant b$, then

$$\begin{aligned} I_2 &= \int_0^{\frac{a+\lambda}{x+\lambda}} K(1,u)u^{-2\frac{\lambda}{r}}du = \left(\frac{a+\lambda}{x+\lambda}\right)^{1-2\frac{\lambda}{r}} h_1\left(\frac{a+\lambda}{x+\lambda}\right) \\ &\geqslant \left(\frac{a+\lambda}{x+\lambda}\right)^{1-2\frac{\lambda}{r}} \int_0^1 K(1,u)u^{-2\frac{\lambda}{r}}du. \end{aligned} \tag{2.9}$$

Let $y = \dfrac{x+\lambda}{b+\lambda}$ in (2.8) for $a \leqslant x \leqslant b$, then

$$\begin{aligned} I_3 &= \int_0^{\frac{x+\lambda}{b+\lambda}} K(1,u)u^{t+2\frac{\lambda}{r}-2}du = \left(\frac{x+\lambda}{b+\lambda}\right)^{t+2\frac{\lambda}{r}-1} h_2\left(\frac{x+\lambda}{b+\lambda}\right) \\ &\geqslant \left(\frac{x+\lambda}{b+\lambda}\right)^{t+2\frac{\lambda}{r}-1} \int_0^1 K(1,u)u^{t+2\frac{\lambda}{r}-2}du. \end{aligned} \tag{2.10}$$

From (2.3), (2.9) and (2.10), we get

$$\omega(r,x,\lambda) = (x+\lambda)^{1-t}[I(r,\lambda) - I_2 - I_3] \leqslant (x+\lambda)^{1-t}[I(r,\lambda) - \varphi_1(r,x,t,\lambda)].$$

Thus Lemma 1 is proved.

LEMMA 2. *Under the same conditions as those of Theorem 2, define the weight function* ω *by*

$$\omega(r,x) = \int_a^b K(x,y)(x/y)^{\frac{1}{r}}dy, \quad r>1,\ a \leqslant x \leqslant b. \tag{2.11}$$

Then

$$\omega(r,x) \leqslant x^{1-t}\left[\int_0^\infty K(1,u)u^{-\frac{1}{r}}du - \varphi_2(r,x)\right], \tag{2.12}$$

where $\varphi_2(r,x)$ *is given by* (1.18).

Proof. Let $u = \dfrac{y}{x}$, then we have

$$
\begin{aligned}
\omega(r,x) &= \int_{\frac{a}{x}}^{\frac{b}{x}} K(x,ux)u^{-\frac{1}{r}}x\,du = x^{1-t}\int_{\frac{a}{x}}^{\frac{b}{x}} K(1,u)u^{-\frac{1}{r}}du \\
&= x^{1-t}\left(\int_0^{\infty} - \int_0^{\frac{a}{x}} - \int_{\frac{b}{x}}^{\infty}\right)K(1,u)u^{-\frac{1}{r}}du \\
&= x^{1-t}(I(r) - I_4 - I_5). \qquad (2.13)
\end{aligned}
$$

Let $v = \dfrac{1}{u}$, then

$$
I_5 = \int_0^{\frac{x}{b}} K(1,u)u^{t+\frac{1}{r}-2}du.
$$

Define h_3, h_4 by

$$
h_3(y) = y^{\frac{1}{r}-1}\int_0^{y} K(1,u)u^{-\frac{1}{r}}du,
$$

$$
h_4(y) = y^{1-t-\frac{1}{r}}\int_0^{y} K(1,u)u^{t+\frac{1}{r}-2}du.
$$

We can similarly show that $h_3'(y) < 0$, $h_4'(y) < 0$. Hence

$$
h_3(y) \geqslant h_3(1) = \int_0^1 K(1,u)u^{-\frac{1}{r}}du, \qquad (2.14)
$$

$$
h_4(y) \geqslant h_4(1) = \int_0^1 K(1,u)u^{t+\frac{1}{r}-2}du. \qquad (2.15)
$$

Let $y = \dfrac{a}{x}$ in (2.14). Then

$$
I_4 = \int_0^{\frac{a}{x}} K(1,u)u^{-\frac{1}{r}}du = \left(\frac{a}{x}\right)^{1-\frac{1}{r}} h_3(y) \geqslant \left(\frac{a}{x}\right)^{1-\frac{1}{r}} \int_0^1 K(1,u)u^{-\frac{1}{r}}du. \qquad (2.16)
$$

Next let $y = \dfrac{x}{b}$ in (2.15). We obtain

$$
I_5 = \int_0^{\frac{x}{b}} K(1,u)u^{t+\frac{1}{r}-2}du = \left(\frac{x}{b}\right)^{t+\frac{1}{r}-1} h_4(y) \geqslant \left(\frac{x}{b}\right)^{t+\frac{1}{r}-1} \int_0^1 K(1,u)u^{t+\frac{1}{r}-2}. \qquad (2.17)
$$

From (2.13), (2.16) and (2.17) we get

$$\omega(r,x) \leqslant x^{1-t}[I(r) - \varphi_2(r,x)],$$

where $I(r)$, $\varphi_2(r,x)$ are given by (1.14), (1.18), respectively. Hence, Lemma 2 is proved.

3. Proof of Theorems

Proof of Theorem 1. By Hölder's inequality, we have

$$\begin{aligned}
&\int_a^b \int_a^b K(x+\lambda, y+\lambda) f(x) g(y) dxdy \\
&= \int_a^b \int_a^b f(x) K(x+\lambda, y+\lambda)^{\frac{1}{p}} \left(\frac{x+\lambda}{y+\lambda}\right)^{\frac{2\lambda}{pq}} g(y) K(x+\lambda, y+\lambda)^{\frac{1}{q}} \left(\frac{y+\lambda}{x+\lambda}\right)^{\frac{2\lambda}{pq}} dxdy \\
&\leqslant \left\{\int_a^b f^p(x)(x+\lambda)^{\frac{2\lambda}{q}} \left(\int_a^b K(x+\lambda, y+\lambda)(y+\lambda)^{-\frac{2\lambda}{q}} dy\right) dx\right\}^{\frac{1}{p}} \times \\
&\qquad \times \left\{\int_a^b g^q(y)(y+\lambda)^{\frac{2\lambda}{p}} \left(\int_a^b K(x+\lambda, y+\lambda)(x+\lambda)^{-\frac{2\lambda}{p}} dx\right) dy\right\}^{\frac{1}{q}} \\
&= \left\{\int_a^b \omega(q,x,\lambda) f^p(x) dx\right\}^{\frac{1}{p}} \left\{\int_a^b \omega(p,x,\lambda) g^q(x) dx\right\}^{\frac{1}{q}}. \qquad (3.1)
\end{aligned}$$

Then, by (2.2), we get (1.7).

Next, to prove (1.9), we take limits as $b \to \infty$ in (1.7) and note that

$$\inf_{r>1} \int_0^1 K(1,u) u^{-2\frac{\lambda}{r}} du = \lim_{r\to\infty} \int_0^1 K(1,u) u^{-2\frac{\lambda}{r}} du = \int_0^1 K(1,u) du.$$

This proves (1.9). We can similarly prove (1.10)–(1.13). Thus, Theorem 1 is proved.

Proof of Theorem 2. By Hölder's inequality, we have

$$\begin{aligned}
&\int_a^b \int_a^b K(x,y) f(x) g(y) dxdy = \int_a^b \int_a^b f(x) K(x,y)^{\frac{1}{p}} \left(\frac{x}{y}\right)^{\frac{1}{pq}} g(y) K(x,y)^{\frac{1}{q}} \left(\frac{y}{x}\right)^{\frac{1}{pq}} dxdy \\
&\leqslant \int_a^b f^p(x) x^{\frac{1}{q}} \left\{\left(\int_a^b K(x,y) y^{-\frac{1}{q}} dy\right) dx\right\}^{\frac{1}{p}} \left\{\int_a^b g^q(y) y^{\frac{1}{p}} \left(\int_a^b K(x,y) x^{-\frac{1}{p}} dx\right) dy\right\}^{\frac{1}{q}} \\
&= \left\{\int_a^b \omega(q,x) f^p(x) dx\right\}^{\frac{1}{p}} \left\{\int_a^b \omega(p,x) g^q(x) dx\right\}^{\frac{1}{q}}. \qquad (3.2)
\end{aligned}$$

Then, by (2.12), we get (1.15). We can similarly prove the rest of Theorem 2. Details are omitted. The proof is completed.

4. Some applications

4.1. Take K to be defined by

$$K(x,y) = (x+y)^{-t}. \tag{4.1}$$

Then by Theorem 1 we get the following

THEOREM 3. *Let* $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $\lambda > 0$, $0 < 1 - 2\frac{\lambda}{r} < t$, $r = p, q$, $0 < a < b$. *If* f *and* g *are nonnegative measurable functions defined on* $[a,b]$, *then*

(1) *If* $0 < \lambda \leqslant \frac{1}{2}$, *then*

$$\int_a^b \int_a^b \frac{f(x)g(y)}{(x+y+2\lambda)^t} dxdy \leqslant \left\{ \int_a^b \left[B\left(1 - 2\frac{\lambda}{q}, t + 2\frac{\lambda}{q} - 1\right) - \varphi_3(q,x,t,\lambda) \right] (x+\lambda)^{1-t} f^p(x) dx \right\}^{\frac{1}{p}} \times \left\{ \int_a^b \left[B\left(1 - 2\frac{\lambda}{p}, t + 2\frac{\lambda}{p} - 1\right) - \varphi_3(q,x,t,\lambda) \right] (x+\lambda)^{1-t} g^q(x) dx \right\}^{\frac{1}{q}}; \tag{4.2}$$

$$\int_a^\infty \int_a^\infty \frac{f(x)g(y)}{(x+y+2\lambda)^t} dxdy \leqslant \left\{ \int_a^\infty \left[B\left(1-2\frac{\lambda}{q}, t+2\frac{\lambda}{q}-1\right) - \left(\frac{a+\lambda}{x+\lambda}\right)^{1-2\frac{\lambda}{q}} H(t) \right] (x+\lambda)^{1-t} f^p(x) dx \right\}^{\frac{1}{p}} \times \left\{ \int_a^\infty \left[B\left(1-2\frac{\lambda}{p}, t+2\frac{\lambda}{p}-1\right) - \left(\frac{a+\lambda}{x+\lambda}\right)^{1-2\frac{\lambda}{p}} H(t) \right] (x+\lambda)^{1-t} g^q(x) dx \right\}^{\frac{1}{q}}; \tag{4.3}$$

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{(x+y+2\lambda)^t} dxdy \leqslant \left\{ \int_0^\infty \left[B\left(1-2\frac{\lambda}{q}, 2\frac{\lambda}{q}\right) - \left(\frac{\lambda}{x+\lambda}\right)^{1-2\frac{\lambda}{q}} \ln 2 \right] f^p(x) dx \right\}^{\frac{1}{p}} \times \left\{ \int_0^\infty \left[B\left(1 - 2\frac{\lambda}{p}, 2\frac{\lambda}{p}\right) - \left(\frac{\lambda}{x+\lambda}\right)^{1-2\frac{\lambda}{p}} \ln 2 \right] g^q(x) dx \right\}^{\frac{1}{q}}, \tag{4.4}$$

where

$$\varphi_3(r,x,t,\lambda)=\left(\frac{a+\lambda}{x+\lambda}\right)^{1-2\frac{\lambda}{r}}\int_0^1\frac{u^{-2\frac{\lambda}{r}}}{(1+u)^t}du+\left(\frac{x+\lambda}{b+\lambda}\right)^{t+2\frac{\lambda}{r}-1}\int_0^1\frac{u^{t+2\frac{\lambda}{r}-2}}{(1+u)^t}du, \tag{4.5}$$

$$H(t)=\begin{cases}\dfrac{2^{1-t}-1}{1-t}, & t\neq 1\\ \ln 2, & t=1,\end{cases} \tag{4.6}$$

and $B(u,v)$ *is the Beta function.*

(2) *If* $\lambda > \frac{1}{2}$, *then*

$$\begin{aligned}&\int_a^b\int_a^b\frac{f(x)g(y)}{(x+y+2\lambda)^t}dxdy \leqslant \int_a^b\int_a^b\frac{f(x)g(y)}{(x+y+1)^t}dxdy\\ &\leqslant\left\{\int_a^b\left[B\left(\frac{1}{p},t-\frac{1}{p}\right)-\varphi_3(q,x,t,\frac{1}{2})\right](x+\frac{1}{2})^{1-t}f^p(x)dx\right\}^{\frac{1}{p}}\times\\ &\times\left\{\int_a^b\left[B\left(\frac{1}{p},t-\frac{1}{q}\right)-\varphi_3(p,x,t,\frac{1}{2})\right](x+\frac{1}{2})^{1-t}g^q(x)dx\right\}^{\frac{1}{q}};\end{aligned} \tag{4.7}$$

$$\begin{aligned}&\int_a^\infty\int_a^\infty\frac{f(x)g(y)}{(x+y+2\lambda)^t}dxdy \leqslant \int_a^\infty\int_a^\infty\frac{f(x)g(y)}{(x+y+1)^t}dxdy\\ &\leqslant\left\{\int_a^\infty\left[B\left(\frac{1}{p},t-\frac{1}{p}\right)-\left(\frac{2a+1}{2x+1}\right)^{\frac{1}{p}}H(t)\right](x+\frac{1}{2})^{1-t}f^p(x)dx\right\}^{\frac{1}{p}}\times\\ &\times\left\{\int_a^\infty\left[B\left(\frac{1}{q},t-\frac{1}{q}\right)-\left(\frac{2a+1}{2x+1}\right)^{\frac{1}{q}}H(t)\right](x+\frac{1}{2})^{1-t}g^q(x)dx\right\}^{\frac{1}{q}};\end{aligned} \tag{4.8}$$

$$\begin{aligned}&\int_0^\infty\int_0^\infty\frac{f(x)g(y)}{x+y+2\lambda}dxdy \leqslant \int_0^\infty\int_0^\infty\frac{f(x)g(y)}{x+y+1}dxdy\\ &\leqslant\left\{\int_0^\infty\left[B\left(\frac{1}{p},\frac{1}{q}\right)-\frac{\ln 2}{(2x+1)^{\frac{1}{p}}}\right]f^p(x)dx\right\}^{\frac{1}{p}}\left\{\int_0^\infty\left[B\left(\frac{1}{p},\frac{1}{q}\right)-\frac{\ln 2}{(2x+1)^{\frac{1}{q}}}\right]g^q(x)dx\right\}^{\frac{1}{q}}.\end{aligned} \tag{4.9}$$

(3) *If* $p=q=2$, *then*

$$\int_0^\infty\int_0^\infty\frac{f(x)g(y)}{x+y+2\lambda}dxdy \leqslant\left(\int_0^\infty\omega(x,\lambda)f^2(x)dx\right)^{\frac{1}{2}}\left(\int_0^\infty\omega(x,\lambda)g^2(x)dx\right)^{\frac{1}{2}}, \tag{4.10}$$

where

$$\omega(x,\lambda)=\begin{cases}\dfrac{\pi}{\sin\lambda\pi}-\left(\dfrac{\lambda}{x+\lambda}\right)^{1-\lambda}\ln 2, & 0<\lambda\leqslant\dfrac{1}{2}\\ \pi-\dfrac{\ln 2}{(2x+1)^{\frac{1}{2}}}, & \lambda>\dfrac{1}{2}.\end{cases} \tag{4.11}$$

Proof. We only note that (4.1) implies that

$$I(r,\lambda)=\int_0^\infty \frac{y^{-2\frac{\lambda}{r}}}{(1+y)^t}dy=B\Big(1-\frac{2\lambda}{r},t+\frac{2\lambda}{r}-1\Big)$$

and

$$H(t)=\int_0^1 K(1,u)du=\int_0^1(1+u)^{-t}du=\begin{cases}\dfrac{2^{1-t}-1}{1-t}, & t\neq 1\\ \ln 2, & t=1.\end{cases}$$

For $p=q=2$, if $0\leqslant\lambda\leqslant\frac{1}{2}$, then by (1.6), (4.1) and (4.5), we get

$$I(2,\lambda)=\int_0^\infty\frac{y^{-\lambda}}{(1+y)^t}dy=B(1-\lambda,t+\lambda-1),$$

$$\varphi_3(2,x,t,\lambda)=\left(\frac{a+\lambda}{x+\lambda}\right)^{1-\lambda}\int_0^1\frac{u^{-\lambda}}{(1+u)^t}du+\left(\frac{x+\lambda}{b+\lambda}\right)^{t+\lambda-1}\int_0^1\frac{u^{t+\lambda-2}}{(1+u)^t}du.$$

In particular, for $t=1$, $\lambda>0$, $a=0$, $b=\infty$, we obtain

$$\omega(x,\lambda)\leqslant B(1-\lambda,\lambda)-\left(\frac{\lambda}{x+\lambda}\right)^{1-\lambda}\int_0^1(1+u)^{-1}du=\frac{\pi}{\sin\lambda\pi}-\left(\frac{\lambda}{x+\lambda}\right)^{1-\lambda}\ln 2;$$

Similarly, if $\lambda>\frac{1}{2}$, we get

$$\omega\big(x,\frac{1}{2}\big)\leqslant\pi-\ln 2(2x+1)^{-\frac{1}{2}}.$$

The rest of the proof can be completed by following the same method as in the proof of Theorem 1 and hence we omit the details.

REMARK 2. Inequality (4.10) is a new improvement of the integral analogue of (1.3).

For $\lambda=0$, we can prove the following:

THEOREM 4. *Let* $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $\max\left\{\frac{1}{p}, \frac{1}{q}\right\} < t$, $0 < a < b$. *If* f *and* g *are nonnegative measurable functions defined on* $[a,b]$, *than*

$$\int_a^b \int_a^b \frac{f(x)g(y)}{(x+y)^t} dxdy \leqslant \left\{ \int_a^b \left[B\left(\frac{1}{p}, t - \frac{1}{p}\right) - \varphi_4(q,x) \right] x^{1-t} f^p(x) dx \right\}^{\frac{1}{p}} \times$$
$$\times \left\{ \int_a^b \left[B\left(\frac{1}{q}, t - \frac{1}{q}\right) - \varphi_4(p,x) \right] x^{1-t} g^q(x) dx \right\}^{\frac{1}{q}}; \tag{4.12}$$

$$\int_a^\infty \int_a^\infty \frac{f(x)g(y)}{(x+y)^t} dxdy \leqslant \left\{ \int_a^\infty \left[B\left(\frac{1}{p}, t - \frac{1}{p}\right) - \left(\frac{a}{x}\right)^{\frac{1}{p}} H(t) \right] x^{1-t} f^p(x) dx \right\}^{\frac{1}{p}} \times$$
$$\times \left\{ \int_a^\infty \left[B\left(\frac{1}{q}, t - \frac{1}{q}\right) - \left(\frac{a}{x}\right)^{\frac{1}{q}} H(t) \right] x^{1-t} g^q(x) dx \right\}^{\frac{1}{q}}; \tag{4.13}$$

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{(x+y)^t} dxdy$$
$$\leqslant B\left(\frac{1}{p}, t - \frac{1}{p}\right)^{\frac{1}{p}} B\left(\frac{1}{q}, t - \frac{1}{q}\right)^{\frac{1}{q}} \left\{ \int_0^\infty x^{1-t} f^p(x) dx \right\}^{\frac{1}{p}} \left\{ \int_0^\infty x^{1-t} g^q(x) dx \right\}^{\frac{1}{q}}, \tag{4.14}$$

where

$$\varphi_4(r,x) = \left(\frac{a}{x}\right)^{1-\frac{1}{r}} \int_0^1 \frac{u^{-\frac{1}{r}}}{(1+u)^t} du + \left(\frac{x}{b}\right)^{t+\frac{1}{r}-1} \int_0^1 \frac{u^{t+\frac{1}{r}-2}}{(1+u)^t} du, \tag{4.15}$$

and $H(t)$ *is given by* (4.6).

If $t = 1$, *then* (4.14) *implies that*

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x+y} dxdy \leqslant \frac{\pi}{\sin \frac{\pi}{p}} \left\{ \int_0^\infty f^p(x) dx \right\}^{\frac{1}{p}} \left\{ \int_0^\infty g^q(x) dx \right\}^{\frac{1}{q}}. \tag{4.16}$$

In particular, for $p = q = 2$, (4.16) *yields* (1.1).

REMARK 3. If we take

$$K(x,y) = (x^t + y^t)^{-1} \tag{4.17}$$

we get similar results (see [5]).

4.2. Take K to be defined by

$$K(x,y) = \frac{\ln \frac{y}{x}}{y - x}, \tag{4.18}$$

then (1.6) implies that

$$I(r,\lambda) = \int_0^\infty \frac{\ln u}{u-1} u^{-2\frac{\lambda}{r}} du = \frac{\pi^2}{\sin \frac{2\lambda\pi}{r}} \tag{4.19}$$

and (1.14) implies that

$$I(r) = \int_0^\infty \frac{\ln u}{u-1} u^{-\frac{1}{r}} du = \frac{\pi^2}{\sin \frac{\pi}{r}}. \tag{4.20}$$

Thus, (1.6) in Theorem 1 is replaced by (4.19), and (1.14) in Theorem 2 is replaced by (4.20). We can write e.g.

$$\int_0^\infty \int_0^\infty \frac{\ln \frac{y}{x}}{y-x} f(x)g(y)dxdy \leqslant \frac{\pi^2}{\sin \frac{\pi}{p}} \left(\int_0^\infty f^p(x)dx \right)^{\frac{1}{p}} \left(\int_0^\infty g^q(x)dx \right)^{\frac{1}{q}}, \tag{4.21}$$

which is Theorem 342 in [1].

4.3. Take K to be defined by

$$K(x,y) = \max\{x,y\}^{-t}, \quad 0 < 1 - 2\frac{\lambda}{r} < t. \tag{4.22}$$

Then $K(1,u) = \max\{1,u\}^{-t}$. By (1.6), we have

$$\begin{aligned} I(r,\lambda) &= \int_0^\infty K(1,u)u^{-2\frac{\lambda}{r}} du = \int_0^1 u^{-2\frac{\lambda}{r}} du + \int_1^\infty u^{-t-2\frac{\lambda}{r}} du \\ &= \frac{1}{1-2\frac{\lambda}{r}} - \frac{1}{1-t-2\frac{\lambda}{r}}, \end{aligned} \tag{4.23}$$

and (1.14) implies that

$$I(r) = \int_0^\infty K(1,u)u^{-\frac{1}{r}} du = \int_0^1 u^{-\frac{1}{r}} du + \int_1^\infty u^{-t-\frac{1}{r}} du = \frac{1}{1-\frac{1}{r}} - \frac{1}{1-t-\frac{1}{r}}. \tag{4.24}$$

Thus, (1.6) in Theorem 1 is replaced by (4.23) and (1.14) in Theorem 2 is replaced by (4.24). We get the desired results, e.g., since for $t=1$, (4.24) implies that

$$I(p) = \frac{p}{p-1} + p = \frac{p^2}{p-1} = pq,$$

thus by (1.19) and (4.25), we get

$$\int_0^\infty \int_0^\infty \frac{f(x)g(y)}{\max\{x,y\}} dxdy \leqslant pq \left(\int_0^\infty f^p(x)dx \right)^{\frac{1}{p}} \left(\int_0^\infty g^q(x)dx \right)^{\frac{1}{q}},$$

which is Theorem 341 in [1].

REFERENCES

[1] G. H. HARDY, J. E. LITTLEWOOD AND G. POLYA, *Inequalities*, Cambridge University Press, Cambridge, MA, 1952.
[2] A. E. INGHAM, *A note on Hilbert's inequality*, J. London Math. Soc. **11** (1936), 237–240.
[3] YANG BICHENG, *On Hilbert's integral inequality*, J. Math. Anal. Appl. **220** (1998), 778–785.
[4] GAO MINGZHE, TAN LI AND L. DEBNATH, *Some improvements on Hilbert's integral inequality*, J. Math. Anal. Appl. **229** (1999), 682–689.
[5] KUANG JICHANG, *On Hilbert's integral inequality*, J. Math. Anal. Appl. **233** (1999).
[6] YANG BICHENG, *On generalizations of Hardy-Hilbert's integral inequalities*, Acta Math. Sinica **41 No. 4** (1998), 839–844.

(Received October 15, 1999)

Kuang Jichang
Department of Mathematics
Hunan Normal University
Changsha
Hunan, 410081
P.R. China
e-mail: jckuang@hotmail.com

Themistocles M. Rassias
Department of Mathematics
National Technical University of Athens
Zografou Campus
15780 Athens
Greece
e-mail: trassias@math.ntua.gr