Mathematical
Inequalities
& Applications
Volume 3, Number 4 (2000), 511–518

CONVERGENCE OF GENERALIZED SINGULAR INTEGRALS TO THE UNIT, UNIVARIATE CASE

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(*communicated by J. Marshall Ash*)

Abstract. In a recent paper, the second author (see [2]) studied the degree of uniform approximation to the unit in terms of uniform moduli of smoothness, by the Jackson-type generalizations of Picard and of Gauss-Weierstrass singular integrals. In this paper we consider the L^p-approximation, $(1 \leqslant p < +\infty)$ by the above singular integrals in terms of the L^p-moduli of smoothness, and both uniform and L^p-approximation (in terms of the corresponding moduli of smoothness) by Jackson-type generalizations of the Poisson-Cauchy singular integrals.

1. Introduction

Let f be a function from $\mathbf{R}$ into itself. For $r \in \mathbf{N}$, the $rth - L_p$-modulus of smoothness over $\mathbf{R}$ $(1 \leqslant p \leqslant +\infty)$ is defined by

$$\omega_r(f;\delta)_X = \sup_{|h|\leqslant\delta} \|\Delta_h^r f\|_X,$$

where

$$\Delta_h^r f(x) = \sum_{i=0}^{r} (-1)^i \binom{r}{i} f(x+ih), \quad r \in \mathbf{N},$$

$$X = L^p(\mathbf{R}) \quad \text{or} \quad X = L^p_{2\pi}(\mathbf{R}),$$

$$\|f\|_{L^p(\mathbf{R})} = \left(\int_{-\infty}^{+\infty} |f(x)|^p dx\right)^{1/p}, \quad \|f\|_{L^p_{2\pi}(\mathbf{R})} = \left(\int_{-\pi}^{\pi} |f(x)|^p dx\right)^{1/p}.$$

Next, for $\xi > 0$ we consider the Jackson-type generalizations of Picard, Poisson-Cauchy and Gauss-Weierstrass singular integrals introduced in [2] by

$$P_{n,\xi}(f;x) = -\frac{1}{2\xi} \sum_{k=1}^{n+1} (-1)^k \binom{n+1}{k} \int_{-\infty}^{+\infty} f(x+kt) e^{-|t|/\xi} dt,$$

$$Q_{n,\xi}(f;x) = \frac{1}{-\left(\frac{2}{\xi}\right)\tan^{-1}\left(\frac{\pi}{\xi}\right)} \sum_{k=1}^{n+1} (-1)^k \binom{n+1}{k} \int_{-\pi}^{\pi} \frac{f(x+kt)}{t^2+\xi^2} dt,$$

Mathematics subject classification (1991): 41A17, 41A25, 41A35.

Key words and phrases: generalized singular integrals, convergence to the unit with rates, moduli of smoothness.

and

$$W_{n,\xi}(f;x) = -\frac{1}{2C(\xi)} \sum_{k=1}^{n+1} (-1)^k \binom{n+1}{k} \int_{-\pi}^{\pi} f(x+kt) e^{-t^2/\xi^2} dt,$$
$$C(\xi) = \int_0^{\pi} e^{-t^2/\xi^2} dt,$$

respectively (the above operators are introduced by generalizing the usual Picard, Poisson-Cauchy and Gauss-Weierstrass singular integrals, by following the same idea which is used to define the Jackson's generalized operator in classical approximation theory).

Here we consider only f such that $P_{n,\xi}(f;x)$, $Q_{n,\xi}(f;x)$, $W_{n,\xi}(f;x) \in \mathbf{R}$, for all $x \in \mathbf{R}$.

Uniform convergences to the unit of $P_{n,\xi}$, $W_{n,\xi}$ operators (as $\xi \to 0$) have been established in [2] and can be stated by the following

THEOREM 1.1. *Let $f \in C_{2\pi}(\mathbf{R})$. We have:*

(i) $\|f - P_{n,\xi}(f)\| \leqslant \left[\sum_{k=0}^{n+1} \binom{n+1}{k} k!\right] \omega_{n+1}(f;\xi),\ \xi > 0;$

(ii) $\|f - W_{n,\xi}\| \leqslant \left[1/\int_0^{\pi} e^{-u^2} du\right] \left[\int_0^{+\infty} (u+1)^{n+1} e^{-u^2} du\right] \omega_{n+1}(f;\xi),$

$0 \leqslant \xi \leqslant 1$, *where* $\|\cdot\|$ *is the uniform norm on* $C_{2\pi}(\mathbf{R})$ *and* $\omega_r(f;\xi)$ *is the rth uniform modulus of smoothness.*

In Section 2 we consider the L^p-approximation, $1 \leqslant p < +\infty$, for $P_{n,\xi}, Q_{n,\xi}, W_{n,\xi}$ operators, while Section 3 contains uniform convergence for $Q_{n,\xi}$ operator (in order to complete somehow the above Theorem 1.1).

2. L^p-approximation, $1 \leqslant p < +\infty$

The first main result of this section is

THEOREM 2.1. *Here take $X = L^1(\mathbf{R})$ (for $P_{n,\xi}$), $X = L^1_{2\pi}(\mathbf{R})$ (for $W_{n,\xi}, Q_{n,\xi}$), $\xi > 0$, $n \in \mathbf{N}$, $f \in X$. Then*

$$\|f - P_{n,\xi}\|_X \leqslant \left[\sum_{k=0}^{n+1} \binom{n+1}{k} k!\right] \omega_{n+1}(f;\xi)_X, \quad \xi > 0, \tag{1}$$

$$\|f - W_{n,\xi}(f)\|_X \leqslant \left[1/\int_0^{\pi} e^{-u^2} du\right] \left[\int_0^{+\infty} (u+1)^{n+1} e^{-u^2} du\right] \omega_{n+1}(f;\xi)_{L^1_{2\pi}(\mathbf{R})},$$
$$0 < \xi \leqslant 1, \tag{2}$$

$$\|f - Q_{n,\xi}(f)\|_X \leqslant K(n,\xi)\omega_{n+1}(f;\xi)_{L^1_{2\pi}(\mathbf{R})}, \quad \xi > 0, \tag{3}$$

where $K(n,\xi) = \left[1/\tan^{-1}\frac{\pi}{\xi}\right] \int_0^{\pi/\xi} \frac{(u+1)^{n+1}}{u^2+1} du.$

Proof. We have (for $X = L^1(\mathbf{R})$)

$$f(x) - P_{n,\xi}(f;x) = (2\xi)^{-1} \int_{-\infty}^{+\infty} (-1)^{n+1} \Delta_t^{n+1} f(x) e^{-|t|/\xi} dt, \tag{4}$$

which implies

$$\begin{aligned} \|f - P_{n,\xi}(f)\|_{L^1(\mathbf{R})} &= \int_{-\infty}^{+\infty} |f(x) - P_{n,\xi}(f;x)| dx \\ &\leqslant (2\xi)^{-1} \int_{-\infty}^{+\infty} \left[\int_{-\infty}^{+\infty} |\Delta_t^{n+1} f(x)| dx \right] e^{-|t|/\xi} dt \\ &\leqslant (2\xi)^{-1} \int_{-\infty}^{+\infty} \omega_{n+1}(f;|t|)_{L^1(\mathbf{R})} e^{-|t|/\xi} dt \\ &= (2\xi)^{-1} \int_{-\infty}^{+\infty} \omega_{n+1}(f;\xi(|t|/\xi))_{L^1(\mathbf{R})} e^{-|t|/\xi} dt \\ &\leqslant (2\xi)^{-1} \omega_{n+1}(f;\xi)_{L^1(\mathbf{R})} \int_{-\infty}^{+\infty} (|t|/\xi + 1)^{n+1} e^{-|t|/\xi} dt \\ &= \xi^{-1} \omega_{n+1}(f;\xi)_{L^1(\mathbf{R})} \int_0^{+\infty} [t/\xi + 1]^{n+1} e^{-t/\xi} dt \\ &= \omega_{n+1}(f;\xi)_{L^1(\mathbf{R})} \int_0^{\infty} [u+1]^{n+1} e^{-u} du \\ &= \left[\sum_{k=0}^{n+1} \binom{n+1}{k} k! \right] \omega_{n+1}(f;\xi)_{L^1(\mathbf{R})}, \end{aligned}$$

which proves (1).

Then,

$$f(x) - W_{n,\xi}(f;x) = [1/2C(\xi)] \int_{-\pi}^{\pi} (-1)^{n+1} \Delta_t^{n+1} f(x) e^{-t^2/\xi^2} dt, \tag{5}$$

and reasoning as above, we get (for $0 < \xi \leqslant 1$)

$$\begin{aligned} \|f - W_{n,\xi}(f)\|_{L^1_{2\pi}(\mathbf{R})} &= \int_{-\pi}^{\pi} |f(x) - W_{n,\xi}(f;x)| dx \\ &\leqslant [1/(2C(\xi))] \omega_{n+1}(f;\xi)_{L^1_{2\pi}(\mathbf{R})} \int_0^{\pi} [t/\xi + 1]^{n+1} e^{-t^2/\xi^2} dt \\ &= \left[\frac{\xi}{C(\xi)} \right] \omega_{n+1}(f;\xi)_{L^1_{2\pi}(\mathbf{R})} \int_0^{\pi/\xi} [u+1]^{n+1} e^{-u^2} du \\ &\qquad \text{(see also e.g. [2, Lemma 3.2])} \\ &\leqslant \left[1 \Big/ \int_0^{\pi} e^{-u^2} du \right] \left[\int_0^{+\infty} (u+1)^{n+1} e^{-u^2} du \right] \omega_{n+1}(f;\xi)_{L^1_{2\pi}(\mathbf{R})}, \end{aligned}$$

which proves (2).

Finally,

$$f(x) - Q_{n,\xi}(f;x) = \frac{1}{\frac{2}{\xi}\tan^{-1}\frac{\pi}{\xi}} \int_{-\pi}^{\pi} \frac{(-1)^{n+1}}{t^2+\xi^2} \Delta_t^{n+1} f(x) dt, \tag{6}$$

and as above, we obtain

$$\begin{aligned} \|f - Q_{n,\xi}(f)\|_{L^1_{2\pi}(\mathbf{R})} &= \int_{-\pi}^{\pi} |f(x) - Q_{n,\xi}(f;x)| dx \\ &\leqslant \frac{\xi}{\tan^{-1}\frac{\pi}{\xi}} \omega_{n+1}(f;\xi)_{L^1_{2\pi}(\mathbf{R})} \int_0^{\pi} \frac{[t/\xi+1]^{n+1}}{t^2+\xi^2} dt \\ &= \frac{1}{\tan^{-1}\frac{\pi}{\xi}} \omega_{n+1}(f;\xi)_{L^1_{2\pi}(\mathbf{R})} \int_0^{\pi/\xi} \frac{(u+1)^{n+1}}{u^2+1} du, \end{aligned}$$

which proves (3).

REMARK. For fixed $n \in \mathbf{N}$, by (1) and (2) it follows that

$$\|f - P_{n,\xi}(f)\|_X \to 0, \quad \|f - W_{n,\xi}(f)\|_X \to 0, \text{ as } \xi \to 0.$$

On the other hand, because $K(n,\xi) \to +\infty$, as $\xi \to 0$, by (3) we do not obtain, in general, the convergence $\|f - Q_{n,\xi}(f)\|_X \to 0$, as $\xi \to 0$. However, in some particular cases the convergence holds, as can be seen by the following.

COROLLARY 2.1. *If $f^{(n+1)} \in L^1_{2\pi}(\mathbf{R})$ and $f^{(n)}$ is absolutely continuous on $\mathbf{R}$, then*

$$\|f - Q_{n,\xi}(f)\|_{L^1_{2\pi}(\mathbf{R})} \leqslant C_n \xi, \quad 0 < \xi \leqslant 1$$

where $C_n > 0$ is a constant independent of f and ξ.

Proof. We have

$$\omega_{n+1}(f;\xi)_{L^1_{2\pi}(\mathbf{R})} \leqslant C_1 \xi^{n+1} \|f^{(n+1)}\|_{L^1_{2\pi}(\mathbf{R})}$$

and for $0 < \xi \leqslant 1$,

$$\begin{aligned} \xi^{n+1} \int_0^{\pi/\xi} \frac{(u+1)^{n+1}}{u^2+1} du &= \xi^{n+1} \left[\int_0^1 \frac{(u+1)^{n+1}}{u^2+1} du + \int_1^{\pi/\xi} \frac{(u+1)^{n+1}}{u^2+1} du \right] \\ &= \xi^{n+1} \left[C_2 + \int_1^{\pi/2} \frac{(u+1)^{n+1}}{u^2+1} du \right] \leqslant \xi^{n+1} \left[C_2 + \int_1^{\pi/\xi} \frac{(u+1)^{n+1}}{u^2} du \right] \\ &\leqslant \xi^{n+1} \left[C_2 + \sum_{k=0}^{n+1} \binom{n+1}{k} \int_1^{\pi/\xi} u^{n-k-1} du \right] \\ &= \xi^{n+1} \left\{ C_2 + \left[\sum_{k=0}^{n-1} \binom{n+1}{k} \frac{u^{n-k}}{n-k} \Big|_1^{\pi/\xi} + (n+1) \ln u \Big|_1^{\pi/\xi} - \frac{1}{u} \Big|_1^{\pi/\xi} \right] \right\} \leqslant C\xi, \end{aligned}$$

which together with relation (3) proves the corollary.

The second main result of the section is

THEOREM 2.2. *Let us consider* $X = L^p(\mathbf{R})$ *(for* $P_{n,\xi}$ *),* $X = L^p_{2\pi}(\mathbf{R})$ *(for* $W_{n,p}, Q_{n,\xi}$ *),* $0 < \xi \leqslant 1,\ n \in \mathbf{N},\ 1 < p < +\infty,\ \frac{1}{p} + \frac{1}{q} = 1,\ f \in X.$

Then

$$\|f - P_{n,\xi}(f)\|_X \leqslant (2/q)^{1/q}\|g\|_{L^p(\mathbf{R}_+)}\omega_{n+1}(f;\xi)_X,$$

where $g(u) = (u+1)^{n+1}e^{-u/2}$,

$$\|f - W_{n,\xi}(f)\|_X \leqslant \left(\sqrt{\frac{\pi}{2q}}\right)^{1/q} \frac{1}{\int_0^{\pi} e^{-u^2}du}\|h\|_{L^p(\mathbf{R}_+)}\omega_{n+1}(f;\xi)_X,$$

where $h(u) = (u+1)^{n+1}e^{-u^2/2}$,

$$\|f - Q_{n,\xi}(f)\|_X \leqslant K_p(n,\xi)\omega_{n+1}(f;\xi)_{L^p_{2\pi}(\mathbf{R})},$$

where $K_p(n,\xi) = \left[\frac{1}{\tan^{-1}\frac{\pi}{\xi}}\int_0^{\pi/\xi}(u+1)^{(n+1)p}\frac{1}{u^2+1}du\right]^{1/p}.$

Proof. Let $X = L^p(\mathbf{R})$, $\frac{1}{p} + \frac{1}{q} = 1$ and $C_1 = \frac{1}{(2\xi)^p}\left(\frac{4\xi}{q}\right)^{p/q}$. By (4) we obtain

$$\begin{aligned}
&\int_{-\infty}^{+\infty}|f(x) - P_{n,\xi}(f;x)|^p dx \\
&= \frac{1}{(2\xi)^p}\int_{-\infty}^{+\infty}\left|\int_{-\infty}^{+\infty}(-1)^{n+1}\Delta_t^{n+1}f(x)e^{-|t|/(2\xi)}e^{-|t|/(2\xi)}dt\right|^p dx \\
&\leqslant \text{(by Hölder's inequality, see [1, proof of Theorem 5])} \\
&\leqslant C_1\int_{-\infty}^{+\infty}\left(\int_{-\infty}^{+\infty}|\Delta_t^{n+1}f(x)|^p dx\right)e^{-|t|p/(2\xi)}dt \\
&\leqslant C_1\int_{-\infty}^{+\infty}[\omega_{n+1}(f;|t|)_X e^{-|t|/(2\xi)}]^p dt \\
&\leqslant 2C_1\omega_{n+1}^p(f;\xi)_X\int_0^{+\infty}[t/\xi + 1]^{(n+1)p}e^{-tp/(2\xi)}dt \\
&= \frac{2^{p-1}}{q^{p/q}}\omega_{n+1}^p(f;\xi)_X\int_0^{+\infty}(u+1)^{(n+1)p}e^{-pu/2}du,
\end{aligned}$$

which implies

$$\|f - P_{n,\xi}(f)\|_X \leqslant \left(\frac{2}{q}\right)^{1/q}\|g\|_{L^p(\mathbf{R}_+)}\omega_{n+1}(f;\xi)_X,$$

with $g(u) = (u+1)^{n+1}e^{-u/2}$, $u \in \mathbf{R}_+$.

Now, let $X = L^p_{2\pi}(\mathbf{R})$, $\frac{1}{p} + \frac{1}{q} = 1$, $Erf\,(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt$. By (5) we get

$$\int_{-\pi}^{\pi} |f\,(x) - W_{n,\xi}(f\,;x)|^p dx$$
$$= \frac{1}{[2C(\xi)]^p} \int_{-\pi}^{\pi} \left| \int_{-\pi}^{\pi} (-1)^{n+1} \Delta_t^{n+1} f\,(x) e^{-t^2/(2\xi^2)} e^{-t^2/(2\xi^2)} dt \right|^p dx$$

(by Hölder's inequality, see [1, proof of Theorem 5])

$$\leqslant \frac{1}{[2C(\xi)]^p} \left(\sqrt{\frac{2\pi}{q}} \xi \right)^{p/q} \left(Erf\, \left(\pi \sqrt{\frac{q}{2}} \cdot \frac{1}{\xi} \right) \right)^{p/q} \times$$
$$\times \left[\int_{-\pi}^{\pi} \left(\int_{-\pi}^{\pi} |\Delta_t^{n+1} f\,(x)|^p dx \right) e^{-t^2 p/(2\xi^2)} dt \right]$$

Denoting $C_2 = \frac{1}{[2C(\xi)]^p} \left(\sqrt{\frac{2\pi}{q}} \xi \right)^{p/q} \left(Erf\, \left(\pi \sqrt{\frac{q}{2}} \cdot \frac{1}{\xi} \right) \right)^{p/q}$, we have

$$\int_{-\pi}^{\pi} |f\,(x) - W_{n,\xi}(f\,;x)|^p dx \leqslant 2C_2 \omega_{n+1}^p (f\,;\xi)_X \int_0^{\pi} [t/\xi + 1]^{(n+1)p} e^{-t^2 p/(2\xi^2)} dt$$
$$\leqslant 2C_2 \omega_{n+1}^p (f\,;\xi)_X \xi \int_0^{\pi/\xi} [u+1]^{(n+1)p} e^{-u^2 p/2} du,$$

i.e.

$$\|f\, - W_{n,\xi}\|_{L^p_{2\pi}(\mathbf{R})} \leqslant \left[2C_2 \xi \int_0^{\pi/\xi} [u+1]^{(n+1)p} e^{-u^2 p/2} du \right]^{1/p} \omega_{n+1}(f\,;\xi)_X.$$

But

$$\left(Erf\, \left(\pi \sqrt{\frac{q}{2}} \cdot \frac{1}{\xi} \right) \right) = \frac{2}{\sqrt{\pi}} \int_0^{\pi\sqrt{\frac{q}{2}}/\xi} e^{-t^2} dt \leqslant \frac{2}{\sqrt{\pi}} \int_0^{+\infty} e^{-t^2} dt = 1$$

and by [2, Lemma 3.2] we have

$$\frac{1}{C(\xi)} \leqslant \frac{1}{\xi \int_0^{\pi} e^{-u^2} du}, \quad 0 < \xi \leqslant 1,$$

which implies

$$2C_2 \xi \leqslant \frac{2}{2^p} \cdot \frac{1}{\xi^p \left(\int_0^{\pi} e^{-u^2} du \right)^p} \left(\sqrt{\frac{2\pi}{q}} \right)^{p/q} \xi^{p/q} \xi$$
$$= \frac{1}{2^{p-1} \left(\int_0^{\pi} e^{-u^2} du \right)^p} \left(\sqrt{\frac{2\pi}{q}} \right)^{p/q},$$

and

$$\|f - W_{n,\xi}(f)\|_{L^p_{2\pi}(\mathbf{R})} \leqslant \left(\sqrt{\frac{\pi}{2q}}\right)^{1/q} \frac{1}{\displaystyle\int_0^{\pi} e^{-u^2} du} \|h\|_{L^p(\mathbf{R}_+)} \omega_{n+1}(f;\xi)_X,$$

where $h(u) = (u+1)^{n+1} e^{-u^2/2}$.

Finally, for $X = L^p_{2\pi}(\mathbf{R})$, $\frac{1}{p} + \frac{1}{q} = 1$, by (6) we get

$$\begin{aligned}
&\int_{-\pi}^{\pi} |f(x) - Q_{n,\xi}(f;x)|^p dx \\
&= \frac{1}{\left[\frac{2}{\xi}\tan^{-1}\frac{\pi}{\xi}\right]^p} \int_{-\pi}^{\pi} \left|\int_{-\pi}^{\pi} (-1)^{n+1} \Delta_t^{n+1} f(x) \frac{1}{(t^2+\xi^2)^{1/p}} \cdot \frac{1}{(t^2+\xi^2)^{1/q}} dt\right|^p dx \\
&\leqslant \text{(by Hölder's inequality, see [1, proof of Theorem 5])} \\
&\leqslant \frac{1}{\left[\frac{2}{\xi}\tan^{-1}\frac{\pi}{\xi}\right]^p} \left(\frac{2}{\xi}\tan^{-1}\frac{\pi}{\xi}\right)^{p/q} \int_{-\pi}^{\pi} \left[\int_{-\pi}^{\pi} |\Delta_t^{n+1} f(x)|^p \frac{1}{t^2+\xi^2} dx\right] dt \\
&\leqslant \frac{1}{\left[\frac{2}{\xi}\tan^{-1}\frac{\pi}{\xi}\right]} \int_{-\pi}^{\pi} \left[\omega_{n+1}(f;|t|)_X \frac{1}{(t^2+\xi^2)^{1/p}}\right]^p dt \\
&= \frac{\xi}{\tan^{-1}\frac{\pi}{\xi}} \int_0^{\pi} \left[\omega_{n+1}(f;t)_X \frac{1}{(t^2+\xi^2)^{1/p}}\right]^p dt \\
&\leqslant \frac{\xi}{\tan^{-1}\frac{\pi}{\xi}} \omega_{n+1}^p(f;\xi)_X \int_0^{\pi} [t/\xi+1]^{(n+1)p} \frac{1}{\xi^2} \cdot \frac{1}{(t/\xi)^2+1} dt \\
&= \frac{1}{\tan^{-1}\frac{\pi}{\xi}} \omega_{n+1}^p(f;\xi)_X \int_0^{\pi/\xi} [u+1]^{(n+1)p} \frac{1}{u^2+1} du,
\end{aligned}$$

which proves the theorem.

REMARK. Theorem 2.2 shows us that

$$\|f - P_{n,\xi}(f)\|_X \leqslant C_1 \omega_{n+1}(f;\xi)_X, \quad \|f - W_{n,\xi}(f)\|_X \leqslant C_2 \omega_{n+1}(f;\xi)_X$$

where $C_1, C_2 > 0$ are independent of f, n and ξ, while $K_p(n,\xi)$ in the third estimation (in Theorem 2.2) tends to $+\infty$ with $\xi \to 0$. In this case, as in Corollary 2.1 we can improve the estimation of $\|f - Q_{n,\xi}(f)\|_X$.

3. Uniform approximation by $Q_{n,\xi}$ operator

By (6) we easily get (for $X = C_{2\pi}(\mathbf{R})$)

$$|f(x) - Q_{n,\xi}(f;x)| \leqslant \frac{1}{\frac{2}{\xi}\tan^{-1}\frac{\pi}{\xi}} \int_{-\pi}^{\pi} |\Delta_t^{n+1} f(x)| \frac{1}{t^2+\xi^2} dt$$
$$\leqslant \frac{1}{\frac{2}{\xi}\tan^{-1}\frac{\pi}{\xi}} \int_{-\pi}^{\pi} \omega_{n+1}(f;|t|)_X \frac{1}{t^2+\xi^2} dt$$
$$= \frac{\xi}{\tan^{-1}\frac{\pi}{\xi}} \omega_{n+1}(f;\xi)_X \int_0^{\pi} [t/\xi+1]^{n+1} \frac{1}{t^2+\xi^2} dt,$$

which immediately implies

THEOREM 3.1. *For $0 < \xi \leqslant 1$, $n \in \mathbf{N}$, $f \in X = C_{2\pi}(\mathbf{R})$, we get the estimation*

$$\|f - Q_{n,\xi}(f)\|_X \leqslant K(n,\xi)\omega_{n+1}(f;\xi)_X,$$

where $K(n,\xi)$ is given by Theorem 2.1.

Reasoning exactly as in Corollary 2.1, we immediately obtain

COROLLARY 3.2. *If $f^{(n+1)} \in C_{2\pi}(\mathbf{R}) = X$, then*

$$\|f - Q_{n,\xi}(f)\|_X \leqslant C_n\xi, \quad 0 < \xi \leqslant 1,$$

where $C_n > 0$ is independent of f and ξ.

REMARK. The results of this paper show us that while the generalized operators $P_{n,\xi}$ and $W_{n,\xi}$ give better estimates than the classical operators of Picard and of Gauss-Weierstrass, the same idea of generalization applied to the Poisson-Cauchy singular integral, which produces the $Q_{n,\xi}$ -operator, does not give a better estimate.

REFERENCES

[1] G. A. ANASTASSIOU AND S. G. GAL, *General theory of global smoothness preservation by singular integrals, univariate case*, J. Comp. Anal. Appl. 1, No. 3(1999), 289–317.

[2] S. G. GAL, *Degree of approximation of continuous functions by some singular integrals*, Rev. Anal. Numér. Théor. Approx. (Cluj), Tome XXVII, No. 2(1998), 251–261.

(Received September 2, 1999)

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